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FUZZY EQUATIONAL CLASSES ARE FUZZY VARIETIES

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Abstract. In the framework of fuzzy algebras with fuzzy equalities and a complete lattice as a structure of membership values, we investigate fuzzy equational classes. They consist of special fuzzy algebras fulfilling the same fuzzy identities, defined with respect to fuzzy equalities. We introduce basic notions and the corresponding operators of universal algebra: construction of fuzzy subalgebras, homomorphisms and direct products. We prove that every fuzzy equational class is closed under these three operators, which means that such a class is a fuzzy variety.

1. Introduction

Our topic is general algebra in the fuzzy framework, so let us mention the chronological development of some relevant fields. At the beginning of fuzzy era, particular algebras have been introduced and investigated: fuzzy groups (Rosenfeld 1971 [20], Das 1981 [10] and then many others), fuzzy semigroups (Malik, Mordeson and Kuroki, the monograph from 2003, [18]), and other best known structures with usually two binary operations (semirings, rings, lattices and connected structures). Basic notions from general and universal algebra have also been investigated in the fuzzy framework, like lattice valued algebras (by Di Nola and Gerla [13] and others), fuzzy congruences (Šešelja 1981 [21], Murali [19] and others). Investigating basic mathematics and algebra in the fuzzy context, Demirci has been dealing with fuzzy functions and related notions ([11, 12]) and also Šostak, ([29]). In the same period fuzzy homomorphisms were investigated, let us mention Chakraborty and Khare, 1993 ([9]), then Šešelja ([22]) and also Kuraoka and Suzuki, ([16]). Though, a systematic approach to fuzzy general algebras should be related to Bělohlávek and Vychodil ([1, 2, 3]). Some recent research in fuzzy general algebra is presented in [5] and [17], and also in [4], the latter for interval valued fuzzy sets.

In dealing with fuzzy structures, we use our previous results ([22, 23, 24, 25, 26, 27, 7]). The topic of our investigation presented here are classes of fuzzy algebras which are defined within a fixed type (language). These are usual fuzzy subalgebras

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of crisp algebras of the same type, equipped with compatible fuzzy equalities, which is introduced instead of the crisp equality. Let us mention the use of fuzzy equality instead of the crisp one started several decades ago e.g., Höhle [15], then more recently Demirci [12], and others. The ordered structure of membership values is, in our case, a complete lattice. If all fuzzy algebras in the class fulfill a set of particular formulas called *fuzzy identities* (which generalize crisp identities), then this class is called an *equational class* of fuzzy algebras. Dealing with fuzzy relations we use a special weak reflexivity. This notion already existed, paper [30] by Yeh and Bang; and also Filep, [14]. Our approach to main fuzzy notions differs from those existing in the literature.

After the definition of fuzzy algebra, we introduce basic universal algebraic constructions. These are fuzzy subalgebras, fuzzy homomorphisms and fuzzy direct products. Each of these notions is appropriately defined with respect to fuzzy equalities. We prove that each equational class of fuzzy algebras is closed under these three constructions. As it is known, a class of algebras of the same type closed under the above constructions is called a variety. Our main result here is that *each fuzzy equational class is a fuzzy variety*.

Our first attempt to deal with fuzzy identities is presented in [28]. Fuzzy algebras with the fuzzy equalities as introduced here, are elaborated in [6].

The structure of the membership values is in our present investigation a complete lattice without additional operations (like e.g. a residuated lattice [1, 2]). Our reason to use meet and join is that these lattice operations are idempotent; this property is naturally connected with reflexivity that we use.

2. Preliminaries

2.1. **Crisp Notions: Algebras, Equational Classes, Varieties.** A **language** or a **type** $\mathcal{L}$, is a set F of functional symbols, together with a set of natural numbers (arities) associated with these symbols. An **algebra** $\mathcal{A} = (A, F^A)$ of a type $\mathcal{L}$ consists of a nonempty set A, and a collection F^A of operations on A. **Terms** in the language $\mathcal{L}$ are usual regular expressions constructed by the variables and operational symbols. If $v(x_1, \ldots, x_n)$ is a term in the language of an algebra $\mathcal{A}$, then $v^A : A^n \to A$ is the corresponding **term-function** on $\mathcal{A}$. An **identity** in $\mathcal{L}$ is a formula $u = v$, where u, v are. A class of algebras of the same type, fulfilling a set of identities is an **equational class**. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a non-empty subset of A and with fundamental operations being restrictions of those on $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are algebras of the same type, then the function $h : A \to B$ compatible with the fundamental operations in the sense that for an n-ary $f \in F$, and $x_1, \ldots, x_n \in A$, $h(f(x_1, \ldots, x_n)) = f(h(x_1), \ldots, h(x_n))$ and for any constant $c \in \mathcal{A}$, $h(c)$ is the corresponding constant in $\mathcal{B}$, is a **homomorphism** of $\mathcal{A}$ into $\mathcal{B}$. The set of images $h(A)$ under h is a subalgebra of $\mathcal{B}$, a **homomorphic image** of $\mathcal{A}$. For a family $\{\mathcal{A}_i \mid i \in I\}$ of algebras of the same type, $\Pi_{i\in I}\mathcal{A}_i$ is their **direct product**, an algebra of the same type with operations defined componentwise. A class of algebras of the same type is a **variety** if it is closed under subalgebras, homomorphic images and direct products. By Birkhoff's theorem, an *equational class is a variety and vice versa*. An equivalence relation ρ on $\mathcal{A}$ which is compatible with respect to all fundamental operations ($x_i \rho y_i, i = 1, \ldots, n$ imply $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$) is a **congruence** relation on $\mathcal{A}$.

For more details in universal algebra, see e.g., [8].

2.2. **Some Basic Notions of Fuzzy Algebra.** Throughout the paper, $(L, \wedge, \vee, 0, 1)$ is supposed to be a fixed, complete lattice and it is used as a set of membership values for all the fuzzy objects.

A **fuzzy set** μ on a nonempty set A is a function $\mu : A \to L$. Consequently, a mapping $\rho : A^2 \to L$ is a fuzzy relation on A.

What we use here are the fuzzy relations on fuzzy sets, defined as follows. If $\mu : A \to L$ is a fuzzy set on a nonempty set A, then a fuzzy relation $\rho : A^2 \to L$ on A is said to be a **fuzzy relation on** μ if for all $x, y \in A$

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y). \tag{1}$$

A fuzzy relation ρ on a fuzzy set μ is **reflexive** if for all $x \in A$,

$$\rho(x, x) = \mu(x). \tag{2}$$

The above notion of reflexivity is known, see e.g., [14]. Observe that in the crisp case, i.e., if μ is the characteristic function of the domain A, then (2) actually determines a classical reflexive relation ($\rho(x, x) = 1$). The following lemma is obvious.

