(T,S)-BASED INTERVAL-VALUED INTUITIONISTIC FUZZY COMPOSITION MATRIX AND ITS APPLICATION FOR CLUSTERING

H. L. HUANG

Abstract. In this paper, the notions of (T,S)-composition matrix and (T,S)-interval-valued intuitionistic fuzzy equivalence matrix are introduced where (T,S) is a dual pair of triangular module. They are the generalization of composition matrix and interval-valued intuitionistic fuzzy equivalence matrix. Furthermore, their properties and characterizations are presented. Then a new method based on $\tilde{\alpha}-$matrix for clustering is developed. Finally, an example is given to demonstrate our method.

1. Introduction

Atanassov [1] introduced the intuitionistic fuzzy set (IFS) which is characterized by a membership function and a non-membership function. Then in [4] , Atanassov and Gargov extended the notion of the IFS to the interval-valued intuitionistic fuzzy set (IVIFS) whose membership function and non-membership function are intervals. The IFSs and IVIFSs are also called double fuzzy sets and interval-valued double fuzzy sets respectively in [10]. In the past few years, the IFS and IVIFS have received much attention from researchers and have been applied in logic programming, medical diagnosis, pattern recognition, decision making, clustering, and other disciplines. Atanassov [2] and Deschrijver [9] introduced some relations and operations of IVIFS. Intuitionistic fuzzy bounded linear operators were taken into account by Narayanan [18]. Some authors [7, 22, 23, 24, 25, 26, 29, 30, 31, 34, 35, 36] proposed different methods for decision making under intuitionistic or interval-valued intuitionistic fuzzy environment. The correlation coefficients of IFS and IVIFS were investigated in [6, 13, 19, 27, 37] from different points of view. IFS theory and IVIFS theory were also considered in the rough set theory and had been used for attribute reduction [14, 40, 41]. By using L-fuzzy set theory [11, 12], Coker [8] proved that fuzzy rough sets are intuitionistic L-fuzzy sets. In Algebra, Borzooei et al. [5, 15, 20] studied different kinds of intuitionistic fuzzy ideals. Topology of interval-valued intuitionistic fuzzy sets was defined by Mondal and some of its properties were presented in [17].

Received: January 2011; Revised: July 2011; Accepted: August 2011

Key words and phrases: Clustering, Interval-valued intuitionistic fuzzy set, Interval-valued intuitionistic fuzzy number, Interval-valued intuitionistic fuzzy matrix, Triangular dual module.

This work was supported by the National Natural Science Foundation of China (10971186), the Natural Science Foundation of Fujian Province, China (2011J05013), the Science Research Foundation of Zhangzhou Normal University, China (SJ1005).

Although there are many researchs on IFSs and IVIFSs, the clustering technique of IFSs and IVIFSs still at an initial stage [24, 28, 32, 33, 39]. Xu [24, 32] gave the notion of interval-valued intuitionistic fuzzy equivalence matrix (IVIFEM) and then presented a method for clustering IVIFSs. The operators involved in the papers above are mainly $\bigwedge$ and $\bigvee$. In a previous paper [16], under the intuitionistic fuzzy environment, we have generalized the operators $\bigwedge$ and $\bigvee$ to $t-$norm and $t-$conorm respectively, and improved the method proposed in [39]. On the other hand, if we finally have a matrix whose elements are interval-valued intuitionistic fuzzy numbers (IVIFNs) for clustering, the method in [32] can not be used directly. Therefore, we discuss more general operators: triangular dual module, and provide a new method for clustering IVIFSs.

The paper is organized as follows: In section 2, some basic concepts, relations, operations and operators are reviewed. In section 3, the notions of $(T,S)-$composition matrix and $(T,S)-$interval-valued intuitionistic fuzzy equivalence matrix are introduced where (T,S) is a dual pair of triangular module. Then their properties are studied. In section 4, a new method for clustering with interval-valued intuitionistic fuzzy information is proposed. In section 5, an illustrative example is given. Finally a conclusion is provided in section 6.

2. Preliminaries

Definition 2.1. [4] Let a Set $X = \{x_1, x_2, \ldots, x_n\}$ be fixed, then an IVIFS $\tilde{A}$ over X is an object having the form: $\tilde{A} = \{< x_i, \tilde{\mu}_{\tilde{A}}(x_i), \tilde{\nu}_{\tilde{A}}(x_i) > \mid x_i \in X\}$ where $\tilde{\mu}_{\tilde{A}}(x_i) \subseteq [0,1]$ and $\tilde{\nu}_{\tilde{A}}(x_i) \subseteq [0,1]$ are intervals, and for every $x_i \in X$, $sup\tilde{\mu}_{\tilde{A}}(x_i) + sup\tilde{\nu}_{\tilde{A}}(x_i) \leq 1$.

Especially if $\inf\tilde{\mu}_{\tilde{A}}(x_i) = \sup\tilde{\mu}_{\tilde{A}}(x_i) = \beta$ and $\inf\tilde{\nu}_{\tilde{A}}(x_i) = \sup\tilde{\nu}_{\tilde{A}}(x_i) = \gamma$, then the IVIFS $\tilde{A}$ reduces to an intuitionistic fuzzy set (IFS) [1]. Furthermore, if $\beta + \gamma = 1$, then the IFS $\tilde{A}$ reduces to a fuzzy set (FS) [38].

For convenience we denote an interval-valued intuitionistic fuzzy number (IVIFN) by $\tilde{\alpha} = ([a,b],[c,d])$ where $[a,b] \subseteq [0,1]$, $[c,d] \subseteq [0,1]$ and $b + d \leq 1$ (for more details see [24]). Then the relationship for any two IVIFNs is defined as follows:

Definition 2.2. [24] Given two IVIFNs $\tilde{\alpha}_i = ([a_i, b_i],[c_i, d_i])$ $(i = 1, 2)$, then

(1) $\tilde{\alpha}_1 \subseteq \tilde{\alpha}_2$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$ and $c_1 \geq c_2$ and $d_1 \geq d_2$;

(2) $\tilde{\alpha}_1 = \tilde{\alpha}_2$ iff $a_1 = a_2$ and $b_1 = b_2$ and $c_1 = c_2$ and $d_1 = d_2$;

(3) $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 = ([a_1 \wedge a_2, b_1 \wedge b_2],[c_1 \vee c_2, d_1 \vee d_2])$;

(4) $\tilde{\alpha}_1 \cup \tilde{\alpha}_2 = ([a_1 \vee a_2, b_1 \vee b_2],[c_1 \wedge c_2, d_1 \wedge d_2])$.

Definition 2.3. [24] Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be an $n \times n$ matrix. If all $\tilde{p}_{ij}$ $(i,j = 1, 2, \ldots, n)$ are IVIFNs, then we call $\tilde{P}$ an interval-valued intuitionistic fuzzy matrix (IVIFM).

