A FIXED POINT APPROACH TO THE INTUITIONISTIC FUZZY STABILITY OF QUINTIC AND SEXTIC FUNCTIONAL EQUATIONS

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Abstract. The fixed point alternative methods are implemented to give Hyers-Ulam stability for the quintic functional equation $f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120f(y)$ and the sextic functional equation $f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)-6f(x-2y)+f(x-3y)=720f(y)$ in the setting of intuitionistic fuzzy normed spaces (IFN-spaces). This method introduces a metrical context and shows that the stability is related to some fixed point of a suitable operator. Furthermore, the interdisciplinary relation among the fuzzy set theory, the theory of intuitionistic spaces and the theory of functional equations are also presented in the paper.

1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g. population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, etc., see e.g. [1, 8, 9, 17, 18, 19, 24, 28, 31, 34, 36, 38]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of an intuitionistic fuzzy metric space and an intuitionistic fuzzy normed space. Atanassov [3, 4] introduced the concept of intuitionistic fuzzy sets which is further studied by Çoker [7]. These problems have been investigated by Park [26] and Saadati and Park [33], respectively, and they introduced and studied a notion of an intuitionistic fuzzy normed space. Hosseini, O'Regan, and Saadati [11] first considered finite dimensional intuitionistic fuzzy normed spaces and proved several theorems about completeness, compactness and weak convergence in these spaces. Applying an intuitionistic fuzzy quasi-metric version of a fixed point theorem, Saadati, Vaezpour, and Cho [37] obtained the existence of solution for a recurrence equation associated with the analysis of Quicksort algorithms.

The generalized Hyers-Ulam stability of different functional equations in intuitionistic fuzzy normed spaces has been recently studied in [20, 23, 32, 33, 41].

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Saadati, Cho, and Vahidi [32] introduced the notation of intuitionistic random normed spaces, and then by virtue of this notation to study the stability of a quartic functional equation in the setting of these spaces under arbitrary triangle norms. Recently, Mursaleen and Mohiuddine [23] linked two different disciplines, namely, the fuzzy spaces and functional equations. They also prove that the existence of a solution for any approximately cubic mapping implies the completeness of IFNS. The purpose of this paper is to establish some interesting results of continuous approximately quintic and sextic mappings in intuitionistic fuzzy normed spaces (IFN-spaces).

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a unique solution, we say the equation is stable (see [22]). The study of stability problems for functional equations is related to a question of Ulam [39] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [12]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [30] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y) = f(x+y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [10], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$ in the spirit of Th. M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see e.g. [5, 6, 17, 19, 20, 21, 23, 25, 27, 29, 40, 41, 42, 43, 44, 45, 46], and references therein).

The functional equation $f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x)$ is said to be the cubic functional equation since the function $f(x) = x^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was solved by Jun and Kim [14, 15] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later, a number of mathematicians have worked on the stability of some types of the cubic equation [5, 21, 23]. The functional equation $f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$ is said to be the quartic functional equation since the function $f(x) = x^4$ is its solution. Every solution of the quartic functional equation is said to be a quartic mapping. The stability problem for the quartic functional equation first was solved by Rassias [29] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space.

Similar to the cubic and quartic functional equations, we may define quintic and sexitic functional equations.

The functional equation

$$f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) - f(x-2y) = 120f(y) \tag{1}$$

is said to be the quintic functional equation since the function $f(x) = x^5$ is its solution. Every solution of the quintic functional equation is said to be a quintic mapping. The functional equation

$$\begin{aligned} f(x+3y) - 6f(x+2y) + 15f(x+y) - 20f(x) + 15f(x-y) \\ -6f(x-2y) + f(x-3y) = 720f(y) \end{aligned} \tag{2}$$

is said to be the sextic functional equation since the function $f(x) = x^6$ is its solution. Every solution of the sextic functional equation is said to be a sextic mapping.

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In 2003, Radu [27] proposed the fixed point alternative method for obtaining the existence of exact solutions and error estimations. Subsequently, Miheţ [17] applied the fixed alternative method to study the fuzzy stability of the Jensen functional equation on the fuzzy space which is defined in [19]. A comparison between the direct method and fixed alternative method for functional equations is given in [27]. The fixed alternative method can be considered as an advantage of this method over direct method in the fact that the range of approximate solutions is much more than the latter.

The stability problems of several various functional equations have been extensively investigated by a number of authors using fixed point methods (see [6, 17, 44]). The aim of this article is to extend the applications of the fixed point alternative method to provide the intuitionistic fuzzy versions of Hyers-Ulam stability for the quintic and sextic functional equations.

2. Preliminaries

In what follows, we shall adopt the usual terminology, notation and some definitions introduced by Saadati et al. [26, 31, 33, 34, 36].

Consider the set L^* and the order relation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : \quad (x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \quad \forall\, (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice ([31, 36]). We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition 2.1. Let U be a non-empty set called the universe. An L^*-fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L^*$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L^*) to which u is an element of A. An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universal set U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : \quad u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ for all $u \in U$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Definition 2.2. A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:

(a) $(\forall\, x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall\, (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
(c) $(\forall\, (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
(d) $(\forall\, (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$ (monotonicity).

