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A NEW ANALYTICAL METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In the literature, several numerical methods are proposed for solving nth-order fuzzy linear differential equations. However, till now there are only two analytical methods for the same. In this paper, the fuzzy Kolmogorov's differential equations, obtained with the help of fuzzy Markov model of piston manufacturing system, are solved by one of these analytical methods and illustrated that the obtained solution does not represent a fuzzy number. To resolve the drawback of existing method, a new analytical method is proposed for solving nth-order fuzzy linear differential equations. Furthermore, the advantage of proposed method over existing method is also discussed.

1. Introduction

In various real-world problems, it is desirable to transform the behavior of a specific phenomenon into a deterministic initial value problem of linear systems of differential equations. However, the model is not usually perfect due to lack of certain information which must be estimated through measurements. In recent years, lot of work has been done by several authors in fuzzy differential equations [1-11,13-33,37-39,46-48,50-52].

The concept of fuzzy derivative was first introduced by Chang and Zadeh [21]. It was followed by Dubois and Prade [28], who defined and applied the extension principle. Goetschel-Voxman derivative was defined in [33]. Seikkala [52] defined fuzzy derivative which is the generalization of Hukuhara derivative [51]. Fuzzy initial value problem has been studied by Kaleva [38, 39] and Seikkala [52], higher order fuzzy differential equations have been studied in [15, 32] and numerical methods have been discussed in [1, 2, 3, 4, 6, 7, 8, 47, 50].

Fuzzy differential equations are utilized to analyze the behavior of phenomena that are subject to imprecise or uncertain factors, ranging from particle physics [46], chaotic systems [30, 31] and engineering [48] to medicine [10] and computational biology [9, 19, 20].

Similar to ordinary differential equations, the analytical solution of fuzzy differential equations is often difficult. Buckley and Feuring [15] introduced two analytical methods for solving nth-order fuzzy linear differential equations. Their first method of solution is to fuzzify the crisp solution and to check that the obtained solution

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satisfies the differential equation or not and in the second method they first solve fuzzy differential equation and then check that if it defines a fuzzy function or not.

In this paper, the fuzzy Kolmogorov's differential equations are solved by an existing analytical method and illustrated that the obtained solution does not define a fuzzy number. To resolve the drawback of existing method, a new analytical method is proposed for solving nth-order fuzzy linear differential equations and the advantage of proposed method over existing method is also demonstrated.

This paper is organized as follows: In section 2, some basic definitions and arithmetic operations are presented. In section 3, an existing analytical method for solving nth-order fuzzy linear differential equations is illustrated. The drawback of existing method is pointed out in section 4 and the drawback of the existing method while solving real life problems is discussed in section 5. In section 6, a new analytical method is proposed to find the solution of nth-order fuzzy linear differential equations with the help of *JMD LR* flat fuzzy numbers. Also, the advantage of proposed method over the existing method as well as the advantage of *JMD LR* flat fuzzy numbers over existing *LR* flat fuzzy numbers are discussed. In section 7, fuzzy reliability of piston manufacturing system is evaluated. The conclusion is drawn in section 8.

2. **Preliminaries**

In this section, some basic definitions and arithmetic operations are presented.

2.1. **Basic Definitions.** In this section, some basic definitions are presented.

2.1.1. α-cut. In this section, definitions of α-cut of a fuzzy number, zero α-cut and equality of α-cuts are presented [40].

Definition 2.1. An α-cut of a fuzzy number $\tilde{A}$ is defined as a crisp set $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X\}$, where $\alpha \in [0, 1]$.

Definition 2.2. An α-cut $A_\alpha = [a, b]$ is said to be zero α-cut if and only if $a = 0$ and $b = 0$.

Definition 2.3. Two α-cuts $A_\alpha = [a_1, b_1]$ and $B_\alpha = [a_2, b_2]$ are said to be equal i.e., $A_\alpha = B_\alpha$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

2.1.2. LR Flat Fuzzy Number. In this section, definitions of LR flat fuzzy number, zero LR flat fuzzy number and equality of LR flat fuzzy numbers are presented [27].

Definition 2.4. A fuzzy number $\widetilde{A} = (m, n, \alpha^L, \alpha^R)_{LR}$ is said to be an LR flat fuzzy number if

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha^L}), & \text{for } x \leq m,\ \alpha^L > 0 \\ R(\frac{x-n}{\alpha^R}), & \text{for } x \geq n,\ \alpha^R > 0 \\ 1, \text{otherwise} & \end{cases}$$

If $m = n$ then $\widetilde{A} = (m, n, \alpha^L, \alpha^R)_{LR}$ will be converted into $\widetilde{A} = (m, \alpha^L, \alpha^R)_{LR}$ and is said to be LR fuzzy number.