Definition 2.4. [24] Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be an IVIFM, where $\tilde{p}_{ij} = ([a_{ij}, b_{ij}],[c_{ij}, d_{ij}])$ $i,j = 1, 2, \ldots, n$. If $\tilde{P}$ satisfies the following conditions:

(1) Reflexivity: $\tilde{p}_{ii} = ([1,1],[0,0])$, $i = 1, 2, \ldots, n$;

(2) Symmetry: $\tilde{p}_{ij} = \tilde{p}_{ji}$, i.e., $([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]) = ([a_{ji}, b_{ji}], [c_{ji}, d_{ji}])$, $i, j = 1, 2, \ldots, n$;

(3) Transitivity: $\tilde{P} \circ \tilde{P} \subseteq \tilde{P}$, i.e., $\bigcup\limits_{k=1}^{n} (\tilde{p}_{ik} \bigcap \tilde{p}_{kj}) \subseteq \tilde{p}_{ij}$, i.e., $\bigvee\limits_{k}(a_{ik} \wedge a_{kj}) \leq a_{ij}$, $\bigvee\limits_{k}(b_{ik} \wedge b_{kj}) \leq b_{ij}$, $\bigwedge\limits_{k}(c_{ik} \vee c_{kj}) \geq c_{ij}$, $\bigwedge\limits_{k}(d_{ik} \vee d_{kj}) \geq d_{ij}$;

then we call $\tilde{P}$ an interval-valued intuitionistic fuzzy equivalence matrix (IVIFEM). If $\tilde{P}$ only satisfies the conditions (1) and (2), then we call $\tilde{P}$ an interval-valued intuitionistic fuzzy similarity matrix (IVIFSM).

Definition 2.5. [21] Mapping $T : [0,1]^2 \to [0,1]$ is called a triangular norm, if $\forall a, b, c, d \in [0,1]$, the following conditions are true:

(1) $T(0,0) = 0$, $T(1,1) = 1$;

(2) if $a \leq c$ and $b \leq d$, then $T(a,b) \leq T(c,d)$;

(3) $T(a,b) = T(b,a)$;

(4) $T(T(a,b),c) = T(a, T(b,c))$.

Furthermore, $\forall a \in [0,1]$, T is a t-norm if $T(a,1) = a$, T is a t-conorm if $T(0,a) = a$.

In this paper, T always denotes the t-norm and S denotes the t-conorm, they have those properties:

Lemma 2.6. [21] $\forall a, b \in [0,1]$*, we have*

(1) $0 \leq T(a,b) \leq a \wedge b \leq 1$*, that is* $T \leq \wedge$ *and* $T(0,a) = 0$*;*

(2) $0 \leq a \vee b \leq S(a,b) \leq 1$*, that is* $\vee \leq S$ *and* $S(a,1) = 1$.

Definition 2.7. [21] Let $a \in [0,1]$, $a' = 1 - a$ is the complement of a, then T and S is called a dual pair of triangular module, if T and S satisfy $T(a,b)' = S(a',b')$, $\forall a, b \in [0,1]$.

Throughout this paper, (T,S) denotes a dual pair of triangular module.

3. Main Results

Definition 3.1. Let $\tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2])$ be any two IVIFNs, we define the (T,S)-union and (T,S)-intersection of IVIFNs as follows:

(1) $\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2 = ([T(a_1,a_2), T(b_1,b_2)], [S(c_1,c_2), S(d_1,d_2)])$;
(2) $\tilde{\alpha}_1 \dot{\cup} \tilde{\alpha}_2 = ([S(a_1,a_2), S(b_1,b_2)], [T(c_1,c_2), T(d_1,d_2)])$.

Remark 3.2. When we choose $T(a,b) = a \wedge b$ and $S(a,b) = a \vee b$, then the (T,S)-union and (T,S)-intersection of IVIFNs reduces to the union and intersection of IVIFNs in [24] (See Definition 2.2).

With Definition 2.5, Definition 2.7 and Definition 3.1, we can have:

Theorem 3.3. *The* (T,S)*-union and* (T,S)*-intersection of any two IVIFNs are still an IVIFN.*

Proof. Let $\tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2])$ be any two IVIFNs, we only give the proof of $\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2$, and $\tilde{\alpha}_1 \dot{\cup} \tilde{\alpha}_2$ is analogous.

By $a_1 \le b_1$ and $a_2 \le b_2$, we have $0 \le T(a_1, a_2) \le T(b_1, b_2) \le 1$. Analogously we can have $0 \le S(c_1, c_2) \le S(d_1, d_2) \le 1$.

From $b_1 + d_1 \le 1$ and $b_2 + d_2 \le 1$, we have $S(d_1, d_2) \le S(1 - b_1, 1 - b_2)$. Furthermore, $S(1 - b_1, 1 - b_2) = S(b_1', b_2') = T(b_1, b_2)' = 1 - T(b_1, b_2)$. These imply that $T(b_1, b_2) + S(d_1, d_2) \le 1$.

So $\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2$ is an IVIFN. $\square$

Theorem 3.4. *Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, 3)$ be any three IVIFNs, then*

(1) $\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2 = \tilde{\alpha}_2 \dot{\cap} \tilde{\alpha}_1$;

(2) $\tilde{\alpha}_1 \dot{\cup} \tilde{\alpha}_2 = \tilde{\alpha}_2 \dot{\cup} \tilde{\alpha}_1$;

(3) $(\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2) \dot{\cap} \tilde{\alpha}_3 = \tilde{\alpha}_1 \dot{\cap} (\tilde{\alpha}_2 \dot{\cap} \tilde{\alpha}_3)$;

(4) $(\tilde{\alpha}_1 \dot{\cup} \tilde{\alpha}_2) \dot{\cup} \tilde{\alpha}_3 = \tilde{\alpha}_1 \dot{\cup} (\tilde{\alpha}_2 \dot{\cup} \tilde{\alpha}_3)$.

Proof. We only give the proof of (3), the others can be easily obtained.

$$\begin{aligned}
(\tilde{\alpha}_1 \dot{\cap} \tilde{\alpha}_2) \dot{\cap} \tilde{\alpha}_3 &= ([T(a_1, a_2), T(b_1, b_2)], [S(c_1, c_2), S(d_1, d_2)]) \dot{\cap} ([a_3, b_3], [c_3, d_3]) \\
&= ([T(T(a_1, a_2), a_3), T(T(b_1, b_2), b_3)], [S(S(c_1, c_2), c_3), S(S(d_1, d_2), d_3)]) \\
&= ([T(a_1, T(a_2, a_3)), T(b_1, T(b_2, b_3))], [S(c_1, S(c_2, c_3)), S(d_1, S(d_2, d_3))]) \\
&= \tilde{\alpha}_1 \dot{\cap} (\tilde{\alpha}_2 \dot{\cap} \tilde{\alpha}_3)
\end{aligned}$$

Therefore, (3) is proved. $\square$

Remark 3.5. In [3], it is shown that the De Morgan's Laws are not valid for a number of negations that can be defined over IFSs (and, respectively, on IVIFSs). These laws are valid only for the classical negation.