A t-norm $\mathcal{T}$ on L^* is said to be continuous if, for any $x, y \in L^*$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y, respectively,

$$\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For examples, let $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, consider

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\}) \quad \text{and} \quad \mathcal{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}).$$

Then $\mathcal{T}(a, b)$ and $\mathcal{M}(a, b)$ are continuous t-norm.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \dots, x^{(n)}), x^{(n+1)})$$

for all $n \geq 2$ and $x^{(i)} \in L^*$.

Definition 2.3. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by $N_s(x) = 1 - x$ for all $x \in [0, 1]$.

The definitions of an intuitionistic fuzzy normed space is given below (see [33]).

Definition 2.4. (1) Let $\mathscr{L} = (L^*, \leq_{L^*})$. The triple $(X, \mathcal{P}, \mathcal{T})$ is said to be an $\mathscr{L}$-fuzzy normed space if X is a vector space, $\mathcal{T}$ is a continuous t-norm on L^* and $\mathcal{P}$ is an L^*-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions for all $x, y \in X$ and $t, s > 0$,

(a) $\mathcal{P}(x, t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{P}(x, t) = 1_{L^*}$ if and only if $x = 0$;

(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;

(d) $\mathcal{P}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s))$;

(e) $\mathcal{P}(x, \cdot) : (0, \infty) \to L^*$ is continuous;

(f) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{L^*}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{L^*}$.

In this case, $\mathcal{P}$ is called an $\mathscr{L}$-fuzzy norm (briefly, L^*-fuzzy norm).

(2) If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set (see Definition 2.1), then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an intuitionistic fuzzy normed space (briefly, IFS-space). In this case, $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is called an intuitionistic fuzzy norm on X.

Note that, if $\mathcal{P}$ is an L^*-fuzzy norm on X, then the following are satisfied:

(i) $\mathcal{P}(x, t)$ is nondecreasing with respect to t for all $x \in X$.

(ii) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$ for all $x, y \in X$ and $t > 0$ (see [33]).

Example 2.5. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = (\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|})$$

for all $t \in \mathbb{R}^+$ in which $m > 1$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFN-space. Here, $\mu(x,t) + \nu(x,t) = 1$ for $x = 0$ and $\mu(x,t) + \nu(x,t) < 1$ for $x \neq 0$.

Let $\mathcal{M}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = (e^{-\|x\|/t}, e^{-\|x\|/t}(e^{\|x\|/t} - 1))$$

for all $x \in X$ and $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an IFN-space.

Definition 2.6. (1) A sequence $\{x_n\}$ in an IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be convergent to a point $x \in X$ (denoted by $x_n \stackrel{IF}{\longrightarrow} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

(2) A sequence $\{x_n\}$ in an IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be Cauchy sequence if, for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall\, n, m \geq n_0,$$

where N_s is the standard negator.

(3) An IFN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be complete if every Cauchy sequence in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is convergent in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$; (2) $d(x,y) = d(y,x)$ for all $x, y \in X$; (3) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 2.7. (The fixed point alternative theorem, see [6, 17]) *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leq Ld(x,y), \quad \forall\, x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall\, n \geq 0,$$

or

$$d(J^n x, J^{n+1} x) < \infty, \quad \forall\, n \geq n_0,$$

for some nonnegative integer n_0. Actually, if the second alternative holds, then the sequence $\{J^n x\}$ converges to a fixed point y^* of J and

(1) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : \ d(J^{n_0} x, y) < \infty\}$;*

(2) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

The next Theorem 2.8 has been proved in [40, Theorems 2.1-2.2]. Some basic facts on n-additive symmetric mappings can be found in [45].

Theorem 2.8. *Let X and Y be vector spaces. Then*

(1) *A function $f : X \to Y$ is a solution of the functional equation* (1) *if and only if f is of the form $f(x) = A^5(x)$ for all $x \in X$, where $A^5(x)$ is the diagonal of the* 5*-additive symmetric map $A_5 : X^5 \to Y$.*

(2) *A function $f : X \to Y$ is a solution of the functional equation* (2) *if and only if f is of the form $f(x) = A^6(x)$ for all $x \in X$, where $A^6(x)$ is the diagonal of the* 6*-additive symmetric map $A_6 : X^6 \to Y$.*

3. Intuitionistic Fuzzy Stability of the Quintic Functional Equation

In this section, we prove the generalized Ulam-Hyers stability of the quintic functional equation in intuitionistic fuzzy normed spaces, based on the fixed point method. For notational convenience, given a function $f : X \to Y$, we define the difference operator

$$\begin{aligned} D_q f(x,y) \quad := \quad & f(x+3y) - 5f(x+2y) + 10f(x+y) \\ & -10f(x) + 5f(x-y) - f(x-2y) - 120f(y) \end{aligned} \tag{3}$$

for all $x, y \in X$.

