ON (L, M)-FUZZY CLOSURE SPACES

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Abstract. The aim of this paper is to introduce (L, M)-fuzzy closure structure where L and M are strictly two-sided, commutative quantales. Firstly, we define (L, M)-fuzzy closure spaces and get some relations between (L, M)-double fuzzy topological spaces and (L, M)-fuzzy closure spaces. Then, we introduce initial (L, M)-fuzzy closure structures and we prove that the category (L, M)-**FC** of (L, M)-fuzzy closure spaces and (L, M)-$\mathcal{C}$-maps is a topological category over the category **SET**. From this fact, we define products of (L, M)-fuzzy closure spaces. Finally, we show that an initial structure of (L, M)-double fuzzy topological spaces can be obtained by the initial structure of (L, M)-fuzzy closure spaces induced by them.

1. Introduction

Kubiak [23] and Sostak [34] introduced the notion of L-fuzzy topological space as a generalization of L-topological spaces introduced by Chang [5]. At the bottom of it lies the degree of openness of an L-fuzzy set. In [38], Ying introduced a fuzzifying topology which is the grade of openness of a set in view of fuzzy logic. A general approach to the study of topological-type structures on fuzzy powersets was developed in [2, 12, 19, 24, 27, 28, 30, 33, 35, 36].

On the other hand, Atanassov [4] introduced the idea of intuitionistic (double graded) fuzzy set. Çoker and coworker [8, 9] introduced the idea of topology of intuitionistic fuzzy sets. Recently, Samanta and Mondal [32] introduced the notion of intuitionistic gradation of openness which is a generalization of both fuzzy topological spaces [34] and the topology of intuitionistic fuzzy sets [8].

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Gutierrez Garcia and Rodabaugh [10]. They argued that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

In this paper, we introduce the notion of (L, M)-fuzzy closure space as a generalization of both [0,1]-fuzzy closure spaces introduced by Chattopadhyay and Samanta [6, 7] and intuitionistic fuzzy closure spaces introduced by Abbas [1] and Lee and Im [25]. We show the existence of initial (L, M)-fuzzy closure structures. From this fact, the category (L, M)-**FC** is a topological category over **SET**. Furthermore, we define products of (L, M)-fuzzy closure spaces. In particular, an

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initial structure of (L, M)-double fuzzy topological spaces is obtained by the initial structure of (L, M)-fuzzy closure spaces induced by them.

2. **Preliminaries**

Throughout this paper, let X be a nonempty set. Let $L = (L, \leq, \vee, \wedge)$ be a complete lattice with the least element 0_L and the greatest element 1_L. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. The second lattice belonging to the context of our work is denoted by M and $M_0 = M - \{0_M\}$ and $M_1 = M - \{1_M\}$.

A complete lattice $L = (L, \leq, \wedge, \vee)$ is called completely distributive, if for any family $\{\{a_{i,j} : j \in J_i\} : i \in I\}$ in L the following identity holds:

$$(CD) \qquad \bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{i,j}) = \bigvee_{\varphi\in\prod_{i\in I} J_i} (\bigwedge_{i\in I} a_{i,\varphi(i)}).$$

Definition 2.1. [16, 17, 29, 31] A triple $L = (L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $(L, \odot)$ is a commutative semigroup.
(L2) $x \odot 1_L = x$, for all $x \in L$.
(L3) $\odot$ is distributive over arbitrary joins:

$$x \odot (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \odot y_i), \forall x \in L, \forall \{y_i\}_{i\in I} \subseteq L.$$

An stsc-quantale $L = (L, \leq, \odot)$ is a $\bigwedge-$distributive quantale if $\odot$ is distributive over non-empty meets:

$$x \odot (\bigwedge_{i\in I} y_i) = \bigwedge_{i\in I}(x \odot y_i), \forall x \in L, \ \forall \{y_i\}_{i\in I} \subseteq L.$$

Remark 2.2. [16, 17, 26, 29, 31, 37] (1) A complete lattice satisfying the infinite distributive law is an stsc-quantale. In particular, the unit interval $([0, 1], \leq, \wedge, 0, 1)$ is an $\wedge-$distributive quantale.

(2) Every left-continuous t-norm T on $[0, 1]$, $([0, 1], \leq, T)$ is an stsc-quantale.

(3) Every continuous t-norm T on $[0, 1]$, $([0, 1], \leq, T)$ is an $\bigwedge-$distributive quantale.

(4) Every GL-monoid is an stsc-quantale.

(5) Let $(L, \leq, \odot)$ be an stsc-quantale. For each $x, y \in L$, we define

$$x \mapsto y = \bigvee\{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence; i.e.

$$x \odot z \leq y \Longleftrightarrow z \leq x \mapsto y, \forall x, y, z \in L.$$

Definition 2.3. [13]-[19],[23, 31, 37] Let $(L, \leq, \odot)$ be an stsc-quantale. A mapping $^\star : L \to L$ is called an order-reversing involution, if it satisfies the following conditions:

(1) $x^{\star\star} = x$, for each $x \in L$,

(2) If $x \leq y$, then $y^{\star} \leq x^{\star}$, for each $x, y \in L$.

An stsc-quantale is called a Girard monoid [19] if $(x \mapsto 0_L) \mapsto 0_L = x,\ \forall x \in L$.

Hence in case L is a Girard monoid, residuation $\mapsto$ induces an order-reversing involution $^{\star} : L \to L$. In this paper, we always assume that $(L, \leq, \odot, \oplus, \star)$ (resp., $(M, \leq, \widetilde{\odot}, \widetilde{\oplus}, \star)$) is a Girard monoid with an order-reversing involution *, and the operation $\oplus$ defined by:

$$x \oplus y = (x^{\star} \odot y^{\star})^{\star}, \quad x^{\star} = x \mapsto 0_L$$

unless otherwise specified, where $\widetilde{\odot}, \widetilde{\oplus}$ denote the quantale operations on M.

Remark 2.4. [20] When the underlying lattice L is the unit interval $[0, 1]$ of the real numbers, the notion of a Girard monoid coincides with the notion of a left continuous t-norm with strong induced negation $\star, (x^{\star} = x \mapsto 0)$.

Lemma 2.5. [21] *Let L be a Girard monoid. For each $x, y, z, x_i, y_i \in L$, we have the following properties:*

(1) *If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \mapsto y \leq x \mapsto z$ and $y \mapsto x \geq z \mapsto x$.*

(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(3) $x \oplus (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I}(x \oplus y_i)$.

(4) $x \mapsto (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I}(x \mapsto y_i)$.

(5) $(\bigvee_{i \in I} x_i) \mapsto y = \bigwedge_{i \in I}(x_i \mapsto y)$.

(6) $x \mapsto (\bigvee_{i \in I} y_i) \geq \bigvee_{i \in I}(x \mapsto y_i)$.

(7) $(\bigwedge_{i \in I} x_i) \mapsto y \geq \bigvee_{i \in I}(x_i \mapsto y)$.

(8) $\bigwedge_{i \in I} x_i^{\star} = (\bigvee_{i \in I} x_i)^{\star}$ *and* $\bigvee_{i \in I} x_i^{\star} = (\bigwedge_{i \in I} x_i)^{\star}$.

(9) $x \mapsto y = y^{\star} \mapsto x^{\star}$.

(10) $x \odot (x^{\star} \oplus y^{\star}) \leq y^{\star}$.

(11) $x \odot y = (x \mapsto y^{\star})^{\star}, \quad x \oplus y = x^{\star} \mapsto y, \quad x \mapsto y = x^{\star} \oplus y$.

(12) $x \mapsto y \leq (x \odot z) \mapsto (y \odot z)$ *and* $(x \mapsto y) \odot (y \mapsto z) \leq x \mapsto z$.

(13) *If $(L, \leq, \odot)$ is $\bigwedge -$distributive, then we have the following (where $I \neq \emptyset$):*

(a) $x \oplus (\bigvee_{i \in I} y_i) = \bigvee_{i \in I}(x \oplus y_i)$.
(b) $x \mapsto (\bigvee_{i \in I} y_i) = \bigvee_{i \in I}(x \mapsto y_i)$.
(c) $(\bigwedge_{i \in I} x_i) \mapsto y = \bigvee_{i \in I}(x_i \mapsto y)$.

Proof. (13) (a) Since $x \odot (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I}(x \odot y_i)$, we have

$x \oplus (\bigvee_{i \in I} y_i) = (x \oplus (\bigvee_{i \in I} y_i))^{\star\star} = (x^{\star} \odot (\bigvee_{i \in I} y_i)^{\star})^{\star} = (x^{\star} \odot (\bigwedge_{i \in I} y_i^{\star}))^{\star} = (\bigwedge_{i \in I}(x^{\star} \odot y_i^{\star}))^{\star} = \bigvee_{i \in I}(x^{\star} \odot y_i^{\star})^{\star} = \bigvee_{i \in I}(x \oplus y_i)$.

(b) $x \mapsto (\bigvee_{i \in I} y_i) = x^{\star} \oplus (\bigvee_{i \in I} y_i) = \bigvee_{i \in I}(x^{\star} \oplus y_i) = \bigvee_{i \in I}(x \mapsto y_i)$.

(c) $(\bigwedge_{i \in I} x_i) \mapsto y = (\bigwedge_{i \in I} x_i)^{\star} \oplus y = (\bigvee_{i \in I} x_i^{\star}) \oplus y = \bigvee_{i \in I}(x_i \mapsto y)$. □

All algebraic operations on L can be extended pointwise to the set L^X as follows: for all $x \in X$,

(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$.

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$.

(3) $(\lambda \mapsto \mu)(x) = \lambda(x) \mapsto \mu(x)$.

Let L be a complete lattice and $\phi : X \longrightarrow Y$ be a function. The Zadeh image and preimage operators $\phi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\phi^{\leftarrow} : L^Y \longrightarrow L^X$ are defined as

$$\phi^{\rightarrow}(\lambda)(y) = \bigvee\{\lambda(x) \mid y = \phi(x)\}, and \quad \phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)), \forall x \in X, y \in Y.$$

respectively

Lemma 2.6. [22] *Let $(L, \leq, \odot)$ be an stsc-quantale and $\phi : X \longrightarrow Y$ be a function. For each $\lambda, \mu \in L^X$ and $\lambda_i \in L^Y$, we have the following properties.*

(1) $\phi^{\rightarrow}(\lambda \odot \mu) \leq \phi^{\rightarrow}(\lambda) \odot \phi^{\rightarrow}(\mu)$, *with equality if ϕ is injective.*
(2) $\phi^{\leftarrow}(\odot_{i=1}^{n} \lambda_i) = \odot_{i=1}^{n} \phi^{\leftarrow}(\lambda_i)$.

3. (L, M)-fuzzy Closure Spaces

Definition 3.1. A map $\mathcal{C} : L^X \times M_0 \times M_1 \rightarrow L^X$ is called an (L, M)-fuzzy closure operator on X iff $\mathcal{C}$ satisfies the following conditions: $\forall\ r \in M_0$ and $s \in M_1$ such that $r \leq s^{\star}$,

(C1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$,

(C2) $\mathcal{C}(\lambda, r, s) \geq \lambda$,

(C3) If $\lambda \leq \mu$, then $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\mu, r, s)$,

(C4) $\mathcal{C}(\lambda \oplus \mu, r \widetilde{\odot} r', s \widetilde{\oplus} s') \leq \mathcal{C}(\lambda, r, s) \oplus \mathcal{C}(\mu, r', s')$,

(C5) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r', s')$, if $r \leq r'$ and $s \geq s'$.

The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy closure space.

An (L, M)-fuzzy closure space $(X, \mathcal{C})$ is called topological if

$$\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) \leq \mathcal{C}(\lambda, r, s), \quad \forall \lambda \in L^X, r \in M_0 \text{ and } s \in M_1 \text{ with } r \leq s^{\star}.$$

Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two (L, M)-fuzzy closure spaces. A map $\Phi : X \rightarrow Y$ is called (L, M)-C-map if $\Phi^{\rightarrow}(\mathcal{C}_1(\lambda, r, s)) \leq \mathcal{C}_2(\Phi^{\rightarrow}(\lambda), r, s)$, $\forall \lambda \in L^X$, $r \in M_0$ and $s \in M_1$ such that $r \leq s^{\star}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be (L, M)-fuzzy closure operators on X. We say that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ ($\mathcal{C}_2$ is coarser than $\mathcal{C}_1$) if $\mathcal{C}_1(\lambda, r, s) \leq \mathcal{C}_2(\lambda, r, s), \forall \lambda \in L^X$, $r \in M_0$ and $s \in M_1$ such that $r \leq s^{\star}$.