Lemma 2.1. *If $\rho : A^2 \to L$ is reflexive on $\mu : A \to L$, then for all $x, y \in A$,*

$$\rho(x, y) \leqslant \rho(x, x) \quad \textit{and} \quad \rho(y, x) \leqslant \rho(x, x).$$

A fuzzy relation on a fuzzy set μ on A is symmetric and transitive if it fulfills these conditions as a fuzzy relation on A:

$$\rho \text{ is } \textbf{symmetric} \text{ if } \rho(x, y) = \rho(y, x) \text{ for all } x, y \in A; \tag{3}$$

$$\rho \text{ is } \textbf{transitive} \text{ if } \rho(x, z) \wedge \rho(z, y) \leqslant \rho(x, y) \text{ for all } x, y, z \in A, \tag{4}$$

$$\text{or equivalently, if } \bigvee_{z \in A} (\rho(x, z) \wedge \rho(z, y)) \leqslant \rho(x, y) \text{ for all } x, y \in A. \tag{5}$$

A reflexive, symmetric and transitive relation ρ on a fuzzy set μ is a **fuzzy equivalence** on μ.

A fuzzy equivalence relation ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$,:

$$\text{if } \rho(x, x) \neq 0, \text{ then } \rho(x, y) < \rho(x, x), \tag{6}$$

is called a **fuzzy equality** relation on μ.

Let $\mathcal{A} = (A, F^A)$ be a crisp algebra. As it is known, a **fuzzy subalgebra** of $\mathcal{A}$ is any mapping $\mu : A \to L$ fulfilling the following:

For any n-ary $f \in F$ with the arity greater than 0, and all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f^A(a_1, \ldots, a_n)).$$

For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1. \tag{7}$$

The following proposition is a fuzzy version of the known property of term operations in universal algebra and it can be easily proved by induction on the complexity of term.

Proposition 2.2. *Let $\mu : A \to L$ be a fuzzy subalgebra of a crisp algebra $\mathcal{A} = (A, F^A)$ and let $t(x_1, \ldots, x_n)$ be a term in the language of $\mathcal{A}$. If t^A is the corresponding term-operation on $\mathcal{A}$ and $a_1, \ldots, a_n \in A$, then the following holds:*

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t^A(a_1, \ldots, a_n)).$$

A fuzzy relation $\rho : A^2 \to L$ on a fuzzy subalgebra $\mu : A \to L$ of $\mathcal{A} = (A, F^A)$ is said to be **compatible** with the operations from F^A, if it is compatible as a fuzzy relation on $\mathcal{A}$, i.e., if for every n-ary operation $f^A \in F^A$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$

$$\bigwedge_{i=1}^{n} \rho(a_i, b_i) \leqslant \rho(f^A(a_1, \ldots, a_n), f^A(b_1, \ldots, b_n)). \tag{8}$$

In particular, for $n = 1$, $\rho(a_1, b_1) \leqslant \rho(f^A(a_1), f^A(b_1))$.

A compatible fuzzy equivalence on a fuzzy subalgebra μ of $\mathcal{A}$ is a **fuzzy congruence** on μ. Obviously, compatible fuzzy equalities on μ are particular fuzzy congruences on this fuzzy subalgebra.

Notation throughout the paper: if f is an operational symbol and u is a term in the given type (language), then the corresponding operation and term-operation on the concrete algebra $\mathcal{A}$ of the same type are denoted respectively by f^A and u^A, and analogously for other objects (e.g., relations).

3. **Results**

In order to deal with classes of fuzzy structures, we introduce fuzzy algebras, and then the corresponding operators which enable the construction of subalgebras, homomorphic images and direct products.

3.1. **Fuzzy Algebra, Identity, Equational Class.** Let $\mathcal{A} = (A, F^A)$ be an algebra of the type $\mathcal{L}$ and let L be a fixed complete lattice, as introduced above. Further, let $\mu : A \to L$ be a fuzzy subalgebra of $\mathcal{A}$ and $E^\mu : A^2 \to L$ a compatible fuzzy equality on μ. Then we say that the four-tuple $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ is a **fuzzy algebra** of the type $\mathcal{L}$.

In other words, a *fuzzy algebra is a fuzzy (lattice valued) subalgebra of a given crisp algebra, endowed with a compatible fuzzy equality.* That is why all relevant parameters are explicitly listed in the defining four-tuple.

Remark 3.1. Let us compare our definition of a fuzzy algebra with the known similar notions. If we take $\mu : A \to L$ to be the characteristic function of A and we keep E^μ (hence with the diagonal having all values equal 1), then the above definition gives an *algebra with fuzzy equality*, [3]. Similarly, if μ is a characteristic function of a proper subalgebra of $\mathcal{A}$, then E^μ can be considered as a fuzzy equality on a subalgebra, as in the previous case. Finally, if both, μ and E^μ take values in $\{0,1\}$, then we deal with classical subalgebras of $\mathcal{A}$, equipped with the corresponding crisp equalities.

Clearly, the above definition implies that two given fuzzy algebras $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ and $\bar{\mathcal{N}} = (\mathcal{B}, \nu, E^\nu, L)$ are equal if $\mathcal{A} = \mathcal{B}$, $\mu = \nu$, $E^\mu = E^\nu$ and both refer to the same lattice L.

As defined in [6], a fuzzy **identity** of the type $\mathcal{L}$ over a set of variables X is the expression $E(u, v)$, where $u(x_1, \ldots, x_n), v(x_1, \ldots, x_n)$ belong to the set $T(X)$ of terms over X and both have at most n variables.

Examples of fuzzy identities (in the suitable languages) are e.g., $E(x*(y*z), (x*y)*z)$, $E(x \cdot (y+z), (x \cdot y) + (x \cdot z))$, $E(f(x, x, y), f(y, x, x))$ and so on.

We say that a fuzzy algebra $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ **satisfies** a fuzzy identity $E(u, v)$ (or that this identity **holds** on $\bar{\mathcal{M}}$), if for all $a_1, \ldots, a_n \in A$

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E^\mu(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)), \tag{9}$$

where u^A, v^A are the corresponding term-operations over $\mathcal{A}$.