Definition 3.6. Let $\tilde{P}_l = (\tilde{p}_{ij}^{(l)})_{n \times n}$ $(l = 1, 2)$ be two IVIFMs, where $\tilde{p}_{ij}^{(l)} = ([a_{ij}^{(l)}, b_{ij}^{(l)}], [c_{ij}^{(l)}, d_{ij}^{(l)}])$, $i, j = 1, 2, \ldots, n$, $l = 1, 2$. We say $\tilde{P} = \tilde{P}_1 \times \tilde{P}_2 = (\tilde{p}_{ij})_{n \times n}$ is a $(T, S)-$composition matrix of $\tilde{P}_1$ and $\tilde{P}_2$ where

$$\begin{aligned}
\tilde{p}_{ij} &= \bigvee_{k=1}^{n} (\tilde{p}_{ik}^{(1)} \dot{\cap} \tilde{p}_{kj}^{(2)}) \\
&= ([\bigvee_{k=1}^{n} T(a_{ik}^{(1)}, a_{kj}^{(2)}), \bigvee_{k=1}^{n} T(b_{ik}^{(1)}, b_{kj}^{(2)})], [\bigwedge_{k=1}^{n} S(c_{ik}^{(1)}, c_{kj}^{(2)}), \bigwedge_{k=1}^{n} S(d_{ik}^{(1)}, d_{kj}^{(2)})])
\end{aligned}$$

Remark 3.7. If T and S are replaced by $\bigwedge$ and $\bigvee$ respectively, then $\tilde{P}_1 \times \tilde{P}_2$ becomes a composition matrix $\tilde{P}_1 \circ \tilde{P}_2$ in [24].

By Definition 3.6, we can obtain further results as follows:

Theorem 3.8. *Let $\tilde{P}_l = (\tilde{p}_{ij}^{(l)})_{n \times n}$ $(l = 1, 2)$ be two IVIFMs, where $\tilde{p}_{ij}^{(l)} = ([a_{ij}^{(l)}, b_{ij}^{(l)}]$, $[c_{ij}^{(l)}, d_{ij}^{(l)}])$, $i, j = 1, 2, \ldots, n$, $l = 1, 2$. Then the $(T, S)-$composition matrix of $\tilde{P}_1$ and $\tilde{P}_2$ is still an IVIFM.*

Proof. By the proof of Theorem 3.3, for $i, j = 1, 2, \ldots, n$, we can easily obtain

$$[\bigvee_{k=1}^{n} T(a_{ik}^{(1)}, a_{kj}^{(2)}), \bigvee_{k=1}^{n} T(b_{ik}^{(1)}, b_{kj}^{(2)})] \subseteq [0,1]$$

and

$$[\bigwedge_{k=1}^{n} S(c_{ik}^{(1)}, c_{kj}^{(2)}), \bigwedge_{k=1}^{n} S(d_{ik}^{(1)}, d_{kj}^{(2)})] \subseteq [0,1].$$

Further, there must exist l_1 and l_2 such that

$$\bigvee_{k=1}^{n} T(b_{ik}^{(1)}, b_{kj}^{(2)}) = T(b_{il_1}^{(1)}, b_{l_1j}^{(2)}), \quad \bigwedge_{k=1}^{n} S(d_{ik}^{(1)}, d_{kj}^{(2)}) = S(d_{il_2}^{(1)}, d_{l_2j}^{(2)}).$$

Either $l_1 = l_2$ or $l_1 \neq l_2$, the following inequation always holds:

$$T(b_{il_1}^{(1)}, b_{l_1j}^{(2)}) + S(d_{il_2}^{(1)}, d_{l_2j}^{(2)}) \leq T(b_{il_1}^{(1)}, b_{l_1j}^{(2)}) + S(d_{il_1}^{(1)}, d_{l_1j}^{(2)}) \leq 1,$$

that is

$$\bigvee_{k=1}^{n} T(b_{ik}^{(1)}, b_{kj}^{(2)}) + \bigwedge_{k=1}^{n} S(d_{ik}^{(1)}, d_{kj}^{(2)}) \leq 1,$$

these complete the proof. □

By Theorem 3.8, Definition 2.2 and Lemma 2.6, we have the following corollaries.

Corollary 3.9. *Let $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$ be any three IVIFMs, and $\tilde{P}_1 \subseteq \tilde{P}_2$, then $\tilde{P}_1 \times \tilde{P}_3 \subseteq \tilde{P}_2 \times \tilde{P}_3 \subseteq \tilde{P}_2 \circ \tilde{P}_3$.*

Corollary 3.10. *The $(T,S)-$composition matrix $\tilde{P}$ of IVIFSMs $\tilde{P}_1$ and $\tilde{P}_2$ is also an IVIFM.*

However, the $(T,S)-$composition matrix of two IVIFSMs may not be an IVIFSM. See the following example:

Example 3.11. Let

$$\tilde{P}_1 = \begin{bmatrix} ([1,1],[0,0]) & ([0.2,0.3],[0.4,0.6]) & ([0.1,0.5],[0.2,0.4]) \\ ([0.2,0.3],[0.4,0.6]) & ([1,1],[0,0]) & ([0.4,0.6],[0.3,0.4]) \\ ([0.1,0.5],[0.2,0.4]) & ([0.4,0.6],[0.3,0.4]) & ([1,1],[0,0]) \end{bmatrix},$$

$$\tilde{P}_2 = \begin{bmatrix} ([1,1],[0,0]) & ([0.1,0.2],[0.3,0.5]) & ([0.2,0.6],[0.1,0.4]) \\ ([0.1,0.2],[0.3,0.5]) & ([1,1],[0,0]) & ([0.5,0.6],[0.2,0.3]) \\ ([0.2,0.6],[0.1,0.4]) & ([0.5,0.6],[0.2,0.3]) & ([1,1],[0,0]) \end{bmatrix}.$$

Then both $\tilde{P}_1$ and $\tilde{P}_2$ are IVIFSMs, and their $(T,S)-$composition matrix is

$$\tilde{P} = \tilde{P}_1 \times \tilde{P}_2 = \begin{bmatrix} ([1,1],[0,0]) & ([0.2,0.3],[0.3,0.5]) & ([0.2,0.6],[0.1,0.4]) \\ ([0.2,0.36],[0.3,0.5]) & ([1,1],[0,0]) & ([0.5,0.6],[0.2,0.3]) \\ ([0.2,0.6],[0.1,0.4]) & ([0.5,0.6],[0.2,0.3]) & ([1,1],[0,0]) \end{bmatrix}$$

if we choose $T(a,b) = ab$ and $S(a,b) = a+b-ab$. Since $\tilde{p}_{12} \neq \tilde{p}_{21}$, $\tilde{P}$ is asymmetrical.