Definition 3.2. The pair $(\mathcal{T}, \mathcal{T}^*)$ of maps $\mathcal{T}, \mathcal{T}^* : L^X \rightarrow M$ is called (L, M)-double fuzzy topology on X if it satisfies the following conditions:

(LO1) $\mathcal{T}(\lambda) \leq (\mathcal{T}^*(\lambda))^{\star}$, $\forall \lambda \in L^X$,

(LO2) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1_M$, $\mathcal{T}^*(\underline{0}) = \mathcal{T}^*(\underline{1}) = 0_M$,

(LO3) $\mathcal{T}(\lambda_1 \odot \lambda_2) \geq \mathcal{T}(\lambda_1) \widetilde{\odot} \mathcal{T}(\lambda_2)$ and $\mathcal{T}^*(\lambda_1 \odot \lambda_2) \leq \mathcal{T}^*(\lambda_1) \widetilde{\oplus} \mathcal{T}^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$,

(LO4) $\mathcal{T}(\bigvee_{i \in \Delta} \lambda_i) \geq \bigwedge_{i \in \Delta} \mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigvee_{i \in \Delta} \lambda_i) \leq \bigvee_{i \in \Delta} \mathcal{T}^*(\lambda_i)$, for each $\lambda_i \in L^X, i \in \Delta$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called an (L, M)-double fuzzy topological space ((L, M)-dfts, for short). $\mathcal{T}$ and $\mathcal{T}^*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Let $(\mathcal{U}, \mathcal{U}^*)$ and $(\mathcal{T}, \mathcal{T}^*)$ be (L, M)-double fuzzy topologies on X. We say $(\mathcal{U}, \mathcal{U}^*)$ is finer than $(\mathcal{T}, \mathcal{T}^*)$ ($(\mathcal{T}, \mathcal{T}^*)$ is coarser than $(\mathcal{U}, \mathcal{U}^*)$) if $\mathcal{T}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*(\lambda) \geq \mathcal{U}^*(\lambda)$ for all $\lambda \in L^X$.

Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{U}, \mathcal{U}^*)$ be (L, M)-dfts's. A function $\Phi : X \to Y$ is called LF-continuous iff $\mathcal{U}(\lambda) \leq \mathcal{T}(\Phi^{\leftarrow}(\lambda))$ and $\mathcal{U}^*(\lambda) \geq \mathcal{T}^*(\Phi^{\leftarrow}(\lambda))$, for all $\lambda \in L^Y$.

Example 3.3. Let $X = \{x, y\}$ be a set, $L = M = [0, 1]$ and $x \odot y = max\{x + y - 1, 0\}, x \oplus y = min\{x + y, 1\}$. Then $([0, 1], \leq, \odot)$ is a left-continuous t-norm with strong induced negation $x \mapsto 0 = min\{1 - x, 1\}$. Let $\mu, \rho \in [0, 1]^X$ be defined as follows: $\mu(x) = 0.6, \mu(y) = 0.3, \rho(x) = 0.5, \rho(y) = 0.7$. Define $\mathcal{T}, \mathcal{T}^\star : [0, 1]^X \to [0, 1]$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1\ , & \text{if } \lambda = \underline{0}, \underline{1}; \\ 0.8, & \text{if } \lambda = \mu; \\ 0.3, & \text{if } \lambda = \rho; \\ 0.7, & \text{if } \lambda = \mu \vee \rho; \\ 0.2, & \text{if } \lambda = \mu \wedge \rho; \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{T}^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1}; \\ 0.2, & \text{if } \lambda = \mu; \\ 0.7, & \text{if } \lambda = \rho; \\ 0.3, & \text{if } \lambda = \mu \vee \rho; \\ 0.8, & \text{if } \lambda = \mu \wedge \rho; \\ 1, & \text{otherwise.} \end{cases}$$

Then, the pair $(\mathcal{T}, \mathcal{T}^*)$ is an $([0, 1], [0, 1])$-dft on X.

Remark 3.4. (1) If $(L = M = [0, 1], \odot = \wedge, \oplus = \vee)$ with an order reversing involution $\star$, $(a^\star = 1 - a)$ (L, M)-dfts is the concept of Samanta and Mondal [32].

(2) If L and M are frames with 0 and 1, (L, M)-dfts is the concept of Gutierrez Garcia [10].

(3) If $(L = M = [0, 1], \odot = \wedge, \oplus = \vee)$ with an order reversing involution $\star$, $(a^\star = 1 - a)$ (L, M)-fuzzy closure space is the concept of Lee and Im [25] and Abbas [1].

Theorem 3.5. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an (L, M)-dfts. For each $\lambda \in L^X, r \in M_0$ and $s \in M_1$ such that $r \leq s^\star$, we define an operator $\mathcal{C}_{\mathcal{T},\mathcal{T}^*} : L^X \times M_0 \times M_1 \to L^X$ as follows:*

$$\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigwedge\{\mu \in L^X \mid \mu \geq \lambda, \mathcal{T}(\mu \mapsto \underline{0}) \geq r \text{ and } \mathcal{T}^*(\mu \mapsto \underline{0}) \leq s\}.$$

Then $(X, \mathcal{C}_{\mathcal{T},\mathcal{T}^*})$ is a topological (L, M)-fuzzy closure space and if $r = \bigvee\{r' \in M_0 \mid \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r', s') = \lambda\}$, then $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \lambda$.

Proof. Firstly, we will show that $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}$ is a topological closure operator on X.

(C1), (C2) and (C3) are clear from the definition.

(C4) Let $\lambda, \mu \in L^X$ and $r, r' \in M_0, s, s' \in M_1$. Since $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \geq \lambda$ and $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu, r', s') \geq \mu$, then we have $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \oplus \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu, r', s') \geq \lambda \oplus \mu$. Moreover,

$$\mathcal{T}((\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\oplus\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu,r',s'))\mapsto\underline{0})=$$
$$=\mathcal{T}((\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\mapsto\underline{0})\odot(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu,r',s')\mapsto\underline{0}))\geq$$
$$\geq\mathcal{T}(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\mapsto\underline{0})\ \widetilde{\odot}\ \mathcal{T}(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu,r',s')\mapsto\underline{0})=$$
$$=\mathcal{T}((\bigwedge\{\nu\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\mapsto\underline{0})\widetilde{\odot}$$
$$\widetilde{\odot}\mathcal{T}((\bigwedge\{\rho\mid\rho\geq\mu,\mathcal{T}(\rho\mapsto\underline{0})\geq r',\mathcal{T}^*(\rho\mapsto\underline{0})\leq s'\})\mapsto\underline{0})=$$
$$=\mathcal{T}(\bigvee\{\nu\mapsto\underline{0}\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\widetilde{\odot}$$
$$\widetilde{\odot}\mathcal{T}(\bigvee\{\rho\mapsto\underline{0}\mid\rho\geq\mu,\mathcal{T}(\rho\mapsto\underline{0})\geq r',\mathcal{T}^*(\rho\mapsto\underline{0})\leq s'\})\geq$$
$$\geq\bigwedge\{\mathcal{T}(\nu\mapsto\underline{0})\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\}\widetilde{\odot}$$
$$\widetilde{\odot}\bigwedge\{\mathcal{T}(\rho\mapsto\underline{0})\mid\rho\geq\mu,\mathcal{T}(\rho\mapsto\underline{0})\geq r',\mathcal{T}^*(\rho\mapsto\underline{0})\leq s'\}\geq$$
$$\geq r\widetilde{\odot}r'.$$

Similarly, $\mathcal{T}^*((\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\oplus\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu,r',s'))\mapsto\underline{0})\leq s\widetilde{\oplus}s'$.

Hence, by the definition $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda\oplus\mu,r\widetilde{\odot}r',s\widetilde{\oplus}s')\leq\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\oplus\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mu,r',s')$ is obtained.

(C5) Let $r\leq r'$ and $s\geq s'$. Since $\{\nu\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\}\supseteq\{\rho\mid\rho\geq\lambda,\mathcal{T}(\rho\mapsto\underline{0})\geq r',\mathcal{T}^*(\rho\mapsto\underline{0})\leq s'\}$, we have $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\leq\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r',s')$.

Since $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\geq\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)$ and also

$$\mathcal{T}(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\mapsto\underline{0})=\mathcal{T}((\bigwedge\{\nu\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\mapsto\underline{0})=$$
$$=\mathcal{T}(\bigvee\{\nu\mapsto\underline{0}\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\geq$$
$$\geq\bigwedge\{\mathcal{T}(\nu\mapsto\underline{0})\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\}\geq r.$$

and

$$\mathcal{T}^*(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)\mapsto\underline{0})=\mathcal{T}^*((\bigwedge\{\nu\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\mapsto\underline{0})=$$
$$=\mathcal{T}^*(\bigvee\{\nu\mapsto\underline{0}\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\})\leq$$
$$\leq\bigvee\{\mathcal{T}^*(\nu\mapsto\underline{0})\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r,\mathcal{T}^*(\nu\mapsto\underline{0})\leq s\}\leq s.$$

We have $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s),r,s)\leq\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)$, i.e., $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}$ is topological.

Secondly, let us show the second claim of the Theorem. Let $r=\bigvee\{r'\in M_0\mid\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r',s')=\lambda\}$ and $s=\bigwedge\{s'\in M_1\mid\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r',s')=\lambda\}$.

$$\lambda=\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r',s')=\bigwedge\{\nu\mid\nu\geq\lambda,\mathcal{T}(\nu\mapsto\underline{0})\geq r',\mathcal{T}^*(\nu\mapsto\underline{0})\leq s'\}$$

if and only if $\mathcal{T}(\lambda\mapsto\underline{0})\geq r'$ and $\mathcal{T}^*(\lambda\mapsto\underline{0})\leq s'$. So, by the definition of r and s, we have $\mathcal{T}(\lambda\mapsto\underline{0})\geq r$ and $\mathcal{T}^*(\lambda\mapsto\underline{0})\leq s$. Therefore, $\mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s)=\lambda$. □

Theorem 3.6. *Let M be completely distributive and $(X, \mathcal{C})$ be an (L, M)-fuzzy closure space. Define the maps $\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}} : L^X \to M$ on X by*

$$\mathcal{T}_{\mathcal{C}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{C}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0} \text{ and } r \leq s^\star\}$$

$$\mathcal{T}^*_{\mathcal{C}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{C}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0} \text{ and } r \leq s^\star\}.$$

Then we have the following properties:

(1) *$(\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}})$ is an (L, M)-double fuzzy topology on X.*

(2) *We have $\mathcal{C} = \mathcal{C}_{\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}}$ iff an (L, M)-fuzzy closure space $(X, \mathcal{C})$ satisfies the following conditions:*

(a) *It is topological.*

(b) *If $r = \bigvee\{r' \in M_0 \mid \mathcal{C}(\lambda, r', s') = \lambda$ and $r' \leq (s')^\star\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{C}(\lambda, r', s') = \lambda$ and $r' \leq (s')^\star\}$, then $\mathcal{C}(\lambda, r, s) = \lambda$.*

Proof. (1) (LO1) $(\mathcal{T}^*_{\mathcal{C}}(\lambda))^\star = \bigvee\{s^\star \mid \mathcal{C}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0}$ and $r \leq s^\star\} \geq \bigvee\{r \mid \mathcal{C}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0}$ and $r \leq s^\star\} = \mathcal{T}_{\mathcal{C}}(\lambda)$.

(LO2) is trivial from the definition.