$L : [0, \infty) \to [0, 1]$ and $R : [0, \infty) \to [0, 1]$ are called reference functions, which are continuous, non-increasing functions that defines the left and right shapes of $\mu_{\widetilde{A}}(x)$ respectively and $L(0) = R(0) = 1$.

Definition 2.5. An LR flat fuzzy number $\tilde{A} = (m, n, \alpha^L, \alpha^R)_{LR}$ is said to be zero LR flat fuzzy number if and only if $m = 0, n = 0, \alpha^L = 0$ and $\alpha^R = 0$.

Definition 2.6. An LR flat fuzzy number $\tilde{A} = (m, n, \alpha^L, \alpha^R)_{LR}$ is said to be non-negative LR flat fuzzy number if and only if $m - \alpha^L \geq 0$.

Definition 2.7. Two LR flat fuzzy numbers $\tilde{A}_1 = (m_1, n_1, \alpha_1^L, \alpha_1^R)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2^L, \alpha_2^R)_{LR}$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $m_1 = m_2, n_1 = n_2, \alpha_1^L = \alpha_2^L$ and $\alpha_1^R = \alpha_2^R$.

Definition 2.8. Let $\tilde{A} = (m, n, \alpha^L, \alpha^R)_{LR}$ be an LR flat fuzzy number and λ be a real number in the interval $[0, 1]$ then the crisp set, $A_\lambda = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\} = [m - \alpha^L L^{-1}(\lambda), n + \alpha^R R^{-1}(\lambda)]$, is said to be λ-cut of $\widetilde{A}$.

Remark 2.9. [52] For fuzzy process, $y : I \to E$ with $y(t, r)$. The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by
$y'(t, r) = [\underline{y}'(t, r), \overline{y}'(t, r)]$, $0 < r \leq 1$
provided that this equation defines a fuzzy number $y'(t) \in E$.

2.2. **Arithmetic Operations.** In this section, some arithmetic operations between intervals and LR flat fuzzy numbers are presented.

2.2.1. **Arithmetic Operations Between Intervals.** In this section, some arithmetic operations between intervals are presented [40].

Let $A = [a_1, b_1]$, $B = [a_2, b_2]$ be two intervals. Then,

(i) $A + B = [a_1 + a_2, b_1 + b_2]$
(ii) $A - B = [a_1 - b_2, b_1 - a_2]$
(iii) $\lambda A = \begin{cases} [\lambda a_1, \lambda b_1], & \lambda \geq 0 \\ [\lambda b_1, \lambda a_1], & \lambda \leq 0 \end{cases}$
(iv) $AB = [a, b]$, where, $a =$ minimum $(a_1a_2, a_1b_2, a_2b_1, b_1b_2)$ and $b =$ maximum $(a_1a_2, a_1b_2, a_2b_1, b_1b_2)$

2.2.2. **Arithmetic Operations Between LR Flat Fuzzy Numbers.** In this section, arithmetic operations between LR flat fuzzy numbers are presented [27].

Let $\tilde{A}_1 = (m_1, n_1, \alpha_1^L, \alpha_1^R)_{LR}$, $\tilde{A}_2 = (m_2, n_2, \alpha_2^L, \alpha_2^R)_{LR}$ be any L-R flat fuzzy numbers and $\tilde{A}_3 = (m_3, n_3, \alpha_3^L, \alpha_3^R)_{RL}$ be any R-L flat fuzzy number. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1^L + \alpha_2^L, \alpha_1^R + \alpha_2^R)_{LR}$
(ii) $\tilde{A}_1 \ominus \tilde{A}_3 = (m_1 - n_3, n_1 - m_3, \alpha_1^L + \alpha_3^R, \alpha_1^R + \alpha_3^L)_{LR}$
(iii) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-negative then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (m_1m_2, n_1n_2, m_1\alpha_2^L + \alpha_1^L m_2 - \alpha_1^L\alpha_2^L, n_1\alpha_2^R + \alpha_1^R n_2 + \alpha_1^R\alpha_2^R)_{LR}$
(iv) If $\tilde{A}_1$ is non-positive and $\tilde{A}_2$ is non-negative then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (m_1n_2, n_1m_2, \alpha_1^L n_2 - m_1\alpha_2^R + \alpha_1^L\alpha_2^R, \alpha_1^R m_2 - n_1\alpha_2^L - \alpha_1^R\alpha_2^L)_{LR}$
(v) If $\tilde{A}_1$ is non-negative and $\tilde{A}_2$ is non-positive then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (n_1m_2, m_1n_2, n_1\alpha_2^L - \alpha_1^R m_2 + \alpha_1^R\alpha_2^L, m_1\alpha_2^R - \alpha_1^L n_2 - \alpha_1^L\alpha_2^R)_{LR}$