Theorem 3.1. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Let $\varphi : X^2 \to Z$ be a function such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x,2y),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x,y),t), \quad \forall\, x \in X, t > 0, \tag{4}$$

for some real number α with $\alpha < 32$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space and $f : X \to Y$ be a mapping such that

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y),t) \tag{5}$$

for all $x \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x),t) \geq_{L^*} M_1(x,(32-\alpha)t) \tag{6}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned} M_1(x,t) \quad := \quad & \mathcal{T}^{16}(\mathcal{P}'_{\mu,\nu}(\varphi(3x,x),\tfrac{5t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x),\tfrac{5t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x),60t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x),60t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0),60t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x),\tfrac{3t}{2}), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(0,0),180t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0),\tfrac{240t}{11}), \mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{600t}{11}), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(x,-x),\tfrac{600t}{11}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x),\tfrac{10t}{11}), \mathcal{P}'_{\mu,\nu}(\varphi(0,x),t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(0,0),12t), \mathcal{P}'_{\mu,\nu}(\varphi(0,x,12t), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x),12t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(0,2x),60t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x),60t)). \end{aligned}$$

Proof. Replacing $x = y = 0$ in (5), we get

$$\mathcal{P}_{\mu,\nu}(f(0),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,0),120t) \tag{7}$$

for all $t > 0$. Replacing x and y by 0 and x in (5), respectively, we get

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(f(3x) - 5f(2x) + 10f(x) - 10f(0) + 5f(-x) \\ & \quad -f(-2x) - 120f(x),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,x),t) \end{aligned} \tag{8}$$

for all $x \in X$ and $t > 0$. Replacing x and y by x and $-x$ in (5), respectively, we have

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(&f(-2x) - 5f(-x) + 10f(0) - 10f(x) + 5f(2x)\\ &-f(3x) - 120f(-x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), t)\end{aligned} \tag{9}$$

for all $x \in X$ and $t > 0$. By (8) and (9), we obtain

$$\mathcal{P}_{\mu,\nu}(f(x) + f(-x), t) \geq_{L^*} \mathcal{T}(\mathcal{P}'_{\mu,\nu}(\varphi(0,x), 60t), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 60t)) \tag{10}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $3x$ and x in (5), respectively, we get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(6x) - 5f(5x) + 10f(4x) - 10f(3x) + 5f(2x) - 121f(x), t)\\ &\quad\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), t)\end{aligned} \tag{11}$$

for all $x \in X$ and $t > 0$. Replacing x and y by 0 and $2x$ in (5), respectively, we find

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(6x) - 5f(4x) - 10f(0) + 5f(-2x) - f(-4x) - 110f(2x), t)\\ &\quad\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), t)\end{aligned} \tag{12}$$

for all $x \in X$ and $t > 0$. By (11) and (12), we obtain

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(&5f(5x) - 14f(4x) + 10f(3x) + 121f(x) - 120f(2x) + 5f(2x)\\ &+5f(-2x) - 10f(0), t) \geq_{L^*} \mathcal{T}(\mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{t}{2}))\end{aligned} \tag{13}$$

for all $x \in X$ and $t > 0$. By (7), (10), and (13), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(5f(5x) - 14f(4x) + 10f(3x) + 121f(x) - 120f(2x), t)\\ &\quad\geq_{L^*} \mathcal{T}^4(\mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{6}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{t}{6}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), 4t),\\ &\qquad \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 4t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 4t))\end{aligned} \tag{14}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (5), respectively, we get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(5x) - 5f(4x) + 10f(3x) - 10f(2x) - 115f(x) - f(0), t)\\ &\quad\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), t)\end{aligned} \tag{15}$$

for all $x \in X$ and $t > 0$. Using (7), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(5x) - 5f(4x) + 10f(3x) - 10f(2x) - 115f(x), t)\\ &\quad\geq_{L^*} \mathcal{T}(\mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 60t))\end{aligned} \tag{16}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(5f(5x) - 25f(4x) + 50f(3x) - 50f(2x) - 575f(x), t)\\ &\quad\geq_{L^*} \mathcal{T}(\mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{10}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 12t))\end{aligned} \tag{17}$$

for all $x \in X$ and $t > 0$. By (14) and (17), we get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(11f(4x) - 40f(3x) - 70f(2x) + 696f(x), t)\\ &\geq_{L^*} \mathcal{T}^6(\mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{12}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{t}{12}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), 2t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 2t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 2t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{20}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 6t))\end{aligned} \tag{18}$$

for all $x \in X$ and $t > 0$. Replacing x and y by x and x in (5), respectively, we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(4x)-5f(3x)+10f(2x)+5f(0)-f(-x)-130f(x),t)\\&\quad\geq_{L^*}\mathcal{P}'_{\mu,\nu}(\varphi(x,x),t)\end{aligned} \tag{19}$$

for all $x \in X$ and $t > 0$. By (8), (11), and (20), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(4x)-5f(3x)+10f(2x)-129f(x),t)\geq_{L^*}\mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\varphi(0,0),8t),\\&\quad\mathcal{P}'_{\mu,\nu}(\varphi(0,x),20t),\mathcal{P}'_{\mu,\nu}(\varphi(x,-x),20t),\mathcal{P}'_{\mu,\nu}(\varphi(x,x),\tfrac{t}{3}))\end{aligned} \tag{20}$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(11f(4x)-55f(3x)+110f(2x)-1419f(x),t)\\&\quad\geq_{L^*}\mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\varphi(0,0),\tfrac{8}{11}t),\mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{20}{11}t),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(x,-x),\tfrac{20}{11}t),\mathcal{P}'_{\mu,\nu}(\varphi(x,x),\tfrac{t}{33}))\end{aligned} \tag{21}$$