Fortunately, the following theorem holds:

Theorem 3.12. *Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be an IVIFSM, then the $(T,S)-$composition matrix $\dot{\tilde{P}}^2 \triangleq \tilde{P} \times \tilde{P}$ is also an IVIFSM.*

Proof. Let $\dot{\tilde{P}}^2 = (\tilde{q}_{ij})_{n\times n}$ and $\tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$.

(1) By Corollary 3.10, we know that $\dot{\tilde{P}}^2$ is an IVIFM.

(2) By Definition 3.6, for $i = 1, 2, \ldots, n$, we can have

$$\tilde{q}_{ii} = ([\bigvee_{k=1}^{n} T(a_{ik}, a_{ki}), \bigvee_{k=1}^{n} T(b_{ik}, b_{ki})], [\bigwedge_{k=1}^{n} S(c_{ik}, c_{ki}), \bigwedge_{k=1}^{n} S(d_{ik}, d_{ki})]).$$

When $k = i$, we have $\tilde{p}_{ii} = ([a_{ii}, b_{ii}], [c_{ii}, d_{ii}]) = ([1, 1], [0, 0])$. Since $T(1, 1) = 1$ and $S(0, 0) = 0$, $\tilde{q}_{ii} = ([1, 1], [0, 0])$ $(i = 1, 2, \ldots, n)$.

(3) Since $\tilde{P}$ is an IVIFSM, $\tilde{p}_{ik} = \tilde{p}_{ki}$. Thus we have

$$\begin{aligned}
\tilde{q}_{ij} &= ([\bigvee_{k=1}^{n} T(a_{ik}, a_{kj}), \bigvee_{k=1}^{n} T(b_{ik}, b_{kj})], [\bigwedge_{k=1}^{n} S(c_{ik}, c_{kj}), \bigwedge_{k=1}^{n} S(d_{ik}, d_{kj})]) \\
&= ([\bigvee_{k=1}^{n} T(a_{kj}, a_{ik}), \bigvee_{k=1}^{n} T(b_{kj}, b_{ik})], [\bigwedge_{k=1}^{n} S(c_{kj}, c_{ik}), \bigwedge_{k=1}^{n} S(d_{kj}, d_{ik})]) \\
&= ([\bigvee_{k=1}^{n} T(a_{jk}, a_{ki}), \bigvee_{k=1}^{n} T(b_{jk}, b_{ki})], [\bigwedge_{k=1}^{n} S(c_{jk}, c_{ki}), \bigwedge_{k=1}^{n} S(d_{jk}, d_{ki})]) \\
&= \tilde{q}_{ji}.
\end{aligned}$$

Therefore, $\dot{\tilde{P}}^2$ is an IVIFSM. $\square$

Definition 3.13. Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be an IVIFM, where $\tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ $i, j = 1, 2, \ldots, n$. If $\tilde{P}$ satisfies the following conditions:

(1) Reflexivity: $\tilde{p}_{ii} = ([1, 1], [0, 0])$, $i = 1, 2, \ldots, n$;

(2) Symmetry: $\tilde{p}_{ij} = \tilde{p}_{ji}$, i.e., $([a_{ij}, b_{ij}], [c_{ij}, d_{ij})] = ([a_{ji}, b_{ji}], [c_{ji}, d_{ji}])$, $i, j = 1, 2, \ldots, n$;

(3) $(T, S)-$Transitivity: $\tilde{P} \times \tilde{P} \subseteq \tilde{P}$, i.e., $\bigcup_{k=1}^{n} T(\tilde{p}_{ik} \dot{\bigcap} \tilde{p}_{kj}) \subseteq \tilde{p}_{ij}$, i.e., $\bigvee_{k} T(a_{ik}, a_{kj}) \leq a_{ij}$, $\bigvee_{k} T(b_{ik}\, b_{kj}) \leq b_{ij}$, $\bigwedge_{k} S(c_{ik}, c_{kj}) \geq c_{ij}$, $\bigwedge_{k} S(d_{ik}, d_{kj}) \geq d_{ij}$;

then we call $\tilde{P}$ a $(T, S)-$interval-valued intuitionistic fuzzy equivalence matrix ($(T, S)-$IVIFEM).

Remark 3.14. If T and S are replaced by $\bigwedge$ and $\bigvee$ respectively, then Definition 3.13 reduces to Definition 2.4. That is, an IVIFEM is a $(T, S)-$IVIFEM. However, a $(T, S)-$IVIFEM may not be an IVIFEM, we can see an example as follows:

Example 3.15. Let

$$\tilde{P} = \begin{bmatrix} ([1, 1], [0, 0]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.1, 0.5], [0.2, 0.4]) \\ ([0.2, 0.3], [0.4, 0.6]) & ([1, 1], [0, 0]) & ([0.4, 0.6], [0.3, 0.4]) \\ ([0.1, 0.5], [0.2, 0.4]) & ([0.4, 0.6], [0.3, 0.4]) & ([1, 1], [0, 0]) \end{bmatrix},$$

then $\tilde{P}$ satisfies conditions (1) and (2) in Definition 3.13, and

$$\tilde{P} \circ \tilde{P} = \begin{bmatrix} ([1,1],[0,0]) & ([0.2,0.5],[0.3,0.4]) & ([0.2,0.5],[0.2,0.4]) \\ ([0.2,0.5],[0.3,0.4]) & ([1,1],[0,0]) & ([0.4,0.6],[0.3,0.4]) \\ ([0.2,0.5],[0.2,0.4]) & ([0.4,0.6],[0.3,0.4]) & ([1,1],[0,0]) \end{bmatrix} \nsubseteq \tilde{P},$$

so $\tilde{P}$ is not an IVIFEM. On the other hand,

$$\tilde{P} \times \tilde{P} = \begin{bmatrix} ([1,1],[0,0]) & ([0.2,0.3],[0.4,0.6]) & ([0.1,0.5],[0.2,0.4]) \\ ([0.2,0.3],[0.4,0.6]) & ([1,1],[0,0]) & ([0.4,0.6],[0.3,0.4]) \\ ([0.1,0.5],[0.2,0.4]) & ([0.4,0.6],[0.3,0.4]) & ([1,1],[0,0]) \end{bmatrix} = \tilde{P}$$

if we choose $T(a,b) = ab$ and $S(a,b) = a+b-ab$, it implies that $\tilde{P}$ is a $(T,S)-$IVIFEM.

In order to clustering, we define the following new notion.

Definition 3.16. Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ be an IVIFM $(i,j = 1,2,\ldots,n)$, $\tilde{\alpha}$ be an IVIFN. We call $\tilde{P}_{\tilde{\alpha}} = (\tilde{p}_{ij} : \tilde{\alpha})_{n\times n}$ is an $\tilde{\alpha}-$matrix of $\tilde{P}$, where

$$\tilde{p}_{ij} : \tilde{\alpha} = ([a_{ij} : \tilde{\alpha}, b_{ij} : \tilde{\alpha}], [c_{ij} : \tilde{\alpha}, d_{ij} : \tilde{\alpha}]) = \begin{cases} ([1,1],[0,0]), & \text{if } \tilde{p}_{ij} \supseteq \tilde{\alpha}, \\ ([0,0],[1,1]), & \text{if } \tilde{p}_{ij} \nsupseteq \tilde{\alpha}. \end{cases}$$

An IVIFM $\tilde{P}$ and its $\tilde{\alpha}-$matrix $\tilde{P}_{\tilde{\alpha}}$ have the following relationships.