(vi) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-positive then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (n_1 n_2, m_1 m_2, -n_1\alpha_2^R - \alpha_1^R n_2 - \alpha_1^R \alpha_2^R, -m_1\alpha_2^L - \alpha_1^L m_2 + \alpha_1^L \alpha_2^L)_{LR}$

(vii) $\lambda \tilde{A}_1 = \begin{cases} (\lambda m_1, \lambda n_1, \lambda \alpha_1^L, \lambda \alpha_1^R)_{LR} & \lambda \geq 0 \\ (\lambda n_1, \lambda m_1, -\lambda \alpha_1^R, -\lambda \alpha_1^L)_{RL} & \lambda < 0 \end{cases}$

3. Existing Method

Buckley and Feuring [15] introduced two analytical methods for solving nth-order fuzzy linear differential equations. In this section, one of these existing methods is illustrated.

The solution of nth-order fuzzy linear differential equation

$$\tilde{a}_n \tilde{y}^{(n)} \oplus \tilde{a}_{n-1} \tilde{y}^{(n-1)} \oplus ... \oplus \tilde{a}_1 \tilde{y}^{(1)} \oplus \tilde{a}_0 \tilde{y} = \tilde{g}(x) \tag{1}$$

with initial conditions $\tilde{y}(0) = \tilde{\gamma}_0, \tilde{y}^{(1)}(0) = \tilde{\gamma}_1, ..., \tilde{y}^{(n-1)}(0) = \tilde{\gamma}_{n-1}$
where, $\tilde{y}^{(i)} = \frac{d^i \tilde{y}}{dx^i}$ for $i = n, n-1, ..., 1$, $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0$ are LR flat fuzzy numbers, can be obtained by using the following steps:

Step 1: Find the α-cuts $[a_{n(1)}(x,\alpha), a_{n(2)}(x,\alpha)]$,$[a_{n-1(1)}(x,\alpha), a_{n-1(2)}(x,\alpha)]$, ..., $[a_{1(1)}(x,\alpha), a_{1(2)}(x,\alpha)]$, $[a_{0(1)}(x,\alpha), a_{0(2)}(x,\alpha)]$,$[y_1^{(n)}(x,\alpha), y_2^{(n)}(x,\alpha)]$, $[y_1^{(n-1)}(x,\alpha), y_2^{(n-1)}(x,\alpha)]$,...,$[y_1^{(1)}(x,\alpha), y_2^{(1)}(x,\alpha)]$,$[y_1(x,\alpha), y_2(x,\alpha)]$ and $[\gamma_{0(1)}(0,\alpha), \gamma_{0(2)}(0,\alpha)]$, $[\gamma_{1(1)}(0,\alpha), \gamma_{1(2)}(0,\alpha)]$, ..., $[\gamma_{n-1(1)}(0,\alpha), \gamma_{n-1(2)}(0,\alpha)]$ corresponding to fuzzy parameters $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0, \tilde{y}^{(n)}, \tilde{y}^{(n-1)}, ..., \tilde{y}^{(1)}, \tilde{y}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}$ respectively.

Step 2: Replacing $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0, \tilde{y}^{(n)}, \tilde{y}^{(n-1)}, ..., \tilde{y}^{(1)}, \tilde{y}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}$ by their α-cuts, obtained in Step 1, the equation (1) can be written as:

$$[a_{n(1)}(x,\alpha), a_{n(2)}(x,\alpha)][y_1^{(n)}(x,\alpha), y_2^{(n)}(x,\alpha)] + [a_{n-1(1)}(x,\alpha), a_{n-1(2)}(x,\alpha)][y_1^{(n-1)}(x,\alpha), y_2^{(n-1)}(x,\alpha)] + ... + [a_{1(1)}(x,\alpha), a_{1(2)}(x,\alpha)][y_1^{(1)}(x,\alpha), y_2^{(1)}(x,\alpha)] + [a_{0(1)}(x,\alpha), a_{0(2)}(x,\alpha)][y_1(x,\alpha), y_2(x,\alpha)] = [g(x), g(x)] \tag{2}$$