Remark 3.2. In paper [6] we proved that *for every fuzzy identity $E(u, v)$ fulfilled on a fuzzy subalgebra μ of a crisp algebra $\mathcal{A}$, there is the smallest compatible fuzzy equality E^m, such that $E^m(u^A, v^A)$ holds on μ.*

Observe that due to the definition of a fuzzy algebra $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$, we say that all properties of μ not referring to some fuzzy equality, are fulfilled also by $\bar{\mathcal{M}}$. In particular, Proposition 2.2 holds for a fuzzy algebra $\bar{\mathcal{M}}$, if it is satisfied by the underlying fuzzy subalgebra μ.

Obviously, it may happen that μ satisfies some property (e.g., a fuzzy identity) with respect to some fuzzy equality, but not with respect to E^μ to which $\bar{\mathcal{M}}$ refers. In this case, $\bar{\mathcal{M}}$ does not fulfill this property.

Let Σ be a set of fuzzy identities of type $\mathcal{L}$ and let L be a fixed complete lattice. Then all fuzzy algebras of this type satisfying all identities in Σ form an **equational class** $\mathfrak{M}$ of fuzzy algebras.

Remark 3.3. Observe that by the above definition, what is fixed in an equational class $\mathfrak{M}$ is the type (language) $\mathcal{L}$ and the lattice of membership values L. Therefore, together with fuzzy algebras, all structures mentioned in Remark 3.1 fulfilling identities in Σ are included in $\mathfrak{M}$. More precisely, a fuzzy equational class consists of three kinds of algebras:

1. Crisp algebras, fulfilling given identities with respect to the crisp equality (and hence also with respect to the corresponding fuzzy ones, see Remark 3.2).

2. Fuzzy subalgebras of algebras mentioned in 1, fulfilling given identities with respect to fuzzy equalities.

3. Fuzzy algebras being fuzzy subalgebras of crisp algebras in the given language, such that the following holds: the crisp algebras do not fulfill given identities, while the fuzzy ones do (with respect to some fuzzy equalities, see Remark 3.2).

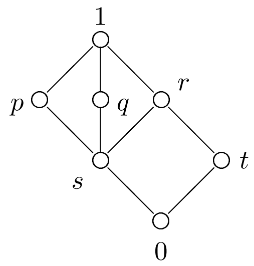

Figure 1. Lattice L

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	b	e	a
c	c	b	a	e

Table 1. Algebra G

In the following two examples we describe fuzzy equational classes of fuzzy groups and fuzzy lattices. One more is the equational class of commutative groupoids, given in Example 3.18, at the end of part 3.5.

Example 3.4. Let us describe the equational class of fuzzy groups, with the membership values lattice given in Figure 1. The language $\mathcal{L}$ contains one binary and one unary operation (denoted by $\cdot$ and $^{-1}$ respectively), and a nullary operation (constant) e. Fuzzy identities defining the equational class of fuzzy groups are

$$E(x \cdot (y \cdot z), (x \cdot y) \cdot z),\ E(x \cdot e, x), E(e \cdot x, x),\ E(x \cdot x^{-1}, e),\ E(x^{-1} \cdot x, e).$$

This equational class consists of all crisp groups (point 1. in Remark 3.3), their fuzzy subgroups, as known in the classical fuzzy algebra (point 2. above), and of fuzzy algebras which are fuzzy subalgebras of algebras in the language $\mathcal{L}$, fulfilling the above fuzzy identities (point 3. above). An example of such fuzzy subalgebra is given bellow. The crisp four-element algebra is $(G, \cdot, ^{-1}, e)$, the binary operation is presented in Table 1, the unary operation is identity ($x^{-1} = x$), and the constant is e. This algebra is not a group (associativity is not satisfied), while its fuzzy subalgebra $\mu : G \to L$ given by

$$\mu(x) = \begin{pmatrix} e & a & b & c \\ 1 & p & q & r \end{pmatrix},$$

fulfils the above identities with respect to the fuzzy equality E^μ (Table 2). Hence, (G, μ, E^μ, L) is a fuzzy group in this equational class.

Example 3.5. Let L be the real interval $[0, 1]$, and let $\mathcal{L} = \{f^2, g^2\}$, where the arities are given as upper indexes (therefore the language consists of two binary symbols). Let the identities be

E^μ	e	a	b	c
e	1	s	s	s
a	s	p	s	s
b	s	s	q	s
c	s	s	s	r

Table 2. Fuzzy equality on μ

$E(f^2(x,y), f^2(y,x))$, $E(g^2(x,y), g^2(y,x))$,
$E(f^2(f^2(x,y),z), f^2(x, f^2(y,z)))$, $E(g^2(g^2(x,y),z), g^2(x, g^2(y,z)))$,
$E(f^2(x, f^2(y,x)), x)$, $E(g^2(x, g^2(y,x)), x)$.

All fuzzy algebras with two binary operations and with membership values in [0,1] satisfying all the above identities with respect to some fuzzy equalities form an equational class of *fuzzy lattices*. Obviously, f and g correspond to meet and join operations in a lattice.

3.2. **Fuzzy Subalgebra of a Fuzzy Algebra.** Here we start with a fuzzy algebra $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ and identify particular fuzzy subalgebras of $\mathcal{A}$ as fuzzy subalgebras of $\bar{\mathcal{M}}$.

The next theorem formalizes our intuition and enables the definition that follows.

Theorem 3.6. *Let $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ be a fuzzy algebra and $\nu : A \to L$ a fuzzy subalgebra of $\mathcal{A}$, fulfilling the following conditions:*

(1) *$\nu(x) \leqslant \mu(x)$ for all $x \in A$.*

(2) *If x and y are distinct elements of A and $\nu(x) > 0$, then $E^\mu(x,y) < \nu(x)$.*

Then, a fuzzy relation E^ν on ν given by

$$E^\nu(x,y) := E^\mu(x,y) \wedge \nu(x) \wedge \nu(y),$$

is a compatible fuzzy equality on ν.

Proof. (1) By the definition, $E^\nu(x,y) \leqslant \nu(x) \wedge \nu(y)$, i.e., $E^\nu(x,y)$ is a fuzzy relation on ν.

(2) Reflexivity of E^ν : $E^\nu(a,a) = E^\mu(a,a) \wedge \nu(a) = \mu(a) \wedge \nu(a) = \nu(a)$.

(3) Symmetry: $E^\nu(a,b) = E^\mu(a,b) \wedge \nu(a) \wedge \nu(b) = E^\mu(b,a) \wedge \nu(b) \wedge \nu(a) = E^\nu(b,a)$.