Theorem 3.17. *Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be an IVIFSM, then $\tilde{P}_{\tilde{\alpha}}$ is also an IVIFSM.*

Proof. (1) Reflexivity: Since $\tilde{p}_{ii} = ([1,1],[0,0])$, we have for all IVIFN $\tilde{\alpha}$,that $\tilde{\alpha} \subseteq \tilde{p}_{ii}$. By Definition 3.16, we know that $\tilde{p}_{ii} : \tilde{\alpha} = ([1,1],[0,0])$ (i=1,2,. . . ,n).

(2) Symmetry: By $\tilde{p}_{ij} = \tilde{p}_{ji}$, we can easily obtain that $\tilde{p}_{ij} : \tilde{\alpha} = \tilde{p}_{ji} : \tilde{\alpha}$ for all IVIFN $\tilde{\alpha}$ $(i,j = 1,2,\ldots,n)$. □

Theorem 3.18. *If $\tilde{P}_{\tilde{\alpha}}$ is a $(T,S)-$IVIFEM, then $\tilde{P}$ is a $(T,S)-$IVIFEM.*

Proof. (1) Reflexivity: Since $\tilde{p}_{ii} : \tilde{\alpha} = ([1,1],[0,0])$, we know that for all IVIFN $\tilde{\alpha} = ([a,b],[c,d])$, $\tilde{\alpha} \subseteq \tilde{p}_{ii} = ([a_{ii}, b_{ii}], [c_{ii}, d_{ii}])$, that is $a \leq a_{ii}, b \leq b_{ii}, c \geq c_{ii}$ and $d \geq d_{ii}$. Let $\tilde{\alpha} = ([1,1],[0,0])$, then $\tilde{p}_{ii} = ([1,1],[0,0])$ $(i = 1,2,\ldots,n)$.

(2) Symmetry: Since $\tilde{P}_{\tilde{\alpha}}$ is a $(T,S)-$IVIFEM, we have $\tilde{p}_{ij} : \tilde{\alpha} = \tilde{p}_{ji} : \tilde{\alpha}(i,j = 1,2,\ldots,n)$.

If $\tilde{p}_{ij} \neq \tilde{p}_{ji}$, then $a_{ij} \neq a_{ji}$ or $b_{ij} \neq b_{ji}$ or $c_{ij} \neq c_{ji}$ or $d_{ij} \neq d_{ji}$. Suppose that $a_{ij} < a_{ji}$, let $\tilde{\alpha} = ([(a_{ij} + a_{ji})/2, b_{ji}], [c_{ji}, d_{ji}])$, then $a_{ij} < (a_{ij} + a_{ji})/2 < a_{ji}$. So $\tilde{p}_{ij} : \tilde{\alpha} = ([0,0],[1,1])$, $\tilde{p}_{ji} : \tilde{\alpha} = ([1,1],[0,0])$. $\tilde{p}_{ij} : \tilde{\alpha} \neq \tilde{p}_{ji} : \tilde{\alpha}$ which is a contradiction. Thus $a_{ij} = a_{ji}$ $(i,j = 1,2,\ldots,n)$.

Analogously we can prove that $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$ and $d_{ij} = d_{ji}$ should be simultaneously by hold. It implies that $\tilde{p}_{ij} = \tilde{p}_{ji}$ $(i,j = 1,2,\ldots,n)$.

(3) $(T,S)-$Transitivity: Since $\tilde{P}_{\tilde{\alpha}} = (\tilde{p}_{ij} : \tilde{\alpha}) = ([a_{ij} : \tilde{\alpha}, b_{ij} : \tilde{\alpha}], [c_{ij} : \tilde{\alpha}, d_{ij} : \tilde{\alpha}])$ satisfies $(T,S)-$Transitivity, that is, for $i,j = 1,2,\ldots,n$, we have

$$\bigvee_{k=1}^{n} T(a_{ik} : \tilde{\alpha}, a_{kj} : \tilde{\alpha}) \leq a_{ij} : \tilde{\alpha}, \quad \bigvee_{k=1}^{n} T(b_{ik} : \tilde{\alpha}, b_{kj} : \tilde{\alpha}) \leq b_{ij} : \tilde{\alpha},$$

$$\bigwedge_{k=1}^{n} S(c_{ik}:\tilde{\alpha}, c_{kj}:\tilde{\alpha}) \geq c_{ij}:\tilde{\alpha}, \quad \bigwedge_{k=1}^{n} S(d_{ik}:\tilde{\alpha}, d_{kj}:\tilde{\alpha}) \geq d_{ij}:\tilde{\alpha}.$$

Now we will prove that for $i, j = 1, 2, \ldots, n$,

$$\bigvee_{k=1}^{n} T(a_{ik}, a_{kj}) \leq a_{ij}, \quad \bigvee_{k=1}^{n} T(b_{ik}, b_{kj}) \leq b_{ij},$$
$$\bigwedge_{k=1}^{n} S(c_{ik}, c_{kj}) \geq c_{ij}, \quad \bigwedge_{k=1}^{n} S(d_{ik}, d_{kj}) \geq d_{ij}.$$

Suppose that there exist i_0 and j_0 such that $\bigvee_{k=1}^{n}(a_{i_0k} \wedge a_{kj_0}) > a_{i_0j_0}$. Further we know that there exists l such that $a_{i_0l} \wedge a_{lj_0} > a_{i_0j_0}$. Assume that $a_{i_0l} \geq a_{lj_0} > a_{i_0j_0}$ and let $\tilde{\alpha} = ([a_{lj_0}, b_{i_0l} \wedge b_{lj_0}], [c_{i_0l} \vee c_{lj_0}, d_{i_0l} \vee d_{lj_0}])$, then

$$\tilde{p}_{i_0l}:\tilde{\alpha} = \tilde{p}_{lj_0}:\tilde{\alpha} = ([1,1],[0,0]) \text{ and } \tilde{p}_{i_0j_0}:\tilde{\alpha} = ([0,0],[1,1])$$

hold. These imply that $\bigvee_{k=1}^{n} T(a_{i_0k}:\tilde{\alpha}, a_{kj_0}:\tilde{\alpha}) = 1 > 0 = a_{i_0j_0}:\tilde{\alpha}$, which produces the contradiction with $\bigvee_{k=1}^{n} T(a_{ik}:\tilde{\alpha}, a_{kj}:\tilde{\alpha}) \leq a_{ij}:\tilde{\alpha}$. So for $i, j = 1, 2, \ldots, n$, we have $\bigvee_{k=1}^{n} T(a_{ik}, a_{kj}) \leq \bigvee_{k=1}^{n}(a_{ik} \wedge a_{kj}) \leq a_{ij}$.