(LO3) Suppose there exist $\lambda_1, \lambda_2 \in L^X$ such that $\mathcal{T}_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \ngeq \mathcal{T}_{\mathcal{C}}(\lambda_1)\widetilde{\odot}\mathcal{T}_{\mathcal{C}}(\lambda_2)$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \nleq \mathcal{T}^*_{\mathcal{C}}(\lambda_1)\widetilde{\oplus}\mathcal{T}^*_{\mathcal{C}}(\lambda_2)$. By the definitions of $\mathcal{T}_{\mathcal{C}}(\lambda_1)$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_1)$, there exist $r_1 \in M_0, s_1 \in M_1$ with $r_1 \leq s_1^\star$ and $\mathcal{C}(\lambda_1 \mapsto \underline{0}, r_1, s_1) = \lambda_1 \mapsto \underline{0}$ such that $\mathcal{T}_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \ngeq r_1\widetilde{\odot}\mathcal{T}_{\mathcal{C}}(\lambda_2)$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \nleq s_1\widetilde{\oplus}\mathcal{T}^*_{\mathcal{C}}(\lambda_2)$. Again, by the definitions of $\mathcal{T}_{\mathcal{C}}(\lambda_2)$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_2)$, there exist $r_2 \in M_0, s_2 \in M_1$ with $r_2 \leq s_2^\star$ and $\mathcal{C}(\lambda_2 \mapsto \underline{0}, r_2, s_2) = \lambda_2 \mapsto \underline{0}$ such that $\mathcal{T}_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \ngeq r_1\widetilde{\odot}r_2$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \nleq s_1\widetilde{\oplus}s_2$. Since by (C4),

$$\begin{aligned}\mathcal{C}((\lambda_1 \odot \lambda_2) \mapsto \underline{0}, r_1\widetilde{\odot}r_2, s_1\widetilde{\oplus}s_2) &= \mathcal{C}((\lambda_1 \mapsto \underline{0}) \oplus (\lambda_2 \mapsto \underline{0}), r_1\widetilde{\odot}r_2, s_1\widetilde{\oplus}s_2) \\ &\leq \mathcal{C}(\lambda_1 \mapsto \underline{0}, r_1, s_1) \oplus \mathcal{C}(\lambda_2 \mapsto \underline{0}, r_2, s_2) \\ &= (\lambda_1 \mapsto \underline{0}) \oplus (\lambda_2 \mapsto \underline{0}) = (\lambda_1 \odot \lambda_2) \mapsto \underline{0}\end{aligned}$$

and by (C2), we have $\mathcal{C}((\lambda_1 \odot \lambda_2) \mapsto \underline{0}, r_1\widetilde{\odot}r_2, s_1\widetilde{\oplus}s_2) = (\lambda_1 \odot \lambda_2) \mapsto \underline{0}$.

Hence, $\mathcal{T}_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \geq r_1\widetilde{\odot}r_2$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda_1 \odot \lambda_2) \leq s_1\widetilde{\oplus}s_2$. This contradicts the assumption.

(LO4) Let us define $A := \{r \mid \mathcal{C}((\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}, r, s) = (\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}\}$ and $B := \{r \mid \mathcal{C}(\lambda_i \mapsto \underline{0}, r, s) = \lambda_i \mapsto \underline{0}$ for every $i \in \Delta\}$. Let $\bigwedge_{i\in\Delta} \mathcal{T}_{\mathcal{C}}(\lambda_i) = \bigwedge_{i\in\Delta} \bigvee\{r \in M_0 \mid \mathcal{C}(\lambda_i \mapsto \underline{0}, r, s) = \lambda_i \mapsto \underline{0}\} = t$ and since M is completely distributive, then $t \leq \bigvee_{f\in F} \bigwedge_{i\in\Delta}\{f(i) \in M_0 \mid \mathcal{C}(\lambda_i \mapsto \underline{0}, f(i), s) = \lambda_i \mapsto \underline{0}\} = t_1$ (F is the set of the respective choice functions). It then follows that $t_1 \leq \bigvee\{r \in M_0 \mid \mathcal{C}(\lambda_i \mapsto \underline{0}, r, s) = \lambda_i \mapsto \underline{0}$ for every $i \in \Delta\} = \bigvee B$. If $r \in B$, then $\mathcal{C}(\lambda_i \mapsto \underline{0}, r, s) = \lambda_i \mapsto \underline{0}$, for each $i \in \Delta$. Therefore, $\mathcal{C}((\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}, r, s) \leq \mathcal{C}(\lambda_i \mapsto \underline{0}, r, s) = \lambda_i \mapsto \underline{0}$, for each $i \in \Delta$. Then, $\mathcal{C}((\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}, r, s) \leq \bigwedge_{i\in\Delta}(\lambda_i \mapsto \underline{0})$ and by (C2), $\mathcal{C}((\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}, r, s) = (\bigvee_{i\in\Delta} \lambda_i) \mapsto \underline{0}$. Since $r \in A$, $B \subset A$. Hence, $\mathcal{T}_{\mathcal{C}}(\bigvee_{i\in\Delta} \lambda_i) \geq \bigwedge_{i\in\Delta} \mathcal{T}_{\mathcal{C}}(\lambda_i)$. Similarly, $\mathcal{T}^*_{\mathcal{C}}(\bigvee_{i\in\Delta} \lambda_i) \leq \bigvee_{i\in\Delta} \mathcal{T}^*_{\mathcal{C}}(\lambda_i)$.

(2) Let $\mathcal{C} = \mathcal{C}_{\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}}$.

(a) $\mathcal{C}_{\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}}(\mathcal{C}(\lambda, r, s), r, s) = \bigwedge\{\nu \mid \nu \geq \mathcal{C}(\lambda, r, s), \mathcal{T}_{\mathcal{C}}(\nu \mapsto \underline{0}) \geq r, \mathcal{T}^*_{\mathcal{C}}(\nu \mapsto \underline{0}) \leq s\}$.

$$\mathcal{T}_{\mathcal{C}}(\mathcal{C}(\lambda,r,s)\mapsto \underline{0}) = \mathcal{T}_{\mathcal{C}}((\bigwedge\{\rho \mid \rho\geq\lambda, \mathcal{T}_{\mathcal{C}}(\rho\mapsto\underline{0})\geq r, \mathcal{T}^*_{\mathcal{C}}(\rho\mapsto\underline{0})\leq s\})\mapsto\underline{0}) =$$

$$= \mathcal{T}_{\mathcal{C}}(\bigvee\{\rho\mapsto\underline{0} \mid \rho\geq\lambda, \mathcal{T}_{\mathcal{C}}(\rho\mapsto\underline{0})\geq r, \mathcal{T}^*_{\mathcal{C}}(\rho\mapsto\underline{0})\leq s\}) \geq$$

$$\geq \bigwedge\{\mathcal{T}_{\mathcal{C}}(\rho\mapsto\underline{0}) \mid \rho\geq\lambda, \mathcal{T}_{\mathcal{C}}(\rho\mapsto\underline{0})\geq r, \mathcal{T}^*_{\mathcal{C}}(\rho\mapsto\underline{0})\leq s\} \geq r.$$

Similarly, $\mathcal{T}^*_{\mathcal{C}}(\mathcal{C}(\lambda,r,s)\mapsto\underline{0})\leq s$. Therefore,

$$\mathcal{C}(\lambda,r,s) = \mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}(\mathcal{C}(\lambda,r,s),r,s) = \mathcal{C}(\mathcal{C}(\lambda,r,s),r,s).$$

(b) Let $r = \bigvee\{r'\in M_0 \mid \mathcal{C}(\lambda,r',s')=\lambda \text{ and } r'\leq (s')^\star\}$ and $s=\bigwedge\{s'\in M_1 \mid \mathcal{C}(\lambda,r',s')=\lambda \text{ and } r'\leq(s')^\star\}$. Since $\mathcal{T}_{\mathcal{C}}(\lambda\mapsto\underline{0}) = \bigvee\{r'\in M_0 \mid \mathcal{C}(\lambda,r',s') = \lambda \text{ and } r'\leq(s')^\star\} = r$ and $\mathcal{T}^*_{\mathcal{C}}(\lambda\mapsto\underline{0}) = \bigwedge\{s'\in M_1 \mid \mathcal{C}(\lambda,r',s')=\lambda \text{ and } r'\leq (s')^\star\} = s$, the hypothesis provides $\mathcal{C}(\lambda,r,s) = \mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}(\lambda,r,s)=\lambda$.

Conversely, let (a) and (b) be satisfied. We know that, by (C2), $\mathcal{C}(\lambda,r,s)\geq\lambda$. Since, $\mathcal{C}$ is topological, $\mathcal{T}_{\mathcal{C}}(\mathcal{C}(\lambda,r,s)\mapsto\underline{0})\geq r$ and $\mathcal{T}^*_{\mathcal{C}}(\mathcal{C}(\lambda,r,s)\mapsto\underline{0})\leq s$. Hence, by the definition of $\mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}$, we have that $\mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}(\lambda,r,s)\leq\mathcal{C}(\lambda,r,s)$.

Now we will show that $\mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}(\lambda,r,s)\geq\mathcal{C}(\lambda,r,s)$. For the proof of this inequality, it is enough to show that $\mathcal{C}(\lambda,r,s)\leq\mu$ for every $\mu\in L^X$ such that $\mu\geq\lambda$ and $\mathcal{T}_{\mathcal{C}}(\mu\mapsto\underline{0})\geq r$, $\mathcal{T}^*_{\mathcal{C}}(\mu\mapsto\underline{0})\leq s$. By the condition (b), $\mathcal{C}(\mu,\mathcal{T}_{\mathcal{C}}(\mu\mapsto\underline{0}),\mathcal{T}^*_{\mathcal{C}}(\mu\mapsto\underline{0}))=\mu$. Since $\mathcal{T}_{\mathcal{C}}(\mu\mapsto\underline{0})\geq r$, $\mathcal{T}^*_{\mathcal{C}}(\mu\mapsto\underline{0})\leq s$ and by (C5), we have that $\mu=\mathcal{C}(\mu,\mathcal{T}_{\mathcal{C}}(\mu\mapsto\underline{0}),\mathcal{T}^*_{\mathcal{C}}(\mu\mapsto\underline{0}))\geq\mathcal{C}(\mu,r,s)\geq\mathcal{C}(\lambda,r,s)$. Hence it is proved.

□

Example 3.7. The unit interval $I=[0,1], \odot=\wedge, \oplus=\vee$ is a completely distributive lattice with an order reversing involution defined as $a^\star = 1-a$. Let $X=\{x,y,z\}$ be a set and χ_A a characteristic function for a subset A of X. A fuzzy point in I^X is a fuzzy set x_t, where $t\in I_0$, such that $x_t(y)=t$ when $y=x$ and $x_t(y)=0$ otherwise.

(A) If $(X,\mathcal{C})$ does not satisfy the condition (a) of Theorem 3.6 (2), then, in general, $\mathcal{C}\neq\mathcal{C}_{\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}}$ as the following example shows. Define $\mathcal{C}: I^X\times I_0\times I_1\longrightarrow I^X$ as follows:

$$\mathcal{C}(\lambda,r,s) = \begin{cases} \underline{0}, & \text{if } \lambda=\underline{0}, r\in I_0, s\in I_1\\ \chi_{\{x,y\}}, & \text{if } \lambda = x_t, 0<r\leq\frac{1}{2}, \frac{1}{2}\leq s<1\\ \chi_{\{z\}}, & \text{if } \lambda=z_k, 0<r\leq\frac{1}{2},\frac{1}{2}\leq s<1\\ \underline{1}, & \text{otherwise.}\end{cases}$$

Then $(X,\mathcal{C})$ is an (I,I)-double fuzzy closure space. Since $\mathcal{C}(x_t,\frac{1}{3},\frac{2}{3})=\chi_{\{x,y\}}$ and $\mathcal{C}(\chi_{\{x,y\}},\frac{1}{3},\frac{2}{3})=\underline{1}$, we have $\mathcal{C}(\mathcal{C}(x_t,\frac{1}{3},\frac{2}{3}),\frac{1}{3},\frac{2}{3})\neq\mathcal{C}(x_t,\frac{1}{3},\frac{2}{3})$. Hence, $(X,\mathcal{C})$ is not a topological (I,I)-double fuzzy closure space. From Theorem 3.6, we can obtain $\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}}: I^X\longrightarrow I$ as follows:

$$\mathcal{T}_{\mathcal{C}}(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ \frac{1}{2}, & \text{if } \lambda=\chi_{\{x,y\}}\\ 0, & \text{otherwise}\end{cases}, \quad \mathcal{T}^*_{\mathcal{C}}(\lambda)=\begin{cases}0, & \text{if } \lambda=\underline{0},\underline{1}\\ \frac{1}{2}, & \text{if } \lambda=\chi_{\{x,y\}}\\ 1, & \text{otherwise}\end{cases}.$$

Thus, $\mathcal{C}_{\mathcal{T}_\mathcal{C},\mathcal{T}^*_\mathcal{C}}(\lambda, r, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, r \in I_0, s \in I_1 \\ \chi_{\{z\}}, & \text{if } \lambda = z_k, 0 < r \le \frac{1}{2}, \frac{1}{2} \le s < 1 \\ \underline{1}, & \text{otherwise} \end{cases}$,

Hence, $\mathcal{C} \neq \mathcal{C}_{\mathcal{T}_\mathcal{C},\mathcal{T}^*_\mathcal{C}}$.