with initial conditions $[y_1(0,\alpha), y_2(0,\alpha)] = [\gamma_{0(1)}(0,\alpha), \gamma_{0(2)}(0,\alpha)]$, $[y_1^{(1)}(0,\alpha), y_2^{(1)}(0,\alpha)] = [\gamma_{1(1)}(0,\alpha), \gamma_{1(2)}(0,\alpha)]$,..., $[y_1^{(n-1)}(0,\alpha), y_2^{(n-1)}(0,\alpha)] = [\gamma_{n-1(1)}(0,\alpha), \gamma_{n-1(2)}(0,\alpha)]$.

Step 3: Using Definition 2.3 and section 2.2.1, the equation (2) can be split into two ordinary differential equations.

Step 4: Solve the ordinary differential equations, obtained in Step 3, to find the values of $y_1(x_0,\alpha)$ and $y_2(x_0,\alpha)$ corresponding to $x = x_0$, where x_0 is any real number.

Step 5: Check that $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ defines the α-cut of a fuzzy number or not i.e., the following conditions are satisfied or not.
(i) $y_1(x_0,\alpha)$ is monotonically increasing function for $\alpha \in [0,1]$
(ii) $y_2(x_0,\alpha)$ is monotonically decreasing function for $\alpha \in [0,1]$

Case 1: If $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ defines α-cut of a fuzzy number then the fuzzy solution $\tilde{y}(x_0)$ of fuzzy differential equation (1) exist and $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ represents the α-cut corresponding to fuzzy solution $\tilde{y}(x_0)$.

Case 2: If $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ does not define α-cut of a fuzzy number then the fuzzy solution $\tilde{y}(x_0)$ of fuzzy differential equation (1) does not exist.

4. **Drawback of Existing Method**

To specify the drawback of existing method [15], the system of fuzzy equations is solved by existing method [15].

Example 4.1. Solve the following system of fuzzy equations:

$$\left.\begin{array}{l} (m_{11}, n_{11}, \alpha_{11}^L, \alpha_{11}^R)_{LR} \oplus (m_{12}, n_{12}, \alpha_{12}^L, \alpha_{12}^R)_{LR} = (20, 30, 10, 10)_{LR} \\ (m_{21}, n_{21}, \alpha_{21}^L, \alpha_{21}^R)_{LR} \oplus (m_{22}, n_{22}, \alpha_{22}^L, \alpha_{22}^R)_{LR} = (20, 35, 5, 5)_{LR} \\ (m_{11}, n_{11}, \alpha_{11}^L, \alpha_{11}^R)_{LR} \oplus (m_{21}, n_{21}, \alpha_{21}^L, \alpha_{21}^R)_{LR} = (15, 25, 3, 5)_{LR} \end{array}\right\} \quad (3)$$

4.1. **Solution of the Chosen Problem.** With the help of the existing method [15], the solution of system of fuzzy equations (3) can be obtained by using the following steps:

Step 1: Replacing $(m_{ij}, n_{ij}, \alpha_{ij}^L, \alpha_{ij}^R)_{LR}$ by its α-cut $[m_{ij} - \alpha_{ij}^L L^{-1}(\alpha), n_{ij} + \alpha_{ij}^R R^{-1}(\alpha)]$ the system of fuzzy equations (3) can be written as:

$$\left.\begin{array}{l} [m_{11} - \alpha_{11}^L L^{-1}(\alpha), n_{11} + \alpha_{11}^R R^{-1}(\alpha)] + [m_{12} - \alpha_{12}^L L^{-1}(\alpha), n_{12}+ \\ \alpha_{12}^R R^{-1}(\alpha)] = [20 - 10L^{-1}(\alpha), 30 + 10R^{-1}(\alpha)] \\ [m_{21} - \alpha_{21}^L L^{-1}(\alpha), n_{21} + \alpha_{21}^R R^{-1}(\alpha)] + [m_{22} - \alpha_{22}^L L^{-1}(\alpha), n_{22}+ \\ \alpha_{22}^R R^{-1}(\alpha)] = [20 - 5L^{-1}(\alpha), 35 + 5R^{-1}(\alpha)] \\ [m_{11} - \alpha_{11}^L L^{-1}(\alpha), n_{11} + \alpha_{11}^R R^{-1}(\alpha)] + [m_{21} - \alpha_{21}^L L^{-1}(\alpha), n_{21}+ \\ \alpha_{21}^R R^{-1}(\alpha)] = [15 - 3L^{-1}(\alpha), 25 + 5R^{-1}(\alpha)] \end{array}\right\} \quad (4)$$