for all $x \in X$ and $t > 0$. By (18) and (21), we obtain

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(15f(3x)-180f(2x)+2115f(x),t)\\&\quad\geq_{L^*}\mathcal{T}^{10}(\mathcal{P}'_{\mu,\nu}(\varphi(3x,x),\tfrac{t}{24}),\mathcal{P}'_{\mu,\nu}(\varphi(0,2x),\tfrac{t}{24}),\mathcal{P}'_{\mu,\nu}(\varphi(0,2x),t),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x),t),\mathcal{P}'_{\mu,\nu}(\varphi(0,0),t),\mathcal{P}'_{\mu,\nu}(\varphi(2x,x),\tfrac{t}{40}),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(0,0),3t),\mathcal{P}'_{\mu,\nu}(\varphi(0,0),\tfrac{4}{11}t),\mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{10}{11}t),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(x,-x),\tfrac{10}{11}t),\mathcal{P}'_{\mu,\nu}(\varphi(x,x,\tfrac{t}{66}))\end{aligned} \tag{22}$$

for all $x \in X$ and $t > 0$. By (7), (8), and (10), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(3x)-4f(2x)-115f(x),t)\\&\quad\geq_{L^*}\mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{t}{4}),\mathcal{P}'_{\mu,\nu}(\varphi(0,0),3t),\mathcal{P}'_{\mu,\nu}(\varphi(0,x),3t),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(x,-x),3t),\mathcal{P}'_{\mu,\nu}(\varphi(0,2x),15t),\mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x),15t))\end{aligned} \tag{23}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(15f(3x)-60f(2x)-1725f(x),t)\\&\quad\geq_{L^*}\mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{t}{60}),\mathcal{P}'_{\mu,\nu}(\varphi(0,0),\tfrac{t}{5}),\mathcal{P}'_{\mu,\nu}(\varphi(0,x),\tfrac{t}{5}),\\&\qquad\mathcal{P}'_{\mu,\nu}(\varphi(x,-x),\tfrac{t}{5}),\mathcal{P}'_{\mu,\nu}(\varphi(0,2x),t),\mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x),t))\end{aligned} \tag{24}$$

for all $x \in X$ and $t > 0$. By (22) and (24), we get

$$\mathcal{P}_{\mu,\nu}(f(2x)-32f(x),t)\geq_{L^*}M_1(x,t) \tag{25}$$

for all $x \in X$ and $t > 0$.

Consider the set $\Omega := \{g : X \to Y\}$ and introduce a complete generalized metric on Ω (see [6])(as usual, $\inf\emptyset = \infty$):

$$d(g,h)=\inf\{K\in\mathbb{R}_+,\mathcal{P}_{\mu,\nu}(g(x)-h(x),Kt)\geq_{L^*}M_1(x,t),\quad\forall x\in X,t>0\}. \tag{26}$$

Now, we consider the mapping $J : \Omega \to \Omega$ such that

$$Jg(x)=\frac{1}{32}g(2x) \tag{27}$$

for all $x \in X$ and we prove that J is a strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha}{32}$.

Let $g, h \in \Omega$ be given such that $d(g,h) < \varepsilon$. Then

$$\mathcal{P}_{\mu,\nu}(g(x) - h(x), \varepsilon t) \geq_{L^*} M_1(x,t) \tag{28}$$

for all $x \in X$ and $t > 0$. Hence

$$\mathcal{P}_{\mu,\nu}(Jg(x) - Jh(x), \varepsilon t) = \mathcal{P}_{\mu,\nu}(g(2x) - h(2x), 32\varepsilon t) \geq_{L^*} M_1(x, \frac{32t}{\alpha}) \tag{29}$$

for all $x \in X$ and $t > 0$. By definition, $d(Jg, Jh) \leq \frac{\alpha}{32}\varepsilon$. Therefore,

$$d(Jg, Jh) \leq \frac{\alpha}{32} d(g,h), \quad \text{for all} \quad g, h \in \Omega. \tag{30}$$

It follows from (25) that $d(f, Jf) \leq \frac{1}{32}$. Therefore, by Theorem 2.7, there exists a mapping $Q : X \to Y$ satisfying:

(1) Q is a fixed point of J, that is

$$Q(2x) = 32Q(x) \tag{31}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $\Delta = \{g \in \Omega : d(g,f) < \infty\}$. This implies that Q is a unique mapping satisfying (31) such that there exists a $K > 0$ satisfying

$$\mathcal{P}_{\mu,\nu}(f(x) - Q(x), Kt) \geq_{L^*} M_1(x,t) \tag{32}$$

for all $x \in X$ and $t > 0$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$Q(x) := \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} \frac{1}{32^n} f(2^n x) \tag{33}$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality

$$d(f, Q) \leq \frac{1}{32 - \alpha}, \tag{34}$$

from which it follows

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), \frac{t}{32-\alpha}) \geq_{L^*} M_1(x,t) \tag{35}$$

for all $x \in X$ and $t > 0$. This implies that the inequality (6) holds.

It remains to show that Q is a quintic map. Replacing x and y by $2^n x$ and $2^n y$ in (5), respectively, it follows that

$$\mathcal{P}_{\mu,\nu}(32^{-n} D_q f(2^n x, 2^n y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 32^n t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), \frac{32^n t}{\alpha^n}).$$

Taking the limit as $n \to \infty$, we find that Q fulfills (1). □

Corollary 3.2. *Let p be a nonnegative real number with $p < 5$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, t)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, \min\{\frac{3}{2(2^p+1)}, \frac{5}{2(3^p+1)}\}(32 - 2^p)t)$$

for all $x \in X$ and $t > 0$.