(B) If $(X, \mathcal{C})$ does not satisfy the condition (b) of Theorem 3.6 (2), then, in general, $\mathcal{C} \neq \mathcal{C}_{\mathcal{T}_\mathcal{C},\mathcal{T}^*_\mathcal{C}}$ as the following example shows. For $\mu \in I^X$ such that $\mu \neq \underline{0}, \underline{1}$, we define an (I, I)-fuzzy closure operator $\mathcal{C} : I^X \times I_0 \times I_1 \longrightarrow I^X$ as follows:

$$\mathcal{C}(\lambda, r, s) = \begin{cases} \underline{0}, & \text{if } \lambda = \underline{0}, r \in I_0, s \in I_1 \\ \mu, & \text{if } \underline{0} \neq \lambda \le \mu, 0 < r < \frac{1}{2}, \frac{1}{2} < s < 1 \\ \underline{1}, & \text{otherwise} \end{cases}.$$

Then $(X, \mathcal{C})$ is a topological (I, I)-fuzzy closure space. From Theorem 3.6, we can obtain $\mathcal{T}_\mathcal{C}, \mathcal{T}^*_\mathcal{C} : I^X \longrightarrow I$ as follows:

$$\mathcal{T}_\mathcal{C}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - \mu \\ 0, & \text{otherwise} \end{cases}, \quad \mathcal{T}^*_\mathcal{C}(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1} \\ \frac{1}{2}, & \text{if } \lambda = \underline{1} - \mu \\ 1, & \text{otherwise} \end{cases}.$$

Since $\mathcal{C}(\mu, r, s) = \mu$ for all $r < \frac{1}{2}, s > \frac{1}{2}$, $\bigvee\{r \in I_0 \mid \mathcal{C}(\mu, r, s) = \mu \text{ and } r \le 1 - s\} = \frac{1}{2}$ and $\bigwedge\{s \in I_1 \mid \mathcal{C}(\mu, r, s) = \mu \text{ and } r \le 1 - s\} = \frac{1}{2}$ but $\mathcal{C}(\mu, \frac{1}{2}, \frac{1}{2}) = \underline{1}$. On the other hand, we have $\mathcal{C}_{\mathcal{T}_\mathcal{C},\mathcal{T}^*_\mathcal{C}}(\mu, \frac{1}{2}, \frac{1}{2}) = \mu$. Hence, $\mathcal{C} \neq \mathcal{C}_{\mathcal{T}_\mathcal{C},\mathcal{T}^*_\mathcal{C}}$.

Theorem 3.8. *Let $(\mathcal{T}, \mathcal{T}^*)$ be an (L, M)-double fuzzy topology and $\mathcal{C}$ be an (L, M)-fuzzy closure operator on X as in Theorem 3.5. Then $(\mathcal{T}_\mathcal{C}, \mathcal{T}^*_\mathcal{C})$ is an (L, M)-double fuzzy topology on X such that $(\mathcal{T}, \mathcal{T}^*) = (\mathcal{T}_\mathcal{C}, \mathcal{T}^*_\mathcal{C})$.*

Proof. Let $\lambda \in L^X$ and $\mathcal{C}$ be an (L, M)-fuzzy closure operator on X as in Theorem 3.5, i.e, $\mathcal{C} = \mathcal{C}_{\mathcal{T},\mathcal{T}^*}$. Since

$$\mathcal{T}_{\mathcal{C}_{\mathcal{T},\mathcal{T}^*}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0} \text{ and } r \le s^\star\}$$

$$\mathcal{T}^*_{\mathcal{C}_{\mathcal{T},\mathcal{T}^*}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda \mapsto \underline{0}, r, s) = \lambda \mapsto \underline{0} \text{ and } r \le s^\star\},$$

$$\lambda \mapsto \underline{0} = \mathcal{C}_{\mathcal{T},\mathcal{T}^*}(\lambda \mapsto \underline{0}, r, s) = \bigwedge\{\nu \mid \nu \ge \lambda \mapsto \underline{0}, \mathcal{T}(\nu \mapsto \underline{0}) \ge r, \mathcal{T}^*(\nu \mapsto \underline{0}) \le s\}$$

if and only if $\mathcal{T}(\lambda) \ge r$ and $\mathcal{T}^*(\lambda) \le s$. Therefore,

$$\mathcal{T}_{\mathcal{C}_{\mathcal{T},\mathcal{T}^*}}(\lambda) = \bigvee\{r \in M_0 \mid \mathcal{T}(\lambda) \ge r\} = \mathcal{T}(\lambda)$$

$$\mathcal{T}^*_{\mathcal{C}_{\mathcal{T},\mathcal{T}^*}}(\lambda) = \bigwedge\{s \in M_1 \mid \mathcal{T}^*(\lambda) \le s\} = \mathcal{T}^*(\lambda).$$

□

Definition 3.9. The pair $(\mathcal{B}, \mathcal{B}^*)$ of maps $\mathcal{B}, \mathcal{B}^* : L^X \to M$ is called (L, M)-double fuzzy base on X if it satisfies the following conditions:

(LB1) $\mathcal{B}(\lambda) \le (\mathcal{B}^*(\lambda))^\star$, $\forall \lambda \in L^X$,

(LB2) $\mathcal{B}(\underline{0}) = \mathcal{B}(\underline{1}) = 1_M$, $\mathcal{B}^*(\underline{0}) = \mathcal{B}^*(\underline{1}) = 0_M$,

(LB3) $\mathcal{B}(\lambda_1 \odot \lambda_2) \ge \mathcal{B}(\lambda_1) \widetilde{\odot} \mathcal{B}(\lambda_2)$ and $\mathcal{B}^*(\lambda_1 \odot \lambda_2) \le \mathcal{B}^*(\lambda_1) \widetilde{\oplus} \mathcal{B}^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$.

Theorem 3.10. *Let $(\mathcal{B},\mathcal{B}^*)$ be an (L,M)-double fuzzy base on X. Define the maps $\mathcal{T}_{\mathcal{B}},\mathcal{T}^*_{\mathcal{B}^*}: L^X \to M$ as follows:*

$$\mathcal{T}_{\mathcal{B}}(\mu)=\bigvee\{\bigwedge_{j\in\Lambda}\mathcal{B}(\mu_j)\mid \mu=\bigvee_{j\in\Lambda}\mu_j\},$$

$$\mathcal{T}^*_{\mathcal{B}^*}(\mu)=\bigwedge\{\bigvee_{j\in\Lambda}\mathcal{B}^*(\mu_j)\mid \mu=\bigvee_{j\in\Lambda}\mu_j\}.$$

If L is completely distributive and M is $\bigwedge$-distributive quantale, then $(\mathcal{T}_{\mathcal{B}},\mathcal{T}^*_{\mathcal{B}^*})$ is the coarsest (L,M)-double fuzzy topology on X such that $\mathcal{T}_{\mathcal{B}}(\lambda)\geq\mathcal{B}(\lambda)$ and $\mathcal{T}^*_{\mathcal{B}^*}(\lambda)\leq\mathcal{B}^*(\lambda)$ for all $\lambda\in L^X$.

Proof. (LO1) By (LB1), $\mathcal{B}(\mu_j)\leq(\mathcal{B}^*(\mu_j))^\star$, for each $j\in\Lambda$. Then $\bigwedge_{j\in\Lambda}\mathcal{B}(\mu_j)\leq(\bigvee_{j\in\Lambda}\mathcal{B}^*(\mu_j))^\star$. So, $\bigvee\{\bigwedge_{j\in\Lambda}\mathcal{B}(\mu_j)\mid\mu=\bigvee_{j\in J}\mu_j\}\leq(\bigwedge\{\bigvee_{j\in\Lambda}\mathcal{B}^*(\mu_j)\mid\mu=\bigvee_{j\in J}\mu_j\})^\star$. Hence, we have $\mathcal{T}_{\mathcal{B}}(\mu)\leq(\mathcal{T}^*_{\mathcal{B}^*}(\mu))^\star$.

(LO2) is trivial from the definition of $(\mathcal{T}_{\mathcal{B}},\mathcal{T}^*_{\mathcal{B}^*})$.

(LO3) For all families $\{\lambda_j\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j\}$ and $\{\mu_k\mid\mu=\bigvee_{k\in\Gamma}\mu_k\}$, there exists a family $\{\lambda_j\odot\mu_k\}$ such that

$$\lambda\odot\mu=(\bigvee_{j\in\Lambda}\lambda_j)\odot(\bigvee_{k\in\Gamma}\mu_k)=\bigvee_{j\in\Lambda,k\in\Gamma}(\lambda_j\odot\mu_k).$$

It implies

$$\begin{aligned}
\mathcal{T}_{\mathcal{B}}(\lambda)\widetilde{\odot}\mathcal{T}_{\mathcal{B}}(\mu)&=(\bigvee\{\bigwedge_{j\in\Lambda}\mathcal{B}(\lambda_j)\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j\})\widetilde{\odot}(\bigvee\{\bigwedge_{k\in\Gamma}\mathcal{B}(\mu_k)\mid\mu=\bigvee_{k\in\Gamma}\mu_k\})=\\
&=\bigvee\{(\bigwedge_{j\in\Lambda}\mathcal{B}(\lambda_j))\widetilde{\odot}(\bigwedge_{k\in\Gamma}\mathcal{B}(\mu_k))\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\leq\\
&\leq\bigvee\{\bigwedge_{j\in\Lambda,k\in\Gamma}(\mathcal{B}(\lambda_j)\widetilde{\odot}\mathcal{B}(\mu_k))\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\leq\\
&\leq\bigvee\{\bigwedge_{j\in\Lambda,k\in\Gamma}\mathcal{B}(\lambda_j\odot\mu_k)\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\leq\\
&\leq\bigvee\{\bigwedge_{j\in\Lambda,k\in\Gamma}\mathcal{B}(\lambda_j\odot\mu_k)\mid\lambda\odot\mu=\bigvee_{j\in\Lambda,k\in\Gamma}(\lambda_j\odot\mu_k)\}\leq\\
&\leq\mathcal{T}_{\mathcal{B}}(\lambda\odot\mu).
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^*_{\mathcal{B}^*}(\lambda)\widetilde{\oplus}\mathcal{T}^*_{\mathcal{B}^*}(\mu)&=(\bigwedge\{\bigvee_{j\in\Lambda}\mathcal{B}^*(\lambda_j)\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j\})\widetilde{\oplus}(\bigwedge\{\bigvee_{k\in\Gamma}\mathcal{B}^*(\mu_k)\mid\mu=\bigvee_{k\in\Gamma}\mu_k\})=\\
&=\bigwedge\{(\bigvee_{j\in\Lambda}\mathcal{B}^*(\lambda_j))\widetilde{\oplus}(\bigvee_{k\in\Gamma}\mathcal{B}^*(\mu_k))\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\geq\\
&\geq\bigwedge\{\bigvee_{j\in\Lambda,k\in\Gamma}(\mathcal{B}^*(\lambda_j)\widetilde{\oplus}\mathcal{B}^*(\mu_k))\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\geq\\
&\geq\bigwedge\{\bigvee_{j\in\Lambda,k\in\Gamma}\mathcal{B}^*(\lambda_j\oplus\mu_k)\mid\lambda=\bigvee_{j\in\Lambda}\lambda_j,\mu=\bigvee_{k\in\Gamma}\mu_k\}\geq\\
&\geq\bigwedge\{\bigvee_{j\in\Lambda,k\in\Gamma}\mathcal{B}^*(\lambda_j\odot\mu_k)\mid\lambda\odot\mu=\bigvee_{j\in\Lambda,k\in\Gamma}(\lambda_j\odot\mu_k)\}\geq\\
&\geq\mathcal{T}^*_{\mathcal{B}^*}(\lambda\odot\mu).
\end{aligned}$$

□

(LO4) Let $\wp_i$ be the collection of all index sets K_i such that $\{\lambda_{i_k} \in L^X \mid \lambda_i = \bigvee_{k \in K_i} \lambda_{i_k}\}$ with $\lambda = \bigvee_{i \in \Gamma} \lambda_i = \bigvee_{i \in \Gamma} \bigvee_{k \in K_i} \lambda_{i_k}$. For each $i \in \Gamma$ and each $\Psi \in \prod_{i \in \Gamma} \wp_i$ with $\Psi(i) = K_i$, we have

$$\mathcal{T}_{\mathcal{B}}(\lambda) \geq \bigwedge_{i \in \Gamma} (\bigwedge_{k \in K_i} \mathcal{B}(\lambda_{i_k})), \tag{A}$$

$$\mathcal{T}^*_{\mathcal{B}^*}(\lambda) \leq \bigvee_{i \in \Gamma} (\bigvee_{k \in K_i} \mathcal{B}^*(\lambda_{i_k})). \tag{B}$$