(4) Transitivity:
$$\begin{aligned} E^\nu(a,b) &= E^\mu(a,b) \wedge \nu(a) \wedge \nu(b) \\ &\geqslant (E^\mu(a,c) \wedge E^\mu(c,b)) \wedge \nu(a) \wedge \nu(b) \wedge \nu(c) \\ &= (E^\mu(a,c) \wedge \nu(a) \wedge \nu(c)) \wedge (E^\mu(c,b) \wedge \nu(c) \wedge \nu(b)) \\ &= E^\nu(a,c) \wedge E^\nu(c,b). \end{aligned}$$

(5) Compatibility with operations: $E^\nu(f^A(a_1,...,a_n), f^A(b_1,...,b_n))$
$$\begin{aligned} &= E^\mu(f^A(a_1,...,a_n), f^A(b_1,...,b_n)) \wedge \nu(f^A(a_1,...,a_n)) \wedge \\ &\quad \nu(f^A(b_1,...,b_n)) \\ &\geqslant \Big(\bigwedge_{i=1}^n E^\mu(a_i,b_i)\Big) \wedge \Big(\bigwedge_{i=1}^n \nu(a_i)\Big) \wedge \Big(\bigwedge_{i=1}^n \nu(b_i)\Big) \\ &= \bigwedge_{i=1}^n \Big(E^\mu(a_i,b_i) \wedge \nu(a_i) \wedge \nu(b_i)\Big) \\ &= \bigwedge_{i=1}^n E^\nu(a_i,b_i). \end{aligned}$$

(6) Finally, if $a \neq b$ and $E^\nu(a,a) = \nu(a) > 0$, then $E^\nu(a,b) = E^\mu(a,b) \wedge \nu(a) \wedge \nu(b) \leqslant E^\mu(a,b) < \nu(a) = E^\nu(a,a)$,

for all $a, b, c, a_1, ..., a_n, b_1, ..., b_n \in A$, $f \in F$.

Therefore, E^ν is a compatible fuzzy equality on ν. □

Now we are ready for the following definition.

Let $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$ be a fuzzy algebra and $\nu : A \to L$ a fuzzy subalgebra of $\mathcal{A}$, fulfilling the following:

(1) $\nu(x) \leqslant \mu(x)$ for all $x \in A$.

(2) If x and y are distinct elements from A and if $\nu(x) > 0$, then $E^\mu(x,y) < \nu(x)$.

(3) $E^\nu(x,y) := E^\mu(x,y) \wedge \nu(x) \wedge \nu(y)$.

Then we say that the fuzzy algebra $\bar{\mathcal{N}} = (\mathcal{A}, \nu, E^\nu, L)$ is a (**fuzzy**) **subalgebra** of the fuzzy algebra $\bar{\mathcal{M}}$.

Theorem 3.7. *Let $\mathfrak{M}$ be an equational class of fuzzy algebras and let $\bar{\mathcal{M}} \in \mathfrak{M}$ where $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^\mu, L)$. If $\bar{\mathcal{N}} = (\mathcal{A}, \nu, E^\nu, L)$ is a fuzzy subalgebra of $\bar{\mathcal{A}}$, then also $\bar{\mathcal{N}} \in \mathfrak{M}$.*

Proof. Let $\bar{\mathcal{M}} \in \mathfrak{M}$ and let $\bar{\mathcal{N}} = (\mathcal{A}, \nu, E^\nu, L)$ be a fuzzy subalgebra of $\bar{\mathcal{M}}$. Let also $\bar{\mathcal{M}}$ fulfils the fuzzy identity $E(u(x_1, ..., x_n), v(x_1, ..., x_n))$, i.e., let

$$E^\mu(u^A(a_1, ..., a_n), v^A(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \mu(a_i)$$

for all $a_1, ..., a_n \in A$. Then we have:

$$\begin{aligned}
& E^\nu(u^A(a_1, ..., a_n), v^A(a_1, ..., a_n)) \\
= \ & E^\mu(u^A(a_1, ..., a_n), v^A(a_1, ..., a_n)) \wedge \nu(u^A(a_1, ..., a_n)) \wedge \\
& \nu(v^A(a_1, ..., a_n)) \\
\geqslant \ & \Big(\bigwedge_{i=1}^{n} \mu(a_i)\Big) \wedge \Big(\bigwedge_{i=1}^{n} \nu(a_i)\Big) \wedge \Big(\bigwedge_{i=1}^{n} \nu(a_i)\Big) \\
= \ & \bigwedge_{i=1}^{n} \nu(a_i)
\end{aligned}$$

Hence, $\bar{\mathcal{N}}$ also fulfils the identity $E(u,v)$, and thus it belongs to $\mathfrak{M}$. □

3.3. **Fuzzy Homomorphism.** Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two algebras and

$$\bar{\mathcal{M}}_1 = (\mathcal{A}_1, \mu_1, E^{\mu_1}, L), \quad \bar{\mathcal{M}}_2 = (\mathcal{A}_2, \mu_2, E^{\mu_2}, L)$$

two fuzzy algebras, all of the same type $\mathcal{L}$. We say that $\varphi : A_1 \to A_2$ is a **fuzzy algebraic mapping** of $\bar{\mathcal{M}}_1$ into $\bar{\mathcal{M}}_2$ if the following conditions hold:

(1) $(\forall a \in A_1)\ \mu_2(\varphi(a)) \geqslant \mu_1(a)$

(2) Let $u(x_1, \ldots, x_n), v(x_1, \ldots, x_n)$ be terms in the language $\mathcal{L}$, let u^{A_1}, v^{A_1} be the corresponding term-operations on $\mathcal{A}_1$ and $a_1, \ldots, a_n \in A_1$. Then from

$$E^{\mu_1}(u^{A_1}(a_1, ..., a_n), v^{A_1}(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \mu_1(a_i),$$

it follows that

$$E^{\mu_2}(\varphi(u^{A_1}(a_1,...,a_n)),\varphi(v^{A_1}(a_1,...,a_n))) \geqslant \mu_2(\varphi(u^{A_1}(a_1,...,a_n))) \wedge \mu_2(\varphi(v^{A_1}(a_1,...,a_n))).$$

Remark 3.8. A fuzzy algebraic mapping defined here differs from the notion of fuzzy mapping defined by Demirci in [11, 12]. In the cited articles the domain is a crisp set equipped with a fuzzy equality and a fuzzy mapping is defined as a particular fuzzy binary relation. In our approach, both the domain and the co-domain are fuzzy sets, while the mapping is a special ordinary function.