Corollary 3.3. *Let r, s be nonnegative real numbers with $\lambda := r + s < 5$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}([\|x\|^r\|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0, t)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t)\\ &\qquad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^\lambda z_0, \min\{\tfrac{3}{2(2^r+2^\lambda+1)}, \tfrac{5}{2(3^r+3^\lambda+1)}\}(32 - 2^\lambda)t)\end{aligned}$$

for all $x \in X$ and $t > 0$.

One can prove a similar result for the case $\alpha > 32$.

Theorem 3.4. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Let $\varphi : X^2 \to Z$ be a function such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(\frac{x}{2}, \frac{y}{2}), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), \alpha t), \quad \forall\, x \in X, t > 0,$$

for some real number α with $\alpha > 32$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space and $f : X \to Y$ be a mapping such that

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), t)$$

for all $x \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} M_1(x, (\alpha - 32)t)$$

for all $x \in X$ and $t > 0$, where $M_1(x,t)$ is defined as in Theorem 3.1.

Corollary 3.5. *Let p be a nonnegative real number with $p > 5$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_q f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, t)$$

for all $x, y \in X$ *and* $t > 0$*, then there exists a unique quintic mapping* $Q : X \to Y$ *such that*

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, \min\{\frac{3}{2(2^p+1)}, \frac{5}{2(3^p+1)}\}(2^p - 32)t)$$

for all $x \in X$ *and* $t > 0$.

Corollary 3.6. *Let* r, s *be nonnegative real numbers with* $\lambda := r + s > 5$*,* X *be a normed space with norm* $\|\cdot\|$*,* $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ *be an intuitionistic fuzzy normed space,* $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ *be a complete intuitionistic fuzzy normed space, and let* $z_0 \in Z$*. If* $f : X \to Y$ *is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_q f(x, y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}([\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})] z_0, t)$$

for all $x, y \in X$ *and* $t > 0$*, then there exists a unique quintic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t)\\ &\qquad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^\lambda z_0, \min\{\tfrac{3}{2(2^r+2^\lambda+1)}, \tfrac{5}{2(3^r+3^\lambda+1)}\}(2^\lambda - 32)t)\end{aligned}$$

for all $x \in X$ *and* $t > 0$.

The following example provides an illustration.

Example 3.7. Let $(X, \|\cdot\|)$ be a Banach algebra, $\mathcal{M}$ a continuous t-norm defined in Example 2.5. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is a complete IRN-space in which $\mathcal{P}_{\mu,\nu}(x, t) = (\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|})$.

Define $f : X \to X$, $f(x) = x^5 + \|x\| x_0$, where x_0 is a unit vector in X. A straightforward computation shows that

$$\begin{aligned}&\|f(x + 3y) - 5f(x + 2y) + 10f(x + y) - 10f(x) + 5f(x - y)\\ &\quad - f(x - 2y) - 120f(y)\| \leq 32\|x\| + 150\|y\|\end{aligned}$$

for all $x, y \in X$, hence

$$\mathcal{P}_{\mu,\nu}(D_q f(x, y), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}([32\|x\| + 150\|y\|] x_0, t)$$

for all $x, y \in X$ and $t > 0$. Let $\varphi(x, y) = (32\|x\| + 150\|y\|)x_0$ for all $x, y \in X$. Moreover, $\mathcal{P}_{\mu,\nu}(\varphi(2x, 2y), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(2\varphi(x, y), t)$ for all $x, y \in X$ and $t > 0$. Therefore, all the conditions of Theorem 3.1 hold by $\alpha = 2 < 32$. It follows that f can be approximated by a quintic mapping. In fact there exists a unique quintic mapping $Q : X \to X$ such that $\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(150\|x\| x_0, t)$ for all $x \in X$ and $t > 0$.

4. Intuitionistic Fuzzy Stability of the Sextic Functional Equation

In this section, we prove the generalized Ulam-Hyers stability of the sextic functional equations in intuitionistic fuzzy normed spaces, based on the fixed point method.

For notational convenience, given a function $f : X \to Y$, we define the difference operator

$$\begin{aligned} D_s f(x,y) \quad := \quad & f(x+3y) - 6f(x+2y) + 15f(x+y) - 20f(x) \\ & +15f(x-y) - 6f(x-2y) + f(x-3y) - 720f(y) \end{aligned} \tag{36}$$

for all $x, y \in X$.

Theorem 4.1. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Let $\varphi : X^2 \to Z$ be a function such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x, 2y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x, y), t) \quad \forall\, x \in X, t > 0, \tag{37}$$

for some real number α with $0 < \alpha < 64$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space and $f : X \to Y$ be a mapping such that

$$\mathcal{P}_{\mu,\nu}(D_s f(x, y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y), t) \tag{38}$$

for all $x \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(S(x) - f(x), t) \geq_{L^*} M_2(x, (64 - \alpha)t) \tag{39}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned} M_2(x,t) \quad := \; & \mathcal{T}^{27}(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 10800t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 216t), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), 6t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x), 2160t), \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x), 2160t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x), 360t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x), 360t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 144t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \\ & 144t), \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), 15t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 300t), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 900t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 900t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{5t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{3t}{2}), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 72t), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 90t), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 90t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 540t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 540t), \mathcal{P}'_{\mu,\nu}(\varphi(0,x), \tfrac{18t}{5}), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{648t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{432t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{432t}{5}), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 216t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 216t), \mathcal{P}'_{\mu,\nu}(\varphi(3x,3x), 1296t), \\ & \mathcal{P}'_{\mu,\nu}(\varphi(3x,-3x), 1296t)). \end{aligned}$$