Put $a_{i,\Psi(i)} = \bigwedge_{k \in K_i} (\mathcal{B}(\lambda_{i_k}))$. From (A),

$$\begin{array}{rcl} \mathcal{T}_{\mathcal{B}}(\lambda) & \geq & \bigvee_{\Psi \in \prod_{i \in \Gamma} \wp_i} (\bigwedge_{i \in \Gamma} a_{i,\Psi(i)}) \\ & = & \bigwedge_{i \in \Gamma} (\bigvee_{M_i \in \wp_i} a_{i,M_i}) \\ & = & \bigwedge_{i \in \Gamma} (\bigvee_{M_i \in \wp_i} (\bigwedge_{m \in M_i} (\mathcal{B}(\lambda_{i_m})))) \\ & = & \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{B}}(\lambda_i). \end{array}$$

Put $b_{i,\Psi(i)} = \bigvee_{k \in K_i} (\mathcal{B}^*(\lambda_{i_k}))$. From (B),

$$\begin{array}{rcl} \mathcal{T}^*_{\mathcal{B}^*}(\lambda) & \leq & \bigwedge_{\Psi \in \prod_{i \in \Gamma} \wp_i} (\bigvee_{i \in \Gamma} b_{i,\Psi(i)}) \\ & = & \bigvee_{i \in \Gamma} (\bigwedge_{M_i \in \wp_i} b_{i,M_i}) \\ & = & \bigvee_{i \in \Gamma} (\bigwedge_{M_i \in \wp_i} (\bigvee_{m \in M_i} (\mathcal{B}^*(\lambda_{i_m})))) \\ & = & \bigvee_{i \in \Gamma} \mathcal{T}^*_{\mathcal{B}^*}(\lambda_i). \end{array}$$

Thus, $(\mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ is an (L, M)-double fuzzy topology on X.
If $\mathcal{T} \geq \mathcal{B}$ and $\mathcal{T}^* \leq \mathcal{B}^*$, for every $\lambda = \bigvee_{j \in \Lambda} \lambda_j$,

$$\mathcal{T}(\lambda) \geq \bigwedge_{j \in \Lambda} \mathcal{T}(\lambda_j) \geq \bigwedge_{j \in \Lambda} \mathcal{B}(\lambda_j),$$

$$\mathcal{T}^*(\lambda) \leq \bigvee_{j \in \Lambda} \mathcal{T}^*(\lambda_j) \leq \bigvee_{j \in \Lambda} \mathcal{B}^*(\lambda_j).$$

Thus, $\mathcal{T} \geq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}^* \leq \mathcal{T}^*_{\mathcal{B}^*}$.
From Theorem 3.10, we easily prove the following lemma:

Lemma 3.11. *Let $(\mathcal{T}, \mathcal{T}^*)$ be an (L, M)-double fuzzy topology on X and $(\mathcal{B}, \mathcal{B}^*)$ be an (L, M)-double fuzzy base on Y. Then a map $\phi : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ is LF-continuous iff $\mathcal{B}(\lambda) \leq \mathcal{T}(\phi^{\leftarrow}(\lambda))$ and $\mathcal{B}^*(\lambda) \geq \mathcal{T}^*(\phi^{\leftarrow}(\lambda))$, for all $\lambda \in L^Y$.*

Proof. ($\Rightarrow$) Let $\phi : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ be an LF-continuous map and $\mu \in L^Y$. Then,

$$\mathcal{T}(\phi^{\leftarrow}(\mu)) \geq \mathcal{T}_{\mathcal{B}}(\mu) \geq \mathcal{B}(\mu) \text{ and } \mathcal{T}^*(\phi^{\leftarrow}(\mu)) \leq \mathcal{T}^*_{\mathcal{B}^*}(\mu) \leq \mathcal{B}^*(\mu).$$

($\Leftarrow$) Let $\mathcal{T}(\phi^{\leftarrow}(\mu)) \geq \mathcal{B}(\mu)$ and $\mathcal{T}^*(\phi^{\leftarrow}(\mu)) \leq \mathcal{B}^*(\mu), \forall\ \mu \in L^Y$. Let $\nu \in L^Y$. For every family of $\{\nu_j \in L^Y \mid \nu = \bigvee_{j \in \Lambda} \nu_j\} \subset L^Y$, we have that

$$\mathcal{T}(\phi^{\leftarrow}(\nu)) = \mathcal{T}(\phi^{\leftarrow}(\bigvee_{j \in \Lambda} \nu_j)) = \mathcal{T}(\bigvee_{j \in \Lambda} \phi^{\leftarrow}(\nu_j)) \geq \bigwedge_{j \in \Lambda} \mathcal{T}(\phi^{\leftarrow}(\nu_j)) \geq \bigwedge_{j \in \Lambda} \mathcal{B}(\nu_j).$$

If we take supremum over $\{\nu_j \in L^Y \mid \nu = \bigvee_{j\in\Lambda} \nu_j\}$, we obtain that $\mathcal{T}(\phi^{\leftarrow}(\nu)) \geq \mathcal{T}_{\mathcal{B}}(\nu), \forall\, \nu \in L^Y$. Similarly, $\mathcal{T}^*(\phi^{\leftarrow}(\nu)) \leq \mathcal{T}^*_{\mathcal{B}^*}(\nu), \forall\, \nu \in L^Y$. □

Theorem 3.12. *Let $\{(X_i, \mathcal{T}_i, \mathcal{T}_i^*)\}_{i\in\Gamma}$ be a family of (L,M)-dfts's, X a set and for each $i \in \Gamma$, $\phi_i : X \to X_i$ a map. Define $\mathcal{B}, \mathcal{B}^* : L^X \to M$ on X by:*

$$\mathcal{B}(\mu) = \bigvee\{\widetilde{\bigodot_{j=1}^{n}}\mathcal{T}_{k_j}(\nu_{k_j}) \mid \mu = \bigodot_{j=1}^{n} \phi_{k_j}^{\leftarrow}(\nu_{k_j})\},$$

$$\mathcal{B}^*(\mu) = \bigwedge\{\widetilde{\bigoplus_{j=1}^{n}}\mathcal{T}^*_{k_j}(\nu_{k_j}) \mid \mu = \bigodot_{j=1}^{n} \phi_{k_j}^{\leftarrow}(\nu_{k_j})\},$$

where $\bigvee$ and $\bigwedge$ are taken over all finite subsets $K = \{k_1, k_2, ..., k_n\} \subset \Gamma$. Then,

(1) *$(\mathcal{B}, \mathcal{B}^*)$ is an (L,M)-double fuzzy base on X.*

(2) *The (L,M)-double fuzzy topology $(\mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ generated by $(\mathcal{B}, \mathcal{B}^*)$ is the coarsest (L,M)-double fuzzy topology on X for which all $\phi_i, i \in \Gamma$ are LF-continuous maps.*

(3) *A map $\phi : (Y, \mathcal{U}, \mathcal{U}^*) \to (X, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ is LF-continuous iff for each $i \in \Gamma$, $\phi_i \circ \phi : (Y, \mathcal{U}, \mathcal{U}^*) \to (X_i, \mathcal{T}_i, \mathcal{T}_i^*)$ is an LF-continuous map.*

Proof. (1) (LB1) Since, by (LO1), $\mathcal{T}_{k_j}(\nu_{k_j}) \leq (\mathcal{T}^*_{k_j}(\nu_{k_j}))^\star$, then $\widetilde{\bigodot}_{j=1}^{n}\mathcal{T}_{k_j}(\nu_{k_j}) \leq \widetilde{\bigodot}_{j=1}^{n}(\mathcal{T}^*_{k_j}(\nu_{k_j}))^\star = (\widetilde{\bigoplus}_{j=1}^{n}\mathcal{T}^*_{k_j}(\nu_{k_j}))^\star$. Then, we have that

$$\bigvee\{\widetilde{\bigodot_{j=1}^{n}}\mathcal{T}_{k_j}(\nu_{k_j}) \mid \mu = \bigodot_{j=1}^{n} \phi_{k_j}^{\leftarrow}(\nu_{k_j})\} \leq (\bigwedge\{\widetilde{\bigoplus_{j=1}^{n}}\mathcal{T}^*_{k_j}(\nu_{k_j}) \mid \mu = \bigodot_{j=1}^{n} \phi_{k_j}^{\leftarrow}(\nu_{k_j})\})^\star.$$

(LB2) Since $\lambda = \phi_i^{\leftarrow}(\lambda)$ for each $\lambda \in \{\underline{0}, \underline{1}\}$, $\mathcal{B}(\underline{1}) = \mathcal{B}(\underline{0}) = 1_M$ and $\mathcal{B}^*(\underline{1}) = \mathcal{B}^*(\underline{0}) = 0_M$.

(LB3) For all finite subsets $K = \{k_1, ..., k_p\}$ and $J = \{j_1, ..., j_q\}$ of Γ such that

$$\lambda = \bigodot_{i=1}^{p} \phi_{k_i}^{\leftarrow}(\lambda_{k_i}) \text{ and } \mu = \bigodot_{i=1}^{q} \phi_{j_i}^{\leftarrow}(\mu_{j_i}),$$

we have

$$\lambda \odot \mu = (\bigodot_{i=1}^{p} \phi_{k_i}^{\leftarrow}(\lambda_{k_i})) \odot (\bigodot_{i=1}^{q} \phi_{j_i}^{\leftarrow}(\mu_{j_i})).$$

Furthermore, we have for each $k \in K \cap J$,

$$\phi_k^{\leftarrow}(\lambda_k) \odot \phi_k^{\leftarrow}(\mu_k) = \phi_k^{\leftarrow}(\lambda_k \odot \mu_k).$$

We have

$$\mathcal{B}(\lambda \odot \mu) \geq (\widetilde{\bigodot_{i=1}^{p}}\mathcal{T}_{k_i}(\lambda_{k_i}))\widetilde{\odot}(\widetilde{\bigodot_{i=1}^{q}}\mathcal{T}_{j_i}(\mu_{j_i})),$$

$$\mathcal{B}^*(\lambda \odot \mu) \leq (\widetilde{\bigoplus_{i=1}^{p}}\mathcal{T}^*_{k_i}(\lambda_{k_i}))\widetilde{\oplus}(\widetilde{\bigoplus_{i=1}^{q}}\mathcal{T}^*_{j_i}(\mu_{j_i})).$$

Then, $\mathcal{B}(\lambda \odot \mu) \geq \mathcal{B}(\lambda) \widetilde{\odot} \mathcal{B}(\mu)$ and $\mathcal{B}^*(\lambda \odot \mu) \leq \mathcal{B}^*(\lambda) \widetilde{\oplus} \mathcal{B}^*(\mu)$.

(2) For each $i \in \Gamma$ and each $\lambda_i \in L^{X_i}$, we have

$$\mathcal{T}_{\mathcal{B}}(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{B}(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i),$$
$$\mathcal{T}^*_{\mathcal{B}^*}(\phi_i^{\leftarrow}(\lambda_i)) \leq \mathcal{B}^*(\phi_i^{\leftarrow}(\lambda_i)) \leq \mathcal{T}_i^*(\lambda_i).$$

Thus, for each $i \in \Gamma$, $\phi_i : (X, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*}) \to (X_i, \mathcal{T}_i, \mathcal{T}_i^*)$ is LF-continuous. Let $\phi_i : (X, \mathcal{T}^\circ, \mathcal{T}^{\circ *}) \to (X_i, \mathcal{T}_i, \mathcal{T}_i^*)$ be LF-continuous, that is, for each $i \in \Gamma$ and $\lambda_i \in L^{X_i}$, $\mathcal{T}^\circ(\phi_i^{\leftarrow}(\lambda_i)) \geq \mathcal{T}_i(\lambda_i)$ and $\mathcal{T}^{\circ *}(\phi_i^{\leftarrow}(\lambda_i)) \leq \mathcal{T}_i^*(\lambda_i)$. For all finite subsets $K = \{k_1, ..., k_p\}$ of Γ such that $\lambda = \bigodot_{i=1}^p \phi_{k_i}^{\leftarrow}(\lambda_{k_i})$ we have

$$\mathcal{T}^\circ(\lambda) \geq \widetilde{\bigodot_{i=1}^{p}} \mathcal{T}^\circ(\phi_{k_i}^{\leftarrow}(\lambda_{k_i})) \geq \widetilde{\bigodot_{i=1}^{p}} \mathcal{T}_{k_i}(\lambda_{k_i}),$$
$$\mathcal{T}^{\circ *}(\lambda) \leq \widetilde{\bigoplus_{i=1}^{p}} \mathcal{T}^{\circ *}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i})) \leq \widetilde{\bigoplus_{i=1}^{p}} \mathcal{T}^*_{k_i}(\lambda_{k_i}).$$

It implies $\mathcal{T}^\circ(\lambda) \geq \mathcal{B}(\lambda)$ and $\mathcal{T}^{\circ *}(\lambda) \leq \mathcal{B}^*(\lambda)$ for each $\lambda \in L^X$. By Theorem 3.10, $\mathcal{T}^\circ \geq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}^{\circ *} \leq \mathcal{T}^*_{\mathcal{B}^*}$.