Lemma 3.9. *Let* $\bar{\mathcal{M}}_1 = (\mathcal{A}_1, \mu_1, E^{\mu_1}, L)$*,* $\bar{\mathcal{M}}_2 = (\mathcal{A}_2, \mu_2, E^{\mu_2}, L)$ *and* $\bar{\mathcal{M}}_3 = (\mathcal{A}_3, \mu_3, E^{\mu_3}, L)$ *be fuzzy algebras of the same type. Let also* φ*,* ψ *be fuzzy algebraic mappings from* $\mathcal{M}_1$ *to* $\mathcal{M}_2$*, and from* $\mathcal{M}_2$ *to* $\mathcal{M}_3$*, respectively. Then also their composition* $\varphi \circ \psi$ *is a fuzzy algebraic mapping from* $\mathcal{M}_1$ *to* $\mathcal{M}_3$*.*

Proof. We prove that the conditions in the definition of a fuzzy algebraic mapping are fulfilled.

(1) for every $a \in A_1$

$$\mu_3(\psi(\varphi(a))) \geqslant \mu_2(\varphi(a)) \geqslant \mu_1(a).$$

(2) Let $u(x_1,...,x_n)$, $v(x_1,...,x_n)$ be terms in the language of the given algebras; let also u^{A_1}, v^{A_1} be the corresponding term operations in $\mathcal{A}_1$ and $a_1,...,a_n \in A_1$. If

$$E^{\mu_1}(u^{A_1}(a_1,...,a_n), v^{A_1}(a_1,...,a_n)) \geqslant \bigwedge_{i=1}^{n} \mu_1(a_i),$$

then

$$E^{\mu_2}(\varphi(u^{A_1}(a_1,...,a_n)),\varphi(v^{A_1}(a_1,...,a_n))) \geqslant$$
$$\geqslant \mu_2(\varphi(u^{A_1}(a_1,...,a_n))) \wedge \mu_2(\varphi(v^{A_1}(a_1,...,a_n))).$$

In the next step we use the following simple consequence of the definition 3.3:

If $\varphi : A_1 \to A_2$ is a fuzzy algebraic mapping of $\bar{\mathcal{M}}_1$ into $\bar{\mathcal{M}}_2$, then the following is fulfilled.

For all $c, d \in A_1$ $E^{\mu_1}(c,d) \geqslant \mu_1(c) \wedge \mu_1(d)$, implies $E^{\mu_2}(\varphi(c),\varphi(d)) \geqslant \mu_2(\varphi(c)) \wedge \mu_2(\varphi(d))$.

Therefore,

$$E^{\mu_3}\Big(\psi\big(\varphi(u^{A_1}(a_1,...,a_n))\big),\psi\big(\varphi(v^{A_1}(a_1,...,a_n))\big)\Big) \geqslant$$
$$\geqslant \mu_3\Big(\psi\big(\varphi(u^{A_1}(a_1,...,a_n))\big)\Big) \wedge \mu_3\Big(\psi\big(\varphi v^{A_1}(a_1,...,a_n))\big)\Big),$$

and $\varphi \circ \psi$ is a fuzzy algebraic mapping. □

Let $\bar{\mathcal{M}}_1 = (\mathcal{A}_1, \mu_1, E^{\mu_1}, L)$ and $\bar{\mathcal{M}}_2 = (\mathcal{A}_2, \mu_2, E^{\mu_2}, L)$ be fuzzy algebras of the same type. We say that the fuzzy algebraic mapping φ of the fuzzy algebra $\bar{\mathcal{M}}_1$ into the fuzzy algebra $\bar{\mathcal{M}}_2$ is a **fuzzy homomorphism** of $\bar{\mathcal{M}}_1$ into $\bar{\mathcal{M}}_2$ if the following holds:

(1) For each n-ary $f \in F$ and for all $a_1,...,a_n \in A_1$, $\varphi(f^{A_1}(a_1,...,a_n)) = f^{A_2}(\varphi(a_1),...,\varphi(a_n))$.

(2) $\varphi(c^{A_1}) = c^{A_2}$, for every nullary operation c in the language, c^{A_1} and c^{A_2} being the corresponding constants in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively.

In the following proposition we investigate a particular type of fuzzy subalgebras, which we need in order to locate the homomorphic image of fuzzy homomorphisms.

Proposition 3.10. *Let* $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^{\mu}, L)$ *be a fuzzy algebra and* $\mathcal{B} = (B, F^B)$ *a crisp subalgebra of* $\mathcal{A}$. *Define* $\nu : A \to L$ *by*

$$\nu(x) := \begin{cases} \mu(x), & x \in B \\ 0, & else \end{cases} \qquad and$$

$$E^{\nu}(x, y) := E^{\mu}(x, y) \wedge \nu(x) \wedge \nu(y),$$

then $\bar{\mathcal{N}} = (\mathcal{A}, \nu, E^{\nu}, L)$ *is a fuzzy subalgebra of* $\bar{\mathcal{M}}$.

Proof. First we prove that ν is a fuzzy subalgebra of $\mathcal{A}$, i.e., that

$$\nu(f^A(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \nu(a_i)$$

for all $a_1, ..., a_n \in A$ and for f^A corresponding to $f \in F$. We analyze two possibilities.

(1) Let $a_1, ..., a_n \in B$. Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, we have $f^A(a_1, ..., a_n) \in B$, and thus $\nu(f^A(a_1, ..., a_n)) = \mu(f^A(a_1, ..., a_n))$.

$$\text{Further, } \nu(f^A(a_1, ..., a_n)) = \mu(f^A(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \mu(a_i) = \bigwedge_{i=1}^{n} \nu(a_i).$$

(2) If some a_j is not in B, then $\nu(x_j) = 0$, hence $\bigwedge_{i=1}^{n} \nu(a_i) = 0$.

In any case $\nu(f^A(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \nu(a_i)$.

Next, since $\mu(c) > 0$ for every constant c, we have $\nu(c) = \mu(c)$.

Finally, let a and b be distinct elements from A and let $\nu(a) > 0$. Then $\nu(a) = \mu(a)$, and $E^{\mu}(a, b) < \mu(a) = \nu(a)$.