Proof. Replacing $x = y = 0$ in (38), we get

$$\mathcal{P}_{\mu,\nu}(f(0), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 720t) \tag{40}$$

for all $t > 0$. Replacing y by $-y$ in (38), we have

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(f(x-3y) - 6f(x-2y) + 15f(x-y) - 20f(x) + 15f(x+y) \\ & \quad -6f(x+2y) + f(x+3y) - 720f(-y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,-y), t) \end{aligned} \tag{41}$$

for all $x, y \in X$ and $t > 0$. By (38) and (41), we get

$$\mathcal{P}_{\mu,\nu}(f(x) - f(-x), t) \geq_{L^*} \mathcal{T}(\mathcal{P}'_{\mu,\nu}(\varphi(x,x), 360t), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 360t)) \tag{42}$$

for all $x \in X$ and $t > 0$. Replacing x and y by 0 and $2x$ in (38), respectively, we get

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(f(6x) - 6f(4x) + 15f(2x) - 20f(0) + 15f(-2x) \\ & \quad -6f(-4x) + f(-6x) - 720f(2x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), t) \end{aligned} \tag{43}$$

for all $x \in X$ and $t > 0$. By (40), (42), and (43), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(6x) - 6f(4x) - 345f(2x), t)\\ &\geq_{L^*} \mathcal{T}^7(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{72t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{2t}{5}, \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x), 144t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x), 144t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x), 24t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x), 24t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{48t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{48t}{5}))\end{aligned} \tag{44}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $3x$ and x in (43), respectively, we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(6x) - 6f(5x) + 15f(4x) - 20f(3x) + 15f(2x)\\ &\quad + f(0) - 726f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), t)\end{aligned} \tag{45}$$

for all $x \in X$ and $t > 0$. By (44), (45), and using (40), we obtain

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(6f(5x) - 21f(4x) + 20f(3x) - 360f(2x) + 726f(x), t)\\ &\geq_{L^*} \mathcal{T}^9(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 240t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{24t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{2t}{15}),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x), 48t), \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x), 48t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x), 8t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x,8t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{16t}{5}),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{16t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{3}))\end{aligned} \tag{46}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (38), respectively, we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(5x) - 6f(4x) + 15f(3x) - 20f(2x) - 6f(0) + f(-x)\\ &\quad -705f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), t)\end{aligned} \tag{47}$$

for all $x \in X$ and $t > 0$. Using (40) and (42), we get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(5x) - 6f(4x) + 15f(3x) - 20f(2x) - 704f(x), t)\\ &\geq_{L^*} \mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 40t), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 120t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 120t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{3}))\end{aligned} \tag{48}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(6f(5x) - 36f(4x) + 90f(3x) - 120f(2x) - 4224f(x), t)\\ &\geq_{L^*} \mathcal{T}^3(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{20t}{3}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 20t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 20t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{18}))\end{aligned} \tag{49}$$

for all $x \in X$ and $t > 0$. By (46) and (49), we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(15f(4x) - 70f(3x) - 240f(2x) + 4950f(x), t)\\ &\geq_{L^*} \mathcal{T}^{13}(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 120t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0, \tfrac{12t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x, \tfrac{t}{15}),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x,24t), \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x,24t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x,4t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x,4t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{8t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{8t}{5}),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(3x,x, \tfrac{t}{6}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0, \tfrac{10t}{3}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x,10t),\\ &\quad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x,10t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x, \tfrac{t}{36}))\end{aligned} \tag{50}$$

for all $x \in X$ and $t > 0$. Replacing x and y by 0 and x in (38), respectively, we get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(3x) - 6f(2x) - 705f(x) - 20f(0) + 15f(-x) - 6f(-2x)\\ &\quad + f(-3x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(0,x), t)\end{aligned} \tag{51}$$

for all $x \in X$ and $t > 0$. By (40), (42), and (51), we have

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(2f(3x) - 12f(2x) - 690f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^7(\mathcal{P}'_{\mu,\nu}(\varphi(0,x), \tfrac{t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{36t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{24t}{5}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{24t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x,12t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 12t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(3x,3x), 72t), \mathcal{P}'_{\mu,\nu}(\varphi(3x,-3x), 72t))
\end{aligned} \tag{52}$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(20f(3x) - 120f(2x) - 6900f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^7(\mathcal{P}'_{\mu,\nu}(\varphi(0,x), \tfrac{t}{50}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{18t}{25}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{12t}{25}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{12t}{25}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{6t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{6t}{5}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(3x,3x), \tfrac{36t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(3x,-3x), \tfrac{36t}{5}))
\end{aligned} \tag{53}$$

for all $x \in X$ and $t > 0$. Replacing x and y by x and x in (38), respectively, and then using (40) and (42), we have