(3) ($\Longrightarrow$) It is straightforward since the composition of LF-continuous maps is LF-continuous.

($\Longleftarrow$) For all finite subsets $K = \{k_1, ..., k_p\}$ of Γ such that $\lambda = \bigodot_{i=1}^p \phi_{k_i}^{\leftarrow}(\lambda_{k_i})$, since $\phi_{k_i} \circ \phi : (Y, \mathcal{U}, \mathcal{U}^*) \to (X_{k_i}, \mathcal{T}_{k_i}, \mathcal{T}^*_{k_i})$ is LF-continuous,

$$\mathcal{U}(\phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \geq \mathcal{T}_{k_i}(\lambda_{k_i}), \tag{C}$$

$$\mathcal{U}^*(\phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \leq \mathcal{T}^*_{k_i}(\lambda_{k_i}). \tag{D}$$

Hence, we have

$$\begin{aligned} \mathcal{U}(\phi^{\leftarrow}(\lambda)) &= \mathcal{U}(\phi^{\leftarrow}(\textstyle\bigodot_{i=1}^p \phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &= \mathcal{U}(\textstyle\bigodot_{i=1}^p \phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &\geq \widetilde{\textstyle\bigodot_{i=1}^p} \mathcal{U}(\phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &\geq \textstyle\bigodot_{i=1}^p \mathcal{T}_{k_i}(\lambda_{k_i}), \quad \text{by (C)} \end{aligned}$$

$$\begin{aligned} \mathcal{U}^*(\phi^{\leftarrow}(\lambda)) &= \mathcal{U}^*(\phi^{\leftarrow}(\textstyle\bigodot_{i=1}^p \phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &= \mathcal{U}^*(\textstyle\bigodot_{i=1}^p \phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &\leq \widetilde{\textstyle\bigoplus_{i=1}^p} \mathcal{U}^*(\phi^{\leftarrow}(\phi_{k_i}^{\leftarrow}(\lambda_{k_i}))) \\ &\leq \textstyle\bigoplus_{i=1}^p \mathcal{T}^*_{k_i}(\lambda_{k_i}). \quad \text{by (D)} \end{aligned}$$

It implies $\mathcal{U}(\phi^{\leftarrow}(\lambda)) \geq \mathcal{B}(\lambda)$ and $\mathcal{U}^*(\phi^{\leftarrow}(\lambda)) \leq \mathcal{B}^*(\lambda)$ for all $\lambda \in L^X$. By Lemma 3.11, $\phi : (Y, \mathcal{U}, \mathcal{U}^*) \to (X, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ is LF-continuous. □

Definition 3.13. [3] (a) Let (**A**,U) be a concrete category over **X**. (**A**,U) is said to be amnestic provided that its fibres are partially ordered classes, i.e., no two different **A**-objects are equivalent.

(b) Let **A** and **B** be categories. A functor $G : \mathbf{A}\to\mathbf{B}$ is called topological provided that every G-structured source $(f_i : B \to GA_i)_{i\in\Gamma}$ has a unique G-initial lift $(\overline{f_i} : A \to A_i)_{i\in\Gamma}$.

Proposition 3.14. [3] *If $G : \mathbf{A}\to\mathbf{B}$ is a functor such that every G-structured source has a G-initial lift, then the following conditions are equivalent:*

(1) *G is topological.*

(2) *$(\mathbf{A}, G)$ is uniquely transportable.*

(3) *$(\mathbf{A}, G)$ is amnestic.*

The category of (L, M)-dfts's and LF-continuous mappings is denoted by (L, M)- **DFTOP**.

Theorem 3.15. *The forgetful functor $V : (L, M)$-**DFTOP**$\to$**SET** defined by $V(X, \mathcal{T}, \mathcal{T}^*) = X$ and $V(\phi) = \phi$ is topological.*

Proof. The proof follows from Definition 3.13, Proposition 3.14 and Theorem 3.12. □

4. Initial (L, M)-fuzzy Closure Structures

In this section, let L be a Girard monoid and $M = (M, \leq, \wedge, \vee, \star)$ be a complete DeMorgan algebra (i.e., a complete lattice with an order-reversing involution).

Theorem 4.1. *Let $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of (L, M)-fuzzy closure spaces, X a set and $\Phi_i : X \to X_i$ a function, for each $i \in \Gamma$ ($\Gamma \neq \emptyset$). Define a function $\mathcal{C} : L^X \times M_0 \times M_1 \to L^X$ on X by*

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\mathcal{C}_{i_j}(\lambda_{i_j}, r, s)) \mid \lambda \leq \bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\lambda_{i_j})\},$$

for all finite subsets $K = \{i_1, i_2, ..., i_p\}$ of Γ. Then we have the following properties:

(1) *$\mathcal{C}$ is the coarsest (L, M)-fuzzy closure operator on X for which all Φ_i, $i \in \Gamma$, are (L, M)-$\mathcal{C}$-maps.*

(2) *If $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ is a family of topological (L, M)-fuzzy closure spaces, then $(X, \mathcal{C})$ is a topological (L, M)-fuzzy closure space.*

(3) *A function $\Phi : (Y, \mathcal{C}') \to (X, \mathcal{C})$ is an (L, M)-$\mathcal{C}$-map iff for each $i \in \Gamma$, $\Phi_i \circ \Phi : (Y, \mathcal{C}') \to (X_i, \mathcal{C}_i)$ is an (L, M)-$\mathcal{C}$-map.*

Proof. (1) First, we will show that $\mathcal{C}$ is an (L, M)-fuzzy closure operator on X.

(C1) Since $\mathcal{C}(\underline{0}, r, s) \leq \Phi_i^{\leftarrow}(\mathcal{C}_i(\underline{0}, r, s)) = \underline{0}$, we have $\mathcal{C}(\underline{0}, r, s) = \underline{0}$.

(C2) For all finite subsets $K = \{i_1, i_2, ..., i_p\}$ of Γ, we have

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\mathcal{C}_{i_j}(\lambda_{i_j}, r, s)) \mid \lambda \leq \bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\lambda_{i_j})\} \geq$$

$$\geq \bigwedge\{\bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\lambda_{i_j}) \mid \lambda \leq \bigoplus_{j=1}^{p} \Phi_{i_j}^{\leftarrow}(\lambda_{i_j})\} \geq \lambda.$$

(C3) and (C5) are easily proved from the definition of $\mathcal{C}$.

(C4) For all finite subsets $K = \{k_1, k_2, ..., k_p\}$ and $J = \{j_1, j_2, ..., j_q\}$ of Γ such that $\lambda \leq \bigoplus_{i=1}^{p} \Phi_{k_i}^{\leftarrow}(\lambda_{k_i})$ and $\mu \leq \bigoplus_{i=1}^{q} \Phi_{j_i}^{\leftarrow}(\mu_{j_i})$, we have

$$\lambda \oplus \mu \leq (\bigoplus_{i=1}^{p} \Phi_{k_i}^{\leftarrow}(\lambda_{k_i})) \oplus (\bigoplus_{i=1}^{q} \Phi_{j_i}^{\leftarrow}(\mu_{j_i})).$$

Furthermore, we have for each $k \in K \cap J$,

$$\Phi_k^{\leftarrow}(\lambda_k) \oplus \Phi_k^{\leftarrow}(\mu_k) = \Phi_k^{\leftarrow}(\lambda_k \oplus \mu_k),$$

$$\Phi_k^{\leftarrow}(\mathcal{C}_k(\lambda_k \oplus \mu_k, r \wedge r', s \vee s')) \leq \Phi_k^{\leftarrow}(\mathcal{C}_k(\lambda_k, r, s)) \oplus \Phi_k^{\leftarrow}(\mathcal{C}_k(\mu_k, r', s')).$$

Put $M = K \cup J = \{m_1, ..., m_r\}$ with

$$\rho_{m_i} = \begin{cases} \lambda_{m_i} \oplus \underline{0} & \text{if } m_i \in K - (K \cap J) \\ \mu_{m_i} \oplus \underline{0} & \text{if } m_i \in J - (K \cap J) \\ \lambda_{m_i} \oplus \mu_{m_i} & \text{if } m_i \in (K \cap J). \end{cases}$$

If $m_i \in K - (K \cap J)$, then we have

$$\begin{aligned} \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\lambda_{m_i}, r, s)) &= \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\lambda_{m_i}, r, s)) \oplus \underline{0} \\ &= \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\lambda_{m_i}, r, s)) \oplus \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\underline{0}, r', s')) \\ &\geq \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\lambda_{m_i} \oplus \underline{0}, r \wedge r', s \vee s')) \\ &= \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\rho_{m_i}, r \wedge r', s \vee s')). \end{aligned}$$

Similarly if $m_i \in J - (K \cap J)$, we have

$$\Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\mu_{m_i}, r', s')) \geq \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\rho_{m_i}, r \wedge r', s \vee s')).$$

Hence,

$$\begin{aligned} &(\bigwedge\{\bigoplus_{i=1}^{p} \Phi_{k_i}^{\leftarrow}(\mathcal{C}_{k_i}(\lambda_{k_i}, r, s)) \mid \lambda \leq \bigoplus_{i=1}^{p} \Phi_{k_i}^{\leftarrow}(\lambda_{k_i})\})\oplus \\ &\oplus(\bigwedge\{\bigoplus_{i=1}^{q} \Phi_{j_i}^{\leftarrow}(\mathcal{C}_{j_i}(\mu_{j_i}, r', s')) \mid \mu \leq \bigoplus_{i=1}^{q} \Phi_{j_i}^{\leftarrow}(\mu_{j_i})\}) \geq \\ &\geq \bigwedge\{\bigoplus_{i=1}^{r} \Phi_{m_i}^{\leftarrow}(\mathcal{C}_{m_i}(\rho_{m_i}, r \wedge r', s \vee s')) \mid \lambda \oplus \mu \leq \bigoplus_{i=1}^{r} \Phi_{m_i}^{\leftarrow}(\rho_{m_i})\} = \\ &= \mathcal{C}(\lambda \oplus \mu, r \wedge r', s \vee s'). \end{aligned}$$

Thus, $\mathcal{C}(\lambda \oplus \mu, r \wedge r', s \vee s') \leq \mathcal{C}(\lambda, r, s) \oplus \mathcal{C}(\mu, r', s')$.

For each $\lambda \in L^X$ and each $i \in \Gamma$, since $\lambda \leq \Phi_i^{\leftarrow}(\Phi_i^{\rightarrow}(\lambda))$, we have

$$\mathcal{C}(\lambda, r, s) \leq \Phi_i^{\leftarrow}(\mathcal{C}_i(\Phi_i^{\rightarrow}(\lambda), r, s)).$$

It implies

$$\Phi_i^{\rightarrow}(\mathcal{C}(\lambda, r, s)) \leq \Phi_i^{\rightarrow}\Phi_i^{\leftarrow}(\mathcal{C}_i(\Phi_i^{\rightarrow}(\lambda), r, s))) \leq \mathcal{C}_i(\Phi_i^{\rightarrow}(\lambda), r, s), \forall\, i \in \Gamma.$$

Hence for each $i \in \Gamma$, $\Phi_i : (X, \mathcal{C}) \to (X_i, \mathcal{C}_i)$ is an (L, M)-$\mathcal{C}$-map.