Step 2: Using section 2.2.1, the system of equations (4) can be written as:

$$\left.\begin{array}{l} [m_{11} - \alpha_{11}^L L^{-1}(\alpha) + m_{12} - \alpha_{12}^L L^{-1}(\alpha), n_{11} + \alpha_{11}^R R^{-1}(\alpha) + n_{12}+ \\ \alpha_{12}^R R^{-1}(\alpha)] = [20 - 10L^{-1}(\alpha), 30 + 10R^{-1}(\alpha)] \\ [m_{21} - \alpha_{21}^L L^{-1}(\alpha) + m_{22} - \alpha_{22}^L L^{-1}(\alpha), n_{21} + \alpha_{21}^R R^{-1}(\alpha) + n_{22}+ \\ \alpha_{22}^R R^{-1}(\alpha)] = [20 - 5L^{-1}(\alpha), 35 + 5R^{-1}(\alpha)] \\ [m_{11} - \alpha_{11}^L L^{-1}(\alpha) + m_{21} - \alpha_{21}^L L^{-1}(\alpha), n_{11} + \alpha_{11}^R R^{-1}(\alpha) + n_{21}+ \\ \alpha_{21}^R R^{-1}(\alpha)] = [15 - 3L^{-1}(\alpha), 25 + 5R^{-1}(\alpha)] \end{array}\right\} \quad (5)$$

Step 3: Using Definition 2.3, the system of equations (5) can be written as:

$$\left.\begin{array}{l} m_{11} - \alpha_{11}^L L^{-1}(\alpha) + m_{12} - \alpha_{12}^L L^{-1}(\alpha) = 20 - 10L^{-1}(\alpha) \\ n_{11} + \alpha_{11}^R R^{-1}(\alpha) + n_{12} + \alpha_{12}^R R^{-1}(\alpha) = 30 + 10R^{-1}(\alpha) \\ m_{21} - \alpha_{21}^L L^{-1}(\alpha) + m_{22} - \alpha_{22}^L L^{-1}(\alpha) = 20 - 5L^{-1}(\alpha) \\ n_{21} + \alpha_{21}^R R^{-1}(\alpha) + n_{22} + \alpha_{22}^R R^{-1}(\alpha) = 35 + 5R^{-1}(\alpha) \\ m_{11} - \alpha_{11}^L L^{-1}(\alpha) + m_{21} - \alpha_{21}^L L^{-1}(\alpha) = 15 - 3L^{-1}(\alpha) \\ n_{11} + \alpha_{11}^R R^{-1}(\alpha) + n_{21} + \alpha_{21}^R R^{-1}(\alpha) = 25 + 5R^{-1}(\alpha) \end{array}\right\} \quad (6)$$

Step 4: On solving the system of equations (6) for $\alpha = 0$, the following optimal solution is obtained:
$m_{11} = 15, n_{11} = 25, \alpha_{11}^L = 15, \alpha_{11}^R = 5$
$m_{12} = 5, n_{12} = 5, \alpha_{12}^L = -5, \alpha_{12}^R = 5$
$m_{21} = 0, n_{21} = 0, \alpha_{21}^L = -12, \alpha_{21}^R = 0$
$m_{22} = 20, n_{22} = 35, \alpha_{22}^L = 17, \alpha_{22}^R = 5$

4.2. **Shortcoming of Obtained Result.** In the fuzzy number $(m_{ij}, n_{ij}, \alpha_{ij}^L, \alpha_{ij}^R)_{LR}$, the conditions $m_{ij} \leq n_{ij}$, $\alpha_{ij}^L \geq 0$ and $\alpha_{ij}^R \geq 0$ should always be satisfied but in the obtained solution $\alpha_{12}^L < 0$ and $\alpha_{21}^L < 0$. Hence, on the basis of obtained solution by applying the existing method [15], one may conclude that the system of fuzzy equations (3) is infeasible.

However, if the system of equations, obtained in Step 3, is solved with the restrictions $m_{ij} \leq n_{ij}$, $\alpha_{ij}^L \geq 0$ and $\alpha_{ij}^R \geq 0$ then the following feasible solution is obtained:
$m_{11} = 0, n_{11} = 10, \alpha_{11}^L = 0, \alpha_{11}^R = 5$
$m_{12} = 20, n_{12} = 20, \alpha_{12}^L = 10, \alpha_{12}^R = 5$