Analogously we can prove that $\bigvee_{k=1}^{n} T(b_{ik}, b_{kj}) \leq b_{ij}$.

On the other hand, suppose that there exist i_1 and j_1 such that $\bigwedge_{k=1}^{n}(c_{i_1k} \vee c_{kj_1}) < c_{i_1j_1}$. Further we know that there exists m such that $c_{i_1m} \vee c_{mj_1} < c_{i_1j_1}$. Assume that $c_{i_1m} \leq c_{mj_1} < c_{i_1j_1}$ and let $\tilde{\alpha} = ([a_{i_1m} \wedge a_{mj_1}, b_{i_1m} \wedge b_{mj_1}], [c_{mj_1}, d_{i_1m} \vee d_{mj_1}])$, then

$$\tilde{p}_{i_1m}:\tilde{\alpha} = \tilde{p}_{mj_1}:\tilde{\alpha} = ([1,1],[0,0]) \text{ and } \tilde{p}_{i_1j_1}:\tilde{\alpha} = ([0,0],[1,1])$$

hold. These imply that $\bigvee_{k=1}^{n} T(a_{i_1k}:\tilde{\alpha}, a_{kj_1}:\tilde{\alpha}) = 1 > 0 = a_{i_1j_1}:\tilde{\alpha}$, which produces the contradiction with $\bigvee_{k=1}^{n} T(a_{ik}:\tilde{\alpha}, a_{kj}:\tilde{\alpha}) \leq a_{ij}:\tilde{\alpha}$. So for $i, j = 1, 2, \ldots, n$, we have $\bigwedge_{k=1}^{n} S(c_{ik}, c_{kj}) \geq \bigwedge_{k=1}^{n}(c_{ik} \vee c_{kj}) \geq c_{ij}$.

Analogously we can prove that $\bigwedge_{k=1}^{n} S(d_{ik}, d_{kj}) \geq d_{ij}$.

Therefore, we can complete the proof. □

If we denote $\dot{\tilde{P}}^{2^k} = \dot{\tilde{P}}^{2^{k-1}} \times \dot{\tilde{P}}^{2,^{k-1}}$ then by the idea of [21], we can easily prove the following theorem.

Theorem 3.19. *Let $\tilde{P}$ be an IVIFSM, then after a finite times of $(T,S)-$compositions: $\tilde{P} \rightarrow \dot{\tilde{P}}^2 \rightarrow \dot{\tilde{P}}^4 \rightarrow \cdots \rightarrow \dot{\tilde{P}}^{2^k} \rightarrow \cdots$, there must exist a positive integer k such that $\dot{\tilde{P}}^{2^k} = \dot{\tilde{P}}^{2^{k+1}}$ and $\dot{\tilde{P}}^{2^k}$ is a $(T,S)-$IVIFEM.*

Definition 3.20. For a given IVIFN $\tilde{\alpha}$, line i and line k of $\tilde{P}_{\tilde{\alpha}}$ are called $\alpha-$congruence if $\tilde{p}_{ij} : \tilde{\alpha} = \tilde{p}_{kj} : \tilde{\alpha}$ for $j = 1, 2, \ldots, n$.

4. A Method for Clustering

In the process of clustering under an intuitionistic fuzzy environment, an expert usually provides his/her preferences with IVIFNs. Then we can use the method in [32] to classify the objects. However, sometimes, experts do not apply their preferences. They just give their opinions denoted by IVIFNs $\tilde{p}_{ij}$ which show the similarity level between the objects A_i and A_j $(i, j = 1, 2, \ldots, n)$. In this case, the method in [32] can not be used directly. So we develop a new method for clustering with interval-valued intuitionistic fuzzy information. The method is described as follows:

Step 1: Experts give information about the similarity level between every two objects A_i and A_j under each factor G_l, characterized by IVIFNs $\tilde{p}^l_{ij}$ $(i, j = 1, 2, \ldots, n, l = 1, 2, \ldots, m)$. Choose an aggregation operator (see [31]), for example, interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator, interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator, and so on. Then we can have an IVIFSM $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ by aggregation.

Step 2: Choose the triangular dual module T and S. Verify that whether $\tilde{P}$ is a $(T, S)-$IVIFEM or not. If not, stop the $(T, S)-$compositions $\tilde{P} \to \dot{\tilde{P}}^2 \to \dot{\tilde{P}}^4 \to \cdots \to \dot{\tilde{P}}^{2^k} \to \cdots$ until $\dot{\tilde{P}}^{2^k} = \dot{\tilde{P}}^{2^{k+1}}$. Then $\dot{\tilde{P}}^{2^k}$ is a $(T, S)-$IVIFEM.

Step 3: Take an appropriate IVIFN $\tilde{\alpha}$, calculate $\dot{\tilde{P}}^{2^k}_{\tilde{\alpha}}$ by Definition 3.16.

Step 4: Classify $\{A_i\}$ by Definition 3.20.

5. Illustrative Example

In Beijing, there are many hotels, and now some experts would like to investigate 5 hotels A_i $(i = 1, 2, 3, 4, 5)$ under six factors G_l $(l = 1, 2, \ldots, 6)$, namely, geographical position, price, security, sanitation, customer service, comfort level. They only provide information about the similarity level between every two hotels A_i and A_j, characterized by IVIFNs $\tilde{p}^l_{ij} = ([a^l_{ij}, b^l_{ij}], [c^l_{ij}, d^l_{ij}])$ under each factor G_l. If we choose the weight vector $\omega = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, then by the interval-valued intuitionistic fuzzy averaging (IIFA) operator which is defined in [31], we can have an IVIFSM $\tilde{P} = (\tilde{p}_{ij})_{5\times 5}$ as follows (in order to make the matrix shorter, we use a simple notation, for example: ".2" means "0.2"):

$$\tilde{P} = \begin{bmatrix} ([1,1],[0,0]) & ([.5,.6],[.2,.3]) & ([.7,.8],[.1,.2]) & ([.3,.6],[.1,.4]) & ([.2,.5],[.3,.4]) \\ ([.5,.6],[.2,.3]) & ([1,1],[0,0]) & ([.3,.4],[.2,.5]) & ([.6,.8],[0,.1]) & ([.2,.4],[.2,.5]) \\ ([.7,.8],[.1,.2]) & ([.3,.4],[.2,.5]) & ([1,1],[0,0]) & ([.4,.5],[.4,.5]) & ([.3,.6],[.2,.3]) \\ ([.3,.6],[.1,.4]) & ([.6,.8],[0,.1]) & ([.4,.5],[.4,.5]) & ([1,1],[0,0]) & ([.5,.6],[.3,.4]) \\ ([.2,.5],[.3,.4]) & ([.2,.4],[.2,.5]) & ([.3,.6],[.2,.3]) & ([.5,.6],[.3,.4]) & ([1,1],[0,0]) \end{bmatrix},$$

where

$$\tilde{p}_{ij} = \mathrm{IIFA}(\tilde{p}_{ij}^1, \tilde{p}_{ij}^2, \ldots, \tilde{p}_{ij}^6) = ([1-\prod_{l=1}^{6}(1-a_{ij}^l)^{\frac{1}{6}}, 1-\prod_{l=1}^{6}(1-b_{ij}^l)^{\frac{1}{6}}], [\prod_{l=1}^{6}(c_{ij}^l)^{\frac{1}{6}}, \prod_{l=1}^{6}(d_{ij}^l)^{\frac{1}{6}}]).$$