By the above and by Theorem 3.6, we have that $\bar{\mathcal{N}} = (\mathcal{A}, \nu, E^{\nu}, L)$ is a subalgebra of the fuzzy algebra $\bar{\mathcal{M}}$. □

Theorem 3.11. *Let* $\bar{\mathcal{M}}_1 = (\mathcal{A}_1, \mu_1, E^{\mu_1}, L)$ *and* $\bar{\mathcal{M}}_2 = (\mathcal{A}_2, \mu_2, E^{\mu_2}, L)$ *be fuzzy algebras and* $\varphi : A_1 \to A_2$ *a fuzzy homomorphism of* $\mathcal{M}_1$ *into* $\mathcal{M}_2$. *Let* $\bar{\mathcal{N}} = (\mathcal{A}_2, \nu, E^{\nu}, L)$ *be an ordered quadruple, where for* $x, y \in A_2$

$$\nu(x) := \begin{cases} \mu_2(x), & x \in \varphi(A_1) \\ 0, & otherwise \end{cases}$$

and

$$E^{\nu}(x, y) = E^{\mu_2}(x, y) \wedge \nu(x) \wedge \nu(y).$$

Then, $\bar{\mathcal{N}}$ *is a fuzzy subalgebra of fuzzy algebra* $\bar{\mathcal{M}}_2$.

Fuzzy subalgebra $\bar{\mathcal{N}} = (\mathcal{A}_2, \nu, E^\nu, L)$ of fuzzy algebra $\bar{\mathcal{M}}_2$, introduced in Theorem 3.11, is said to be the **homomorphic image** of fuzzy algebra $\bar{\mathcal{M}}_1$.

We are now ready to prove that equational classes of fuzzy algebras are closed under homomorphic images.

Theorem 3.12. *Let $\mathfrak{M}$ be an equational class of fuzzy algebras. If $\bar{\mathcal{M}}_1 \in \mathfrak{M}$ and $\bar{\mathcal{N}}$ is a homomorphic image of $\bar{\mathcal{M}}_1$, then also $\bar{\mathcal{N}} \in \mathfrak{M}$.*

Proof. Let $\bar{\mathcal{M}}_1, \bar{\mathcal{M}}_2$ and $\bar{\mathcal{N}}$ be as in Theorem 3.11. Assuming that $\bar{\mathcal{M}}_1$ fulfils fuzzy identity $E(u(x_1, ..., x_n), v(x_1, ..., x_n))$, we prove that also its homomorphic image $\bar{\mathcal{N}}$ satisfies the same identity.

Indeed, suppose that for $a_1, ..., a_n \in A_1$

$$E^{\mu_1}(u^{A_1}(a_1, ..., a_n), v^{A_1}(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \mu_1(a_i),$$

where u^{A_1} and v^{A_1} are the term-operations on A_1, and similarly u^{A_2} and v^{A_2} the term-operations on A_2, corresponding to u and v respectively. Now let $b_1, \ldots, b_n \in A_2$. In order to prove that the same fuzzy identity holds on $\bar{\mathcal{N}}$, we analyze two cases.

(1) If $b_i \in \varphi(A_1)$ for every $i \in \{1, ..., n\}$, then for every i there exists $a_i \in A_1$ such that $\varphi(a_i) = b_i$.

By Proposition 2.2 and by the definition of fuzzy homomorphism, we have

$$\begin{aligned}
& E^{\nu}(u^{A_2}(b_1, ..., b_n), v^{A_2}(b_1, ..., b_n)) = \\
= \;& E^{\mu_2}(u^{A_2}(b_1, ..., b_n), v^{A_2}(b_1, ..., b_n)) \wedge \nu(u^{A_2}(b_1, ..., b_n)) \wedge \\
& \nu(v^{A_2}(b_1, ..., b_n)) \\
\geqslant \;& E^{\mu_2}(u^{A_2}(\varphi(a_1), ..., \varphi(a_n)), v^{A_2}(\varphi(a_1), ..., \varphi(a_n))) \wedge \Big(\bigwedge_{i=1}^{n} \nu(b_i)\Big) \\
= \;& E^{\mu_2}(\varphi(u^{A_1}(a_1, ..., a_n)), \varphi(v^{A_1}(a_1, ..., a_n))) \wedge \Big(\bigwedge_{i=1}^{n} \nu(b_i)\Big) \\
\geqslant \;& \mu_2(\varphi(u^{A_1}(a_1, ..., a_n))) \wedge \mu_2(\varphi(v^{A_1}(a_1, ..., a_n))) \wedge \Big(\bigwedge_{i=1}^{n} \nu(b_i)\Big) \\
= \;& \mu_2(u^{A_2}(\varphi(a_1), ..., \varphi(a_n))) \wedge \mu_2(v^{A_2}(\varphi(a_1), ..., \varphi(a_n))) \wedge \Big(\bigwedge_{i=1}^{n} \nu(b_i)\Big) \\
\geqslant \;& \Big(\bigwedge_{i=1}^{n} \mu_2(\varphi(a_i))\Big) \wedge \Big(\bigwedge_{i=1}^{n} \nu(b_i)\Big) \\
= \;& \bigwedge_{i=1}^{n} \nu(b_i).
\end{aligned}$$

(2) If for some $i \in \{1, ..., n\}$ we have that $b_i \notin \varphi(A_1)$, then $\nu(b_i) = 0$, hence

$$E^{\nu}(u^{A_2}(b_1, ..., b_n), v^{A_2}(b_1, ..., b_n)) \geqslant \bigwedge_{i=1}^{n} \nu(b_i) = 0.$$

Therefore, the homomorphic image of a fuzzy algebra in an equational class $\mathfrak{M}$ belongs to $\mathfrak{M}$ as well.

□

3.4. **Direct Product of Fuzzy Algebras.** Here we deal with the third algebraic construction in the fuzzy framework, namely with the fuzzy direct products.

Theorem 3.13. *Let* $\{\bar{\mathcal{M}}_i = (\mathcal{A}_i, \mu_i, E^{\mu_i}, L) \mid i \in I\}$ *be a family of fuzzy algebras of the same type,* $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ *the direct product of algebras* $\mathcal{A}_i$ *and let the following holds for all* $g_1, g_2 \in \prod_{i\in I} \mathcal{A}_i$*,* $g_1 \neq g_2$*:*

$$\text{If } \bigwedge_{i\in I} \mu_i(g_1(i)) \neq 0, \quad \text{then } \bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) \neq \bigwedge_{i\in I} \mu_i(g_1(i)). \tag{10}$$

Now, if $\mu : \prod_{i\in I} A_i \to L$ *and* $E^\mu : (\prod_{i\in I} A_i)^2 \to L$ *are defined by*