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(f(4x) - 6f(3x) + 16f(2x) - 746f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{4}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), 12t), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 15t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 15t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 90t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 90t))
\end{aligned} \tag{54}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(20f(3x) - 480f(2x) + 16140f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^5(\mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{60}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{4t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 6t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 6t))
\end{aligned} \tag{55}$$

for all $x \in X$ and $t > 0$. By (50) and (55), we get

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(20f(3x) - 480f(2x) + 16140f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^{19}(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 60t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{6t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{t}{30}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x), 12t), \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x, 12t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x), 2t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x), 2t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{4t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{4t}{5}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{12}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{5t}{3}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), 5t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), 5t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{72}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{120}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{2t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{t}{2}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), 3t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), 3t))
\end{aligned} \tag{56}$$

for all $x \in X$ and $t > 0$. By (53) and (56), we obtain

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(360f(2x) - 23040f(x), t) \\
&\quad \geq_{L^*} \mathcal{T}^{27}(\mathcal{P}'_{\mu,\nu}(\varphi(0,0), 30t), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{3t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(0,2x), \tfrac{t}{60}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(6x,6x), 6t), \mathcal{P}'_{\mu,\nu}(\varphi(6x,-6x), 6t), \mathcal{P}'_{\mu,\nu}(\varphi(4x,4x), t), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(4x,-4x), t), \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{2t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{2t}{5}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(3x,x), \tfrac{t}{24}), \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{5t}{6}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{5t}{2}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{5t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,x), \tfrac{t}{144}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{240}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{t}{4}), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{t}{4}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{3t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{3t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0,x), \tfrac{t}{100}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(0,0), \tfrac{9t}{25}), \mathcal{P}'_{\mu,\nu}(\varphi(x,x), \tfrac{6t}{25}), \mathcal{P}'_{\mu,\nu}(\varphi(x,-x), \tfrac{6t}{25}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(2x,2x), \tfrac{3t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(2x,-2x), \tfrac{3t}{5}), \\
&\qquad \mathcal{P}'_{\mu,\nu}(\varphi(3x,3x), \tfrac{18t}{5}), \mathcal{P}'_{\mu,\nu}(\varphi(3x,-3x), \tfrac{18t}{5}))
\end{aligned} \tag{57}$$

for all $x \in X$ and $t > 0$. Therefore,

$$\mathcal{P}_{\mu,\nu}(f(2x) - 2^6 f(x), t) \geq_{L^*} M_2(x,t) \tag{58}$$

for all $x \in X$ and $t > 0$.

Consider the set $\Omega := \{g : X \to Y\}$ and introduce a complete generalized metric on Ω (as usual, $\inf \emptyset = \infty$):

$$d(g,h) = \inf\{K \in \mathbb{R}_+, \mathcal{P}_{\mu,\nu}(g(x) - h(x), Kt) \geq_{L^*} M_2(x,t), \quad \forall x \in X, t > 0\}.$$

Now, we consider the mapping $J : \Omega \to \Omega$ such that

$$Jg(x) = \frac{1}{64} g(2x) \tag{59}$$

for all $x \in X$ and prove that J is a strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha}{64}$.

Let $g, h \in \Omega$ be given such that $d(g,h) < \varepsilon$. Then

$$\mathcal{P}_{\mu,\nu}(g(x) - h(x), \varepsilon t) \geq_{L^*} M_2(x,t) \tag{60}$$

for all $x \in X$ and $t > 0$. Hence

$$\mathcal{P}_{\mu,\nu}(Jg(x) - Jh(x), \varepsilon t) = \mathcal{P}_{\mu,\nu}(g(2x) - h(2x), 64\varepsilon t) \geq_{L^*} M_2(x, \frac{64t}{\alpha}) \tag{61}$$

for all $x \in X$ and $t > 0$. By definition, $d(Jg, Jh) \leq \frac{\alpha}{64}\varepsilon$. Therefore,

$$d(Jg, Jh) \leq \frac{\alpha}{64} d(g,h), \quad \text{for all} \quad g, h \in \Omega. \tag{62}$$

It follows from (58) that $d(f, Jf) \leq \frac{1}{64}$. Therefore, by Theorem 2.7, there exists a mapping $S : X \to Y$ satisfying:

(1) S is a fixed point of J, that is

$$S(2x) = 64S(x) \tag{63}$$

for all $x \in X$. The mapping S is a unique fixed point of J in the set $\Delta = \{g \in \Omega : d(g,f) < \infty\}$. This implies that S is a unique mapping satisfying (4.28) such that there exists a $K > 0$ satisfying

$$\mathcal{P}_{\mu,\nu}(f(x) - S(x), Kt) \geq_{L^*} M_2(x,t) \tag{64}$$

for all $x \in X$ and $t > 0$;

(2) $d(J^n f, S) \to 0$ as $n \to \infty$. This implies the equality

$$S(x) := \lim_{n\to\infty} J^n f(x) = \lim_{n\to\infty} \frac{1}{64^n} f(2^n x) \tag{65}$$

for all $x \in X$;

(3) $d(f,S) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality

$$d(f,S) \leq \frac{1}{64 - \alpha}, \tag{66}$$

from which it follows

$$\mathcal{P}_{\mu,\nu}(S(x) - f(x), \frac{t}{64-\alpha}) \geq_{L^*} M_2(x,t) \tag{67}$$

for all $x \in X$ and $t > 0$. This implies that the inequality (39) holds.