If $\Phi_i : (X, \mathcal{C}') \to (X_i, \mathcal{C}_i)$ is an (L, M)-$\mathcal{C}$-map for every $i \in \Gamma$, then we have

$$\Phi_i^{\to}(\mathcal{C}'(\lambda, r, s)) \leq \mathcal{C}_i(\Phi_i^{\to}(\lambda), r, s)).$$

It implies that

$$\mathcal{C}'(\lambda, r, s) \leq \Phi_i^{\leftarrow}(\Phi_i^{\to}(\mathcal{C}'(\lambda, r, s))) \leq \Phi_i^{\leftarrow}(\mathcal{C}_i(\Phi_i^{\to}(\lambda), r, s)).$$

Hence for all finite subsets $K = \{i_1, ..., i_n\}$ of Γ, we have

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s)) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \geq$$

$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\Phi_{i_k}^{\to}(\Phi_{i_k}^{\leftarrow}(\lambda_{i_k})), r, s)) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \geq$$

$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \mathcal{C}'(\Phi_{i_k}^{\leftarrow}(\lambda_{i_k}), r, s) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \geq$$

$$\geq \bigwedge\{\mathcal{C}'(\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k}), \bigwedge_{k=1}^{n} r, \bigvee_{k=1}^{n} s) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} =$$

$$= \bigwedge\{\mathcal{C}'(\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k}), r, s) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \geq \mathcal{C}'(\lambda, r, s).$$

(2) We will show that $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) \leq \mathcal{C}(\lambda, r, s)$, for all $\lambda \in L^X$, $r \in M_0$ and $s \in M_1$ such that $r \leq s^\star$. For all finite subsets $K = \{i_1, ..., i_n\}$ of Γ, we have

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s)) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} =$$

$$= \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s), r, s)) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \geq$$

$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s), r, s)) \mid \mathcal{C}(\lambda, r, s) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s))\} \geq$$

$$\geq \mathcal{C}(\mathcal{C}(\lambda, r, s), r, s).$$

because $\lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})$ implies $\mathcal{C}(\lambda, r, s) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s))$.

(3) Necessity of the composition condition is clear since the composition of (L, M)-$\mathcal{C}$-maps is an (L, M)-$\mathcal{C}$-map.

Conversely, since $\Phi_i \circ \Phi$ is an (L, M)-$\mathcal{C}$-map, for each $\mu \in L^X$, we have

$$\Phi_i^{\to}(\Phi^{\to}(\mathcal{C}'(\Phi^{\leftarrow}(\mu), r, s))) \leq \mathcal{C}_i(\Phi_i^{\to}(\Phi^{\to}(\Phi^{\leftarrow}(\mu))), r, s) \leq \mathcal{C}_i(\Phi_i^{\to}(\mu), r, s).$$

It follows that for all $i \in \Gamma$,

$$\Phi^{\to}(\mathcal{C}'(\Phi^{\leftarrow}(\mu), r, s)) \leq \Phi_i^{\leftarrow}(\mathcal{C}_i(\Phi_i^{\to}(\mu), r, s)).$$

For all finite subsets $K = \{i_1, ..., i_n\}$ of Γ, we have

$$\mathcal{C}(\Phi^{\to}(\lambda), r, s) = \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\mu_{i_k}, r, s)) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\} \geq$$
$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\Phi_{i_k}^{\to}(\Phi_{i_k}^{\leftarrow}(\mu_{i_k})), r, s)) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\} \geq$$
$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \Phi^{\to}(\mathcal{C}'(\Phi^{\leftarrow}(\Phi_{i_k}^{\leftarrow}(\mu_{i_k})), r, s)) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\}.$$

It implies that

$$\Phi^{\leftarrow}(\mathcal{C}(\Phi^{\to}(\lambda), r, s)) \geq$$
$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \Phi^{\leftarrow}\Phi^{\to}(\mathcal{C}'(\Phi^{\leftarrow}(\Phi_{i_k}^{\leftarrow}(\mu_{i_k})), r, s)) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\} \geq$$
$$\geq \bigwedge\{\bigoplus_{k=1}^{n} \mathcal{C}'(\Phi^{\leftarrow}(\Phi_{i_k}^{\leftarrow}(\mu_{i_k})), r, s) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\} \geq$$
$$\geq \bigwedge\{\mathcal{C}'(\Phi^{\leftarrow}(\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})), r, s) \mid \Phi^{\to}(\lambda) \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mu_{i_k})\} \geq$$
$$\geq \mathcal{C}'(\Phi^{\leftarrow}(\Phi^{\to}(\lambda)), r, s) \geq \mathcal{C}'(\lambda, r, s).$$

It follows that

$$\Phi^{\to}(\mathcal{C}'(\lambda, r, s)) \leq \Phi^{\to}(\Phi^{\leftarrow}(\mathcal{C}(\Phi^{\to}(\lambda), r, s))) \leq \mathcal{C}(\Phi^{\to}(\lambda), r, s).$$

Therefore, $\Phi^{\to}(\mathcal{C}'(\lambda, r, s)) \leq \mathcal{C}(\Phi^{\to}(\lambda), r, s)$, for each $\lambda \in L^Y$, $r \in M_0$ and $s \in M_1$ such that $r \leq s^{\star}$. Hence $\Phi : (Y, \mathcal{C}') \to (X, \mathcal{C})$ is an (L, M)-$\mathcal{C}$-map.

The category of (L, M)-fuzzy closure spaces and (L, M)-$\mathcal{C}$-maps is denoted by (L, M)-**FC**. □

Theorem 4.2. *The forgetful functor* $W : (L, M)$-**FC**→**SET** *defined by* $W(X, \mathcal{C}) = X$ *and* $W(\Phi) = \Phi$ *is topological.*

Proof. From Theorem 4.1, every W-structured source $(\Phi_i : X \to W(X_i, \mathcal{C}_i))_{i\in\Gamma}$ has a unique W-initial lift $(\Phi_i : (X, \mathcal{C}) \to W(X_i, \mathcal{C}_i))_{i\in\Gamma}$, where $\mathcal{C}$ defined in Theorem 4.1. □

Using Theorems 4.1 and 4.2, we obtain the following definition:

Definition 4.3. Let $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ be a family of (L, M)-fuzzy closure spaces, X a set and for each $i \in \Gamma$, $\Phi_i : X \to X_i$ a function. The initial (L, M)-fuzzy closure structure $\mathcal{C}$ on X with respect to $(X, \Phi_i, (X_i, \mathcal{C}_i), \Gamma)$ is the coarsest (L, M)-fuzzy closure operator on X for which all Φ_i, $i \in \Gamma$ are (L, M)-$\mathcal{C}$-maps.

Definition 4.4. Let $X = \prod_{i\in\Gamma} X_i$ be the product of the sets from a family $\{(X_i, \mathcal{C}_i) \mid i \in \Gamma\}$ of (L, M)-fuzzy closure spaces. The initial (L, M)-fuzzy closure structure $\mathcal{C} = \bigotimes \mathcal{C}_i$ on X with respect to the family $\{\pi_i : X \to (X_i, \mathcal{C}_i) \mid i \in \Gamma\}$ of all projection maps is called the product (L, M)-fuzzy closure structure of $\{\mathcal{C}_i \mid i \in \Gamma\}$ and the pair $(X, \mathcal{C})$ is called the product (L, M)-fuzzy closure space.

Using Theorem 4.1, we have the following corollary.

Corollary 4.5. *Let* $\{(X_i, \mathcal{C}_i)\}_{i\in\Gamma}$ *be a family of* (L,M)*-fuzzy closure spaces,* $X = \prod_{i\in\Gamma} X_i$ *be the product set, for each* $i \in \Gamma$ *and* $\pi_i : X \to X_i$ *a projection. The structure* $\mathcal{C} = \bigotimes \mathcal{C}_i$ *on* X *is defined by*

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{j=1}^{p} \pi_{i_j}^{\leftarrow}(\mathcal{C}_{i_j}(\lambda_{i_j}, r, s)) \mid \lambda \leq \bigoplus_{j=1}^{p} \pi_{i_j}^{\leftarrow}(\lambda_{i_j})\},$$

for all finite subsets $K = \{i_1, i_2, ..., i_p\}$ *of* Γ*. Then the followings are satisfied.*

(1) $\mathcal{C}$ *is the coarsest* (L,M)*-fuzzy closure operator on* X *for which all* π_i*,* $i \in \Gamma$*, are* (L,M)*-*$\mathcal{C}$*-maps.*

(2) *A function* $\Phi : (Y, \mathcal{C}') \to (X, \mathcal{C})$ *is an* (L,M)*-*$\mathcal{C}$*-map iff for each* $i \in \Gamma$*,* $\pi_i \circ \Phi : (Y, \mathcal{C}') \to (X_i, \mathcal{C}_i)$ *is an* (L,M)*-*$\mathcal{C}$*-map.*

From the following theorem, an initial structure of (L,M)-dfts's can be obtained by the initial structure of (L,M)-fuzzy closure spaces induced by them.

Theorem 4.6. *Let* $\{(X_i, \mathcal{T}_i, \mathcal{T}_i^*)\}_{i\in\Gamma}$ *be a family of* (L,M)*-dfts's. Let* X *be a set and for each* $i \in \Gamma$*,* $\Phi_i : X \to X_i$ *be a function. Define the map* $\mathcal{C} : L^X \times M_0 \times M_1 \to L^X$ *on* X *by*

$$\mathcal{C}(\lambda, r, s) = \bigwedge\{\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r, s)) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\},$$

for all finite subsets $K = \{i_1, i_2, ..., i_n\}$ *of* Γ *and* $\mathcal{C}_i$ *defined as in Theorem 3.5 (i.e.,* $\mathcal{C}_i = \mathcal{C}_{\mathcal{T}_i, \mathcal{T}_i^*}$*). Then we have*

$$(\mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*}) = (\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}),$$

where the (L,M)*-double fuzzy topology* $(\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}})$ *is induced by* $\mathcal{C}$ *and* $(\mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ *is defined by Theorem 3.10, where the* (L,M)*-double fuzzy base* $(\mathcal{B}, \mathcal{B}^*)$ *on* X *is obtained as in Theorem 3.12.*

Proof. Suppose that there exist $\lambda, \mu \in L^X$ such that

$$\mathcal{T}_{\mathcal{B}}(\lambda) \ngeq \mathcal{T}_{\mathcal{C}}(\lambda) \text{ or } \mathcal{T}^*_{\mathcal{B}^*}(\mu) \nleq \mathcal{T}^*_{\mathcal{C}}(\mu).$$

Without loss of generality, let us assume that $\mathcal{T}_{\mathcal{B}}(\lambda) \ngeq \mathcal{T}_{\mathcal{C}}(\lambda)$. By the definition of $\mathcal{T}_{\mathcal{C}}$ from Theorem 3.6, there exist $r' \in M_0$ and $s' \in M_1$ with $r' \leq (s')^{\star}$ such that $\mathcal{C}(\lambda \mapsto \underline{0}, r', s') = \lambda \mapsto \underline{0}$ and $\mathcal{T}_{\mathcal{B}}(\lambda) \ngeq r'$. Since $\mathcal{C}(\lambda \mapsto \underline{0}, r', s') = (\lambda \mapsto \underline{0})$, we have

$$\begin{aligned} \lambda &= \mathcal{C}(\lambda \mapsto \underline{0}, r', s') \mapsto \underline{0} \\ &= (\bigwedge\{\textstyle\bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s')) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\}) \mapsto \underline{0} \\ &= \bigvee\{\textstyle\bigodot_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s') \mapsto \underline{0}) \mid \lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})\} \end{aligned}$$

for all finite subsets $K = \{i_1, ..., i_n\}$ of Γ.

Since $\mathcal{C}_{i_k}(\lambda_{i_k}, r', s') = \mathcal{C}_{i_k}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s'), r', s')$, using the fact $\mathcal{T}_i = \mathcal{T}_{\mathcal{C}_i}$ from Theorem 3.8, we have

$$\mathcal{T}_{i_k}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s') \mapsto \underline{0}) \geq r'.$$

Put $\mu_k = \Phi_{i_k}^{\leftarrow}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s') \mapsto \underline{0})$. From Theorem 3.12, we have

$$\mathcal{B}(\mu_k) \geq \mathcal{T}_{i_k}(\mathcal{C}_{i_k}(\lambda_{i_k}, r', s') \mapsto \underline{0}) \geq r'.$$

Put $\mu_K = \bigodot_{k=1}^{n} \mu_k$. For each finite set $K \subset \Gamma$ such that $\lambda \leq \bigoplus_{k=1}^{n} \Phi_{i_k}^{\leftarrow}(\lambda_{i_k})$, by the definition of $\mathcal{T}_{\mathcal{B}}$ from Theorem 3.10, we have

$$\mathcal{T}_{\mathcal{B}}(\lambda) = \mathcal{T}_{\mathcal{B}}(\bigvee \mu_K) \geq \bigwedge \mathcal{B}(\mu_K) \geq r'.$$

$$(\mathcal{B}(\mu_K) = \mathcal{B}(\bigodot_{k=1}^{n} \mu_k) \geq \bigwedge_{k=1}^{n} \mathcal{B}(\mu_k) \geq r')$$

Hence $\mathcal{T}_{\mathcal{B}}(\lambda) \geq r'$. It is a contradiction. Therefore, $\mathcal{T}_{\mathcal{B}}(\mu) \geq \mathcal{T}_{\mathcal{C}}(\mu)$ and $\mathcal{T}^*_{\mathcal{B}^*}(\mu) \leq \mathcal{T}^*_{\mathcal{C}}(\mu)$ for all $\mu \in L^X$.