$\mu(g) := \bigwedge_{i\in I} \mu_i(g(i)),\ g \in \prod_{i\in I} A_i$ *and*

$E^\mu(g_1, g_2) := \bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i));\ g_1, g_2 \in \prod_{i\in I} A_i$ *respectively,*

then $\bar{\mathcal{M}} = \prod_{i\in I} \bar{\mathcal{M}}_i := (\mathcal{A}, \mu, E^\mu, L)$ *is a fuzzy algebra.*

Proof. First we prove that μ is a fuzzy subalgebra of the product algebra $\mathcal{A}$. For an n-ary operational symbol F in the given language, F^{A_i} denotes the corresponding operation in algebra $\mathcal{A}_i$:

$$\begin{aligned} \mu(F(g_1, ..., g_n)) &= \bigwedge_{i\in I} \mu_i\Big(F^{A_i}(g_1(i), ..., g_n(i))\Big) \\ &\geqslant \bigwedge_{i\in I} \Big(\bigwedge_{j=1}^{n} \mu_i(g_j(i))\Big) \\ &= \bigwedge_{j=1}^{n} \Big(\bigwedge_{i\in I} \mu_i(g_j(i))\Big) \\ &= \bigwedge_{j=1}^{n} \mu(g_j) \end{aligned}$$

Next we prove that E^μ is a fuzzy equality on μ.

a) E^μ is a fuzzy relation on μ:

$E^\mu(f, g) = \bigwedge_{i\in I} E^{\mu_i}(f(i), g(i)) \leqslant \bigwedge_{i\in I} \Big(\mu_i(f(i)) \wedge \mu_i(g(i))\Big)$

$= \Big(\bigwedge_{i\in I} \mu_i(f(i))\Big) \wedge \Big(\bigwedge_{i\in I} \mu_i(g(i))\Big) = \mu(f) \wedge \mu(g).$

b) Reflexivity: $E^\mu(g, g) = \bigwedge_{i\in I} E^{\mu_i}(g(i), g(i)) = \bigwedge_{i\in I} \mu_i(g(i)) = \mu(g)$

c) Symmetry: $E^\mu(g_1, g_2) = \bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i))$

$= \bigwedge_{i\in I} E^{\mu_i}(g_2(i), g_1(i)) = E^\mu(g_2, g_1)$

d) Transitivity: $E^\mu(g_1, g_3) = \bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_3(i))$

$\geqslant \bigwedge_{i\in I} \Big(E^{\mu_i}(g_1(i), g_2(i)) \wedge E^{\mu_i}(g_2(i), g_3(i))\Big)$

$= \Big(\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i))\Big) \bigwedge \Big(\bigwedge_{i\in I} E^{\mu_i}(g_2(i), g_3(i))\Big)$

$= E^{\mu}(g_1, g_2) \wedge E^{\mu}(g_2, g_3)$

e) Compatibility with the operations:

$$E^{\mu}(F^A(g_1, ..., g_n), F^A(f_1, ..., f_n))$$
$$= \bigwedge_{i\in I} E^{\mu_i}\Big(F^A(g_1(i), ..., g_n(i)), F^A(f_1(i), ..., f_n(i))\Big)$$
$$\geqslant \bigwedge_{i\in I}\bigwedge_{j=1}^{n} E^{\mu_i}(g_j(i), f_j(i)) = \bigwedge_{j=1}^{n}\bigwedge_{i\in I} E^{\mu_i}(g_j(i), f_j(i)) = \bigwedge_{j=1}^{n} E^{\mu}(g_j, f_j)$$

f) E^{μ} is a fuzzy equality:

Let $g_1 \neq g_2$ and $\mu(g_1) \neq 0$. Since $E^{\mu_i}(g_1(i), g_2(i)) \leqslant \mu_i(g_1(i))$, we have

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) \leqslant \bigwedge_{i\in I}(\mu_i(g_1(i)).$$

By

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) \neq \bigwedge_{i\in I}(\mu_i(g_1(i)),$$

we get

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) < \bigwedge_{i\in I} \mu_i(g_1(i)).$$

i.e., $$E^{\mu}(g_1, g_2) < \mu(g_1) = E^{\mu}(g_1, g_2).$$

□

The fuzzy algebra $\bar{\mathcal{M}} := (\mathcal{A}, \mu, E^{\mu}, L)$ introduced in Theorem 3.13 is said to be the **direct product** of fuzzy algebras $\bar{\mathcal{M}}_i, i \in I$.

Remark 3.14. In the proof of Theorem 3.13 we have

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) \leqslant \bigwedge_{i\in I}(\mu_i(g_1(i))).$$

From $E^{\mu_i}(g_1(i), g_2(i)) < \mu_i(g_1(i)), i \in I$, it is not possible to deduce

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) < \bigwedge_{i\in I}(\mu_i(g_1(i))).$$

Therefore, it is necessary to include condition

$$\bigwedge_{i\in I} E^{\mu_i}(g_1(i), g_2(i)) \neq \bigwedge_{i\in I}(\mu_i(g_1(i)))$$

in the definition of fuzzy direct product, as it is done by (10) in Theorem 3.13.

Observe that the corresponding condition for the crisp algebras is trivially fulfilled. Indeed, let us take the contraposition of the implication in (10) and formulate crisp claims corresponding to the given fuzzy formulas. It means that we have two different members $g_1 \neq g_2$ of some direct product, and the assumption (of the implication obtained by contraposition) is the following: the components of g_1 and g_2 coincide iff i-th component in g_1 belongs to i-th algebra in the product for every $i \in I$. The consequence is that at least one component is not in the corresponding algebra. Since $g_1 \neq g_2$, the assumption (being logical equivalence) is false, and our implication trivially holds.

Next we prove that the direct product of fuzzy algebras satisfies a fuzzy identity fulfilled by all the fuzzy algebras forming this product.