Choose $T(a,b) = ab$ and $S(a,b) = a + b - ab$, then calculate $\dot{\tilde{P}}^2 = \tilde{P} \times \tilde{P}$ below:

$$\dot{\tilde{P}}^2 = \begin{bmatrix} ([1,1],[0,0]) & ([.5,.6],[.2,.3]) & ([.7,.8],[.1,.2]) & ([.3,.6],[.1,.37]) & ([.21,.5],[.28,.4]) \\ ([.5,.6],[.2,.3]) & ([1,1],[0,0]) & ([.35,.48],[.2,.44]) & ([.6,.8],[0,.1]) & ([.3,.48],[.2,.46]) \\ ([.7,.8],[.1,.2]) & ([.35,.48],[.2,.44]) & ([1,1],[0,0]) & ([.4,.5],[.2,.5]) & ([.3,.6],[.2,.3]) \\ ([.3,.6],[.1,.37]) & ([.6,.8],[0,.1]) & ([.4,.5],[.2,.5]) & ([1,1],[0,0]) & ([.5,.6],[.2,.4]) \\ ([.21,.5],[.28,.4]) & ([.3,.48],[.2,.46]) & ([.3,.6],[.2,.3]) & ([.5,.6],[.2,.4]) & ([1,1],[0,0]) \end{bmatrix}.$$

Obviously $\dot{\tilde{P}}^2 \not\subseteq \tilde{P}$, continue calculating, then we can have

$$\dot{\tilde{P}}^4 = \begin{bmatrix} ([1,1],[0,0]) & ([.5,.6],[.1,.3]) & ([.7,.8],[.1,.2]) & ([.3,.6],[.1,.37]) & ([.21,.5],[.28,.4]) \\ ([.5,.6],[.1,.3] & ([1,1],[0,0]) & ([.35,.48],[.19,.44]) & ([.6,.8],[0,.1]) & ([.3,.48],[.2,.46]) \\ ([.7,.8],[.1,.2]) & ([.35,.48],[.19,.44]) & ([1,1],[0,0]) & ([.4,.5],[.19,.496]) & ([.3,.6],[.2,.3]) \\ ([.3,.6],[.1,.37]) & ([.6,.8],[0,.1]) & ([.4,.5],[.19,.496]) & ([1,1],[0,0]) & ([.5,.6],[.2,.4]) \\ ([.21,.5],[.28,.4]) & ([.3,.48],[.2,.46]) & ([.3,.6],[.2,.3]) & ([.5,.6],[.2,.4]) & ([1,1],[0,0]) \end{bmatrix}.$$

It is clear that $\dot{\tilde{P}}^4 \not\subseteq \dot{\tilde{P}}^2$. However, $\dot{\tilde{P}}^8 = \dot{\tilde{P}}^4$, that is, $\dot{\tilde{P}}^4$ is a $(T,S)-$IVIFEM.

Let $\tilde{\alpha}_1 = ([0.2, 0.4], [0.3, 0.5])$, by Definition 3.16, we have

$$\dot{\tilde{P}}^4_{\tilde{\alpha}_1} = \begin{bmatrix} ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) \end{bmatrix}.$$

By Definition 3.20, $\{A_i\}$ $(i = 1, 2, 3, 4, 5)$ can be divided into one category $\{A_1, A_2, A_3, A_4, A_5\}$.

Let $\tilde{\alpha}_2 = ([0.3, 0.4], [0.19, 0.5])$, then

$$\dot{\tilde{P}}^4_{\tilde{\alpha}_2} = \begin{bmatrix} ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([1,1],[0,0]) \end{bmatrix}.$$

So $\{A_i\}$ can be divided into two categories $\{A_1, A_2, A_3, A_4\}$, $\{A_5\}$.

Let $\tilde{\alpha}_3 = ([0.35, 0.4], [0.1, 0.2])$, then

$$\dot{\tilde{P}}^4_{\tilde{\alpha}_3} = \begin{bmatrix} ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([1,1],[0,0]) \end{bmatrix}.$$

So $\{A_i\}$ can be divided into three categories $\{A_1, A_3\}$, $\{A_2, A_4\}$, $\{A_5\}$.

Let $\tilde{\alpha}_4 = ([0.35, 0.48], [0.19, 0.44])$, then

$$\dot{\tilde{P}}^4_{\tilde{\alpha}_4} = \begin{bmatrix} ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) & [0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([1,1],[0,0]) \end{bmatrix}.$$

So $\{A_i\}$ can be divided into four categories $\{A_1, A_3\}$, $\{A_2\}$, $\{A_4\}$, $\{A_5\}$.

Let $\tilde{\alpha}_5 = ([0.5, 0.6], [0.1, 0.3])$, then

$$\dot{P}^4_{\tilde{\alpha}_4} = \begin{bmatrix} ([1,1],[0,0]) & ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) & [0,0],[1,1]) \\ ([1,1],[0,0]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) & ([1,1],[0,0]) & ([0,0],[1,1]) \\ ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([0,0],[1,1]) & ([1,1],[0,0]) \end{bmatrix}.$$

So $\{A_i\}$ can be divided into five categories $\{A_1\}$, $\{A_2\}$, $\{A_3\}$, $\{A_4\}$, $\{A_5\}$.

Remark 5.1. In a previous paper [16], in order to compare with the algorithm in reference [39], we used the same example given by Zhang. In [39], under the intuitionistic fuzzy environment, the object $\{A_i\}$ $(i = 1, 2, 3, 4, 5)$ only can be divided into one, three, five categories with different IFN α. But when we choose appropriate IFN α, the object $\{A_i\}$ can be divided into one, three, four, five categories by our method which was introduced in [16]. In this paper, we consider the clustering for IVIFSs, and it can be seen that our method is more rational and thoughtful when we use the more general operators: $t-$norm and $t-$conorm. Furthermore, our method is also applicable to the papers mentioned above.

6. **Conclusion**

Today, the applications of interval-valued intuitionistic fuzzy sets are taken into account with more and more experts and scholars. However, the integration of interval-valued intuitionistic fuzzy sets has only just begun, there are many issues worth discussing in depth. In this paper, the notions of composition matrix and interval-valued intuitionistic fuzzy equivalence matrix are generalized to $(T, S)-$composition matrix and $(T, S)-$interval-valued intuitionistic fuzzy equivalence matrix when (T, S) is a dual pair of triangular module. As shown, (T, S)-composition matrix and $\tilde{\alpha}-$matrix could be used to categories some objects when the similarity level (characterized by IVIFNs) between every two objects are only available.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[2] K. Atanassov, *Operators over interval-valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **64** (1994), 159-174.