We will show that $\mathcal{T}_{\mathcal{B}}(\mu) \leq \mathcal{T}_{\mathcal{C}}(\mu)$ and $\mathcal{T}^*_{\mathcal{B}^*}(\mu) \geq \mathcal{T}^*_{\mathcal{C}}(\mu)$ for all $\mu \in L^X$, equivalently, the identity function $id_X : (X, \mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}) \to (X, \mathcal{T}_{\mathcal{B}}, \mathcal{T}^*_{\mathcal{B}^*})$ is LF-continuous. From Theorem 3.12 (3), we will only show that $\Phi_i \circ id_X : (X, \mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}}) \to (X_i, \mathcal{T}_i, \mathcal{T}^*_i)$ is LF-continuous, that is,

$$\mathcal{T}_i(\nu_i) \leq \mathcal{T}_{\mathcal{C}}(\Phi_i^{\leftarrow}(\nu_i)) \text{ and } \mathcal{T}^*_i(\nu_i) \geq \mathcal{T}^*_{\mathcal{C}}(\Phi_i^{\leftarrow}(\nu_i)) \text{ for all } \nu_i \in L^{X_i}.$$

If $\mathcal{T}_i(\nu_i) \geq r$ and $\mathcal{T}^*_i(\nu_i) \leq s$ for $r \in M_0$ and $s \in M_1$, then, by Theorem 3.5, we have

$$\mathcal{C}_i(\nu_i \mapsto \underline{0}, r, s) = \nu_i \mapsto \underline{0}.$$

From the definition of $\mathcal{C}$, it follows that:

$$\mathcal{C}(\Phi_i^{\leftarrow}(\nu_i) \mapsto \underline{0}, r, s) \leq \Phi_i^{\leftarrow}(\mathcal{C}_i(\nu_i \mapsto \underline{0}, r, s)) = \Phi_i^{\leftarrow}(\nu_i \mapsto \underline{0}) = \Phi_i^{\leftarrow}(\nu_i) \mapsto \underline{0}.$$

From (C2) of Definition 3.1, we have $\mathcal{C}((\Phi_i^{\leftarrow}(\nu_i)) \mapsto \underline{0}, r, s) = (\Phi_i^{\leftarrow}(\nu_i)) \mapsto \underline{0}$. Hence, by Theorem 3.6, $\mathcal{T}_{\mathcal{C}}(\Phi_i^{\leftarrow}(\nu_i)) \geq r$ and $\mathcal{T}^*_{\mathcal{C}}(\Phi_i^{\leftarrow}(\nu_i)) \leq s$. □

Theorem 4.7. (a) *Let* $(X_1, \mathcal{T}_1, \mathcal{T}^*_1)$, $(X_2, \mathcal{T}_2, \mathcal{T}^*_2)$ *be* (L, M)*-dfts's and let* $\phi : X_1 \to X_2$ *be a function. If* $\phi : (X_1, \mathcal{T}_1, \mathcal{T}^*_1) \longrightarrow (X_2, \mathcal{T}_2, \mathcal{T}^*_2)$ *is LF-continuous, then* $\phi : (X_1, \mathcal{C}_{\mathcal{T}_1, \mathcal{T}^*_1}) \longrightarrow (X_2, \mathcal{C}_{\mathcal{T}_2, \mathcal{T}^*_2})$ *is an* $(L, M) - \mathcal{C}$*-map.*

(b) *Let* $(X_1, \mathcal{C}_1), (X_2, \mathcal{C}_2)$ *be* (L, M)*-fuzzy closure spaces and let* $\phi : X_1 \to X_2$ *be a function. If* $\phi : (X_1, \mathcal{C}_1) \longrightarrow (X_2, \mathcal{C}_2)$ *is an* $(L, M) - \mathcal{C}$*-map, then* $\phi : (X_1, \mathcal{T}_{\mathcal{C}_1}, \mathcal{T}^*_{\mathcal{C}_1}) \longrightarrow (X_2, \mathcal{T}_{\mathcal{C}_2}, \mathcal{T}^*_{\mathcal{C}_2})$ *is LF-continuous.*

Proof. (a) Let $\phi : (X_1, \mathcal{T}_1, \mathcal{T}^*_1) \to (X_2, \mathcal{T}_2, \mathcal{T}^*_2)$ be an LF-continuous map.

$$\begin{aligned}
&\phi^{\leftarrow}(\mathcal{C}_{\mathcal{T}_2, \mathcal{T}^*_2}(\mu \mapsto \underline{0}, r, s) \mapsto \underline{0}) = \\
&= \phi^{\leftarrow}((\bigwedge\{\rho \in L^{X_2} \mid \rho \geq \mu \mapsto \underline{0}, \mathcal{T}_2(\rho \mapsto \underline{0}) \geq r \text{ and } \mathcal{T}^*_2(\rho \mapsto \underline{0}) \leq s\}) \mapsto \underline{0}) = \\
&= \phi^{\leftarrow}(\bigvee\{\rho \mapsto \underline{0} \mid \rho \geq \mu \mapsto \underline{0}, \mathcal{T}_2(\rho \mapsto \underline{0}) \geq r \text{ and } \mathcal{T}^*_2(\rho \mapsto \underline{0}) \leq s\}) \leq \\
&\leq \bigvee\{\phi^{\leftarrow}(\rho \mapsto \underline{0}) \mid \phi^{\leftarrow}(\rho) \geq \phi^{\leftarrow}(\mu \mapsto \underline{0}), \mathcal{T}_1(\phi^{\leftarrow}(\rho \mapsto \underline{0})) \geq r, \mathcal{T}^*_1(\phi^{\leftarrow}(\rho \mapsto \underline{0})) \leq s\} \leq \\
&\leq \bigvee\{\nu \mapsto \underline{0} \mid \nu \geq \phi^{\leftarrow}(\mu \mapsto \underline{0}), \mathcal{T}_1(\nu \mapsto \underline{0}) \geq r \text{ and } \mathcal{T}^*_1(\nu \mapsto \underline{0}) \leq s\} = \\
&= (\bigwedge\{\nu \mid \nu \geq \phi^{\leftarrow}(\mu \mapsto \underline{0}), \mathcal{T}_1(\nu \mapsto \underline{0}) \geq r \text{ and } \mathcal{T}^*_1(\nu \mapsto \underline{0}) \leq s\}) \mapsto \underline{0} = \\
&= \mathcal{C}_{\mathcal{T}_1, \mathcal{T}^*_1}(\phi^{\leftarrow}(\mu \mapsto \underline{0}), r, s) \mapsto \underline{0}.
\end{aligned}$$

$$\begin{array}{rcl}\phi^{\leftarrow}(\mathcal{C}_{\mathcal{T}_2,\mathcal{T}_2^*}(\phi^{\rightarrow}(\lambda),r,s)) & = & \phi^{\leftarrow}((\mathcal{C}_{\mathcal{T}_2,\mathcal{T}_2^*}(\phi^{\rightarrow}(\lambda),r,s)\mapsto \underline{0})\mapsto \underline{0}) \\ & \geq & (\mathcal{C}_{\mathcal{T}_1,\mathcal{T}_1^*}(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda)),r,s)\mapsto \underline{0})\mapsto \underline{0} \\ & \geq & \mathcal{C}_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda,r,s).\end{array}$$

It implies that

$$\phi^{\rightarrow}(\mathcal{C}_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda,r,s)) \leq \phi^{\rightarrow}\phi^{\leftarrow}(\mathcal{C}_{\mathcal{T}_2,\mathcal{T}_2^*}(\phi^{\rightarrow}(\lambda),r,s)) \leq \mathcal{C}_{\mathcal{T}_2,\mathcal{T}_2^*}(\phi^{\rightarrow}(\lambda),r,s).$$

(b) Since ϕ is an $(L,M)-\mathcal{C}$-map, $\phi^{\rightarrow}(\mathcal{C}_1(\lambda,r,s)) \leq \mathcal{C}_2(\phi^{\rightarrow}(\lambda),r,s)$. Then, $\mathcal{C}_1(\lambda,r,s) \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mathcal{C}_1(\lambda,r,s))) \leq \phi^{\leftarrow}(\mathcal{C}_2(\phi^{\rightarrow}(\lambda),r,s))$. Let $A := \{r \in M_0 \mid \mathcal{C}_2(\mu \mapsto \underline{0},r,s) = \mu \mapsto \underline{0} \text{ and } r \leq s^{\star}\}$ and $B := \{r \in M_0 \mid \mathcal{C}_1(\phi^{\leftarrow}(\mu)\mapsto \underline{0},r,s) = \phi^{\leftarrow}(\mu)\mapsto \underline{0}$ and $r \leq s^{\star}\}$.

Since $\mathcal{C}_2(\mu \mapsto \underline{0},r,s) = \mu \mapsto \underline{0}$, for $r \in A$, $\phi^{\leftarrow}(\mathcal{C}_2(\mu \mapsto \underline{0},r,s)) = \phi^{\leftarrow}(\mu)\mapsto \underline{0}$.

$$\begin{array}{rcl}\phi^{\leftarrow}(\mu)\mapsto \underline{0} & = & \phi^{\leftarrow}(\mathcal{C}_2(\mu\mapsto \underline{0},r,s)) \\ & \geq & \phi^{\leftarrow}(\mathcal{C}_2(\phi^{\rightarrow}(\phi^{\leftarrow}(\mu\mapsto \underline{0})),r,s)) \\ & \geq & \mathcal{C}_1(\phi^{\leftarrow}(\mu\mapsto \underline{0}),r,s) \\ & = & \mathcal{C}_1(\phi^{\leftarrow}(\mu)\mapsto \underline{0},r,s).\end{array}$$

It implies that, and by Definition 3.1 (C2), $\mathcal{C}_1(\phi^{\leftarrow}(\mu)\mapsto \underline{0},r,s) = \phi^{\leftarrow}(\mu)\mapsto \underline{0}$. So, $r \in B$, i.e., $A \subset B$. Thus, $\mathcal{T}_{\mathcal{C}_2}(\mu) \leq \mathcal{T}_{\mathcal{C}_1}(\phi^{\leftarrow}(\mu))$. Similarly, $\mathcal{T}^*_{\mathcal{C}_2}(\mu) \geq \mathcal{T}^*_{\mathcal{C}_1}(\phi^{\leftarrow}(\mu)), \forall\ \mu \in L^{X_2}$. Therefore, $\phi : (X_1,\mathcal{T}_{\mathcal{C}_1},\mathcal{T}^*_{\mathcal{C}_1}) \longrightarrow (X_2,\mathcal{T}_{\mathcal{C}_2},\mathcal{T}^*_{\mathcal{C}_2})$ is LF-continuous. □

Corollary 4.8. (1) *Define* $F : (L,M)$-**DFTOP** $\longrightarrow (L,M)$-**FC** *by* $F(X,\mathcal{T},\mathcal{T}^*) = (X,\mathcal{C}_{\mathcal{T},\mathcal{T}^*})$ *and* $F(\phi) = \phi$. *Then* F *is an embedding functor, i.e.,* (L,M)*-fuzzy closure operators extend the concept of* (L,M)*-double fuzzy topology.*

(2) *Define* $G : (L,M)$-**FC** $\longrightarrow (L,M)$-**DFTOP** *by* $G(X,\mathcal{C}) = (X,\mathcal{T}_{\mathcal{C}},\mathcal{T}^*_{\mathcal{C}})$ *and* $G(\phi) = \phi$. *Then* G *is a functor.*

(3) $G \circ F = id_{(L,M)-\mathbf{DFTOP}}$.

(4) *Let* (L,M)-**TFC** *denote the full subcategory of* (L,M)-**FC** *whose objects* $(X,\mathcal{C})$ *satisfy conditions (a), (b) of Theorem 3.6. If we restrict the functors* F *and* G*, accordingly, then* $F \circ G = id_{(L,M)-\mathbf{TFC}}$.

5. **Conclusion**

In this study, we introduced the notion of (L,M)-fuzzy closure space where L and M are stsc-quantales as an extension of frames. We showed the existence of initial (L,M)-fuzzy closure structures. We also proved that the category (L,M)-**FC** is a topological category over **SET**. By using these concepts, we obtained products of (L,M)-fuzzy closure spaces. Finally, we showed that an initial structure of (L,M)-dfts's can be obtained by the initial structure of (L,M)-fuzzy closure spaces induced by them.

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