Theorem 3.15. *If a fuzzy identity $E(u(x_1,...,x_n), v(x_1,...,x_n))$ holds on all fuzzy algebras of the family $\{\bar{\mathcal{M}}_i = (\mathcal{A}_i, \mu_i, E^{\mu_i}, L) \mid i \in I\}$ and all these algebras are of a fixed type, then also this fuzzy identity holds on the product $\bar{\mathcal{M}} = (\mathcal{A}, \mu, E^{\mu}, L)$.*

Proof. Let $u(x_1,...,x_n), v(x_1,...,x_n)$ be terms in the language of fuzzy algebras $\bar{\mathcal{M}}_i,\ i \in I$, and suppose that the fuzzy identity $E(u,v)$ holds on all of these, i.e., that for all $a_1^i,\dots,a_n^i \in A_i,\ i \in I$

$$E^{\mu_i}(u^{A_i}(a_1^i,\dots,a_n^i), v^{A_i}(a_1^i,\dots,a_n^i)) \geqslant \bigwedge_{j=1}^{n} \mu_i(a_j^i).$$

Let $f_1,...,f_n$ be arbitrary elements from $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$. Then

$$\begin{aligned}
&E^{\mu}(u^{A}(f_1,...,f_n), v^{A}(f_1,...,f_n))\\
&= \bigwedge_{i\in I} E^{\mu_i}(u^{A_i}(f_1,...,f_n)(i), v^{A_i}(f_1,...,f_n)(i))\\
&= \bigwedge_{i\in I} E^{\mu_i}(u^{A_i}(f_1(i),...,f_n(i)), v^{A_i}(f_1(i),...,f_n(i)))\\
&\geqslant \bigwedge_{i\in I} \Big(\bigwedge_{j=1}^{n} \mu_i(f_j(i))\Big)\\
&= \bigwedge_{j=1}^{n} \Big(\bigwedge_{i\in I} \mu_i(f_j(i))\Big)\\
&= \bigwedge_{j=1}^{n} \mu(f_j)
\end{aligned}$$

Therefore, the direct product of these fuzzy algebras also satisfies the identity $E(u,v)$. □

3.5. **Fuzzy Variety.** Now we give the final result, i.e., we prove that any fuzzy equational class is closed under fuzzy subalgebras, homomorphic images and products.

Theorem 3.16. *Let $\mathfrak{M}$ be an equational class of fuzzy algebras. Then the following hold:*

(1) *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{B}}$ is a fuzzy subalgebra of $\bar{\mathcal{A}}$, then $\bar{\mathcal{B}} \in \mathfrak{M}$.*
(2) *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{D}}$ is a homomorphic image of $\bar{\mathcal{A}}$, then $\bar{\mathcal{D}} \in \mathfrak{M}$.*
(3) *If for every $i \in I$, $\bar{\mathcal{A}}_i$ belongs to $\mathfrak{M}$, then also $\prod_{i\in I} \bar{\mathcal{A}}_i \in \mathfrak{M}$.*

Proof. Straightforward by Theorems 3.7, 3.12, 3.15. □

In order to formulate the above result appropriately in terms of general algebra, let us define the following notion. For a fixed language and a complete lattice L, a class $\mathcal{V}$ of fuzzy algebras closed under fuzzy subalgebras, fuzzy homomorphic images and fuzzy direct products is a **fuzzy variety**.

In terms of fuzzy varieties, Theorem 3.7 can be formulated as follows.

$\cdot$	a	b	c	d
a	b	b	c	c
b	b	a	d	d
c	c	c	d	d
d	d	d	d	c

TABLE 3. Groupoid G

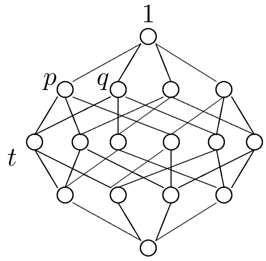

FIGURE 2. Lattice L

Corollary 3.17. *Every fuzzy equational class is a fuzzy variety.*

As illustrations of Corollary 3.17, let us mention that Example 3.4 describes a fuzzy variety of fuzzy groups and Example 3.5 a fuzzy variety of fuzzy lattices, both with particular lattices of membership values.

Under the present conditions, the converse of Corollary 3.17 is not true in general, as witnessed by the following example.

Example 3.18. The class generated by crisp commutative groupoids with the usual equality is a class of fuzzy groupoids in our approach, for any complete lattice L of membership values: groupoids themselves represented by characteristic functions, and the crisp equality taking values 0 and 1 from L. In addition, by the fuzzy subalgebra operator, this class contains also fuzzy subgroupoids of crisp commutative groupoids. Hence, with the operation represented by $\cdot$, the fuzzy identity $E(x \cdot y, y \cdot x)$ holds. The class is closed under formation of fuzzy subalgebras, homomorphisms and products, hence being a fuzzy variety. However, fuzzy commutative groupoids which are subgroupoids of crisp non-commutative groupoids (point 3., Remark 3.3) could not be obtained as described here, though they do belong to the corresponding fuzzy equational class. An example is given in the sequel (formulated in [6], but in a different context).

G is a four-element groupoid $(\{a, b, c, d\}, \cdot)$, whose operation is given in Table 3, and L is a 16-element Boolean lattice, given in Figure 2.

Consider the fuzzy groupoid (G, μ, E^{μ}, L), where $\mu : G \to L$ is given by

$$\mu(x) = \begin{pmatrix} a & b & c & d \\ p & p & q & q \end{pmatrix},$$

and a fuzzy equality E^{μ} on μ presented in Table 4.

Then (G, μ, E^{μ}, L) fulfils the fuzzy identity $E(x \cdot y, y \cdot x)$, i.e., the formula $E^{\mu}(x \cdot y, y \cdot x) \geqslant \mu(x) \wedge \mu(y)$ is satisfied for all $x, y \in \{a, b, c, d\}$. This fuzzy commutative

E^μ	a	b	c	d
a	p	0	0	0
b	0	p	0	0
c	0	0	q	t
d	0	0	t	q

Table 4. Fuzzy Equality on μ

groupoid belongs to the fuzzy equational class determined by L and a fuzzy identity $E(x\cdot y, y\cdot x)$, though it can not be generated by crisp commutative groupoids, using operators S, H and P. Hence, it does not belong to the corresponding fuzzy variety.

4. Conclusion

As presented above, a one way round of the connection among equational classes and fuzzy varieties in fuzzy framework is established. Namely, our notion of fuzzy equality enable identification of fuzzy equational classes. Then we have introduced and investigated particular fuzzy algebras and their subalgebras, as well as fuzzy homomorphisms and products. The final result is that each fuzzy equational class is a fuzzy variety. For the converse, in order to capture equational classes in general, it should be necessary to introduce some additional requirements on fuzzy homomorphisms, though the present notion seems quite natural. This is a problem we would deal with as a continuation of our research elaborated here.

What seems to be the most important result of the present work, are possible *applications* of fuzzy equational classes to concrete known algebraic structures and their usage in mathematics, computer science, automata theory etc. Namely, using our approach, equational classes of fuzzy semigroups, fuzzy groups and other important algebraic structures could be introduced in the new way, capturing fuzziness by fuzzy identities. This would lead to new directions in dealing with e.g, fuzzy automata and fuzzy languages, or in applications of fuzzy quotient groups (e.g., coding theory in the fuzzy framework) etc.

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