[3] K. Atanassov, *On intuitionistic fuzzy negations and De Morgan Laws, Proceedings of Eleventh International Conference*, IPMU 2006, Paris, July 2-7, (2006), 2399-2404.

[4] K. Atanassov and G. Gargov, *Interval-valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **31** (1989), 343-349.

[5] R. A. Borzooei and Y. B. Jun, *Intuitionistic fuzzy hyper BCK-ideals of hyper BCK-algebras*, Iranian Journal of Fuzzy Systems, **1** (2004), 65-78.

[6] H. Bustince and P. Burillo, *Correlation of interval-valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **74** (1995), 237-244.

[7] Q. Chen, Z. S. Xu, S. S. Liu and X. H. Yu, *A method based on interval-valued intuitionistic fuzzy entropy for multiple attribute decision making*, Information: An International Interdisciplinary Journal, **13** (2010), 67-77.

[8] D. Coker, *Fuzzy rough sets are intuitionistic L-fuzzy sets*, Fuzzy Sets and Systems, **96** (1998), 381-383.

[9] G. Deschrijver, *Arithmetic operators in interval-valued fuzzy set theory*, Information Sciences, **177** (2007), 2906-2924.

[10] J. Garcia and S. E. Rodabaugh, *Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued intuitionistic sets, intuitionistic fuzzy sets and topologies*, Fuzzy Sets and Systems, **156** (2005), 445-484.

[11] J. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18** (1967), 145-174.

[12] H. L. Huang and F. G. Shi, *L-fuzzy numbers and their properties*, Information Sciences, **178** (2008), 1141-1151.

[13] W. L. Hung and J. W. Wu, *Correlation of intuitionistic fuzzy sets by centroid method*, Information Sciences, **144** (2002), 219-225.

[14] Y. Jiang, Y. Tang, J. Wang and S. Tang, *Reasoning within intuitionistic fuzzy rough description logics*, Information Sciences, **189** (2009), 2362-2378.

[15] A. Khan, Y. B. Jun and M. Shabir, *Ordered semigroups characterized by their intuitionistic fuzzy BI-ideals*, Iranian Journal of Fuzzy Systems, **7** (2010), 55-69.

[16] M. L. Lin and H. L. Huang, *(T,S)-based intuitionistic fuzzy composite matrix and its application*, International Journal of Applied Mathematics and Statistics, **23** (2011), 54-63.

[17] T. K. Mondal and S. K. Samanta, *Topology of interval-valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **119** (2001), 483-494.

[18] A. Narayanan, S. Vijayabalaji and N. Thillaigovindan, *Intuitionistic fuzzy bounded linear operators*, Iranian Journal of Fuzzy Systems, **4** (2007), 89-101.

[19] D. G. Park, Y. C. Kwun, J. H. Park and et al., *Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems*, Mathematical and Computer Modelling, **50** (2009), 1279-1293.

[20] L. Torkzadeh, M. Abbasi and M. M. Zahedi, *Some results of intuitionistic fuzzy weak dual hyper K-ideals*, Iranian Journal of Fuzzy Systems, **5** (2008), 65-78.

[21] P. Z. Wang, *Fuzzy set theory and its application*, Shanghai Science and Technology Press, Shanghai, 1983.

[22] Z. Wang, K. W. Li and W. Wang, *An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights*, Information Sciences, **179** (2009), 3026-3040.

[23] G. Wei, *Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making*, Applied Soft Computing, **10** (2010), 423-431.

[24] Z. S. Xu, *A method based on distance measure for interval-valued intuitionistic fuzzy group decision making*, Information Sciences, **180** (2010), 181-190.

[25] Z. S. Xu, *Choquet integrals of weighted intuitionistic fuzzy information*, Information Sciences, **180** (2010), 726-736.

[26] Z. S. Xu, *A deviation-based approach to intuitionistic fuzzy multiple attribute group decision making*, Group Decision and Negotiation, **19** (2010), 57-76.

[27] Z. S. Xu, *On correlation measures of intuitionistic fuzzy sets*, Lecture Notes in Computer Science, **4224** (2006), 16-24.

[28] Z. S. Xu, *Intuitionistic fuzzy hierarchical clustering algorithms*, Journal of Systems Engineering and Electronics, **20** (2009), 90-97.

[29] Z. S. Xu and X. Cai, *Incomplete interval-valued intuitionistic preference relations*, International Journal of General Systems, **38** (2009), 871-886.

[30] Z. S. Xu and X. Q. Cai, *Nonlinear optimization models for multiple attribute group decision making with intuitionistic fuzzy information*, International Journal of Intelligent Systems, **25** (2010), 489-513.

[31] Z. S. Xu and J. Chen, *Approach to group decision making based on interval-valued intuitionistic judgment matrices*, Systems Engineering-Theory and Practice, **27(4)** (2007), 126-133.

[32] Z. S. Xu, J. Chen and J. Wu, *Clustering algorithm for intuitionistic fuzzy sets*, Information Sciences, **178** (2008), 3775-3790.

[33] Z. S. Xu and J. J. Wu, *Intuitionistic fuzzy c-means clustering algorithms*, Journal of Systems Engineering and Electronics, **21** (2010), 580-590.

[34] Z. S. Xu and R. R. Yager, *Dynamic intuitionistic fuzzy multi-attribute decision making*, International Journal of Approximate Reasoning, **48** (2008), 246-262.

[35] Z. S. Xu and R. R. Yager, *Intuitionistic and interval-valued intutionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group*, Fuzzy Optimization and Decision Making, **8** (2009), 123-139.

[36] J. Ye and Multicriteria, *Fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment*, Expert Systems with Applications, **36** (2009), 6899-6902.

[37] J. Ye and Multicriteria, *Fuzzy decision-making method using entropy weights-based correlation coefficients of interval-valued intuitionistic fuzzy sets*, Applied Mathematical Modelling, **34** (2010), 3864-3870.

[38] L. A. Zadeh, *Fuzzy sets*, Information Control, **8** (1965), 338-353.

[39] H. M. Zhang, Z. S. Xu and Q. Chen, *On clustering approach to intuitionistic fuzzy sets*, Control and Decision, **22** (2007), 882-888.

[40] H. Y. Zhang, W. X. Zhang and W. Z. Wu, *On characterization of generalized interval-valued fuzzy rough sets on two universes of discourse*, International Journal of Approximate Reasoning, **51** (2009), 56-70.

[41] L. Zhou, W. Z. Wu and W. X. Zhang, *On characterization of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators*, Information Sciences, **179** (2009), 883-898.

Han-Liang Huang, Department of Mathematics and Information Science, Zhangzhou Normal University, Zhangzhou 363000, China

E-mail address: hl_huang1980.student@sina.com