It remains to show that S is a sextic map. Replacing x and y by $2^n x$ and $2^n y$ in (38), respectively, it follows that

$$\mathcal{P}_{\mu,\nu}(64^{-n} D_s f(2^n x, 2^n y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 64^n t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), \frac{64^n t}{\alpha^n}).$$

Taking the limit as $n \to \infty$, we find that S fulfills (2). □

Corollary 4.2. *Let p be a nonnegative real number with $p < 6$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that $\mathcal{P}_{\mu,\nu}(D_s f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p) z_0, t)$ for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \to Y$ such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(S(x) - f(x), t)\\ &\quad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, \min\{\tfrac{1080}{6^p}, \tfrac{180}{4^p}, \tfrac{15}{3^p+1}, \tfrac{3}{4}, \tfrac{5}{2(2^p+1)}\}(64 - 2^p)t)\end{aligned}$$

for all $x \in X$ and $t > 0$.

Corollary 4.3. *Let r, s be nonnegative real numbers with $\lambda := r + s < 6$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_s f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}([\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})] z_0, t)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \to Y$ such that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(S(x) - f(x), t)\\ &\quad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^\lambda z_0, \min\{\tfrac{720}{6^\lambda}, \tfrac{120}{4^\lambda}, \tfrac{15}{3^r+3^\lambda+1}, \tfrac{1}{2}, \tfrac{5}{2(2^r+2^\lambda+1)}\}(64 - 2^\lambda)t)\end{aligned}$$

for all $x \in X$ and $t > 0$.

One can prove a similar result for the case $\alpha > 64$.

Theorem 4.4. *Let X be a linear space and let $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{T}')$ be an intuitionistic fuzzy normed space. Let $\varphi : X^2 \to Z$ be a function such that*

$$\mathcal{P}'_{\mu,\nu}(\varphi(\frac{x}{2}, \frac{y}{2}), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), \alpha t), \quad \forall\, x \in X, t > 0,$$

for some real number α with $\alpha > 64$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete intuitionistic fuzzy normed space and $f : X \to Y$ be a mapping such that $\mathcal{P}_{\mu,\nu}(D_s f(x,y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,y), t)$ for all $x \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(S(x) - f(x), t) \geq_{L^*} M_2(x, (\alpha - 64)t)$$

for all $x \in X$ and $t > 0$, where $M_2(x,t)$ is defined as in Theorem 4.1.

Corollary 4.5. *Let p be a nonnegative real number with $p > 6$, X be a normed space with norm $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ be an intuitionistic fuzzy normed space, $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ be a complete intuitionistic fuzzy normed space, and let $z_0 \in Z$. If $f : X \to Y$ is a* *mapping such that* $\mathcal{P}_{\mu,\nu}(D_s f(x,y),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, t)$ *for all* $x, y \in X$ *and* $t > 0$, *then there exists a unique sextic mapping* $S : X \to Y$ *such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(S(x) - f(x), t)\\ &\quad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, \min\{\tfrac{1080}{6^p}, \tfrac{180}{4^p}, \tfrac{15}{3^p+1}, \tfrac{3}{4}, \tfrac{5}{2(2^p+1)}\}(2^p - 64)t)\end{aligned}$$

for all $x \in X$ *and* $t > 0$.

Corollary 4.6. *Let* r, s *be nonnegative real numbers with* $\lambda := r + s > 6$, X *be a normed space with norm* $\|\cdot\|$, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ *be a intuitionistic fuzzy normed space,* $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ *be a complete intuitionistic fuzzy normed space, and let* $z_0 \in Z$. *If* $f : X \to Y$ *is a mapping such that*

$$\mathcal{P}_{\mu,\nu}(D_s f(x,y),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}([\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0, t)$$

for all $x, y \in X$ *and* $t > 0$, *then there exists a unique sextic mapping* $S : X \to Y$ *such that*

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(S(x) - f(x), t)\\ &\quad \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^\lambda z_0, \min\{\tfrac{720}{6^\lambda}, \tfrac{120}{4^\lambda}, \tfrac{15}{3^r+3^\lambda+1}, \tfrac{1}{2}, \tfrac{5}{2(2^r+2^\lambda+1)}\}(2^\lambda - 64)t)\end{aligned}$$

for all $x \in X$ *and* $t > 0$.

Similar to the Example 3.7, we have the following example.

Example 4.7. Let $(X, \|\cdot\|)$ be a Banach algebra, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is a complete IRN-space defined in Example 3.7.

Let $f : X \to X$, $f(x) = x^6 + \|x\|x_0$, where x_0 is a unit vector in X. A straightforward computation shows that $\|D_s f(x,y)\| \leq 64\|x\| + 780\|y\|$ for all $x, y \in X$, hence

$$\mathcal{P}_{\mu,\nu}(D_s f(x,y),t) \geq_{L^*} \mathcal{P}_{\mu,\nu}([64\|x\| + 780\|y\|]x_0, t)$$

for all $x, y \in X$ and $t > 0$. Let $\varphi(x,y) = (64\|x\| + 780\|y\|)x_0$ for all $x, y \in X$. Moreover, $\mathcal{P}_{\mu,\nu}(\varphi(2x,2y),t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(2\varphi(x,y),t)$ for all $x, y \in X$ and $t > 0$. Therefore, all the conditions of Theorem 4.1 hold by $\alpha = 2 < 64$. It follows that f can be approximated by a sextic mapping. In fact there exists a unique sextic mapping $S : X \to X$ such that $\mathcal{P}_{\mu,\nu}(S(x) - f(x), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(\|x\|x_0, \frac{3}{1688}t)$ for all $x \in X$ and $t > 0$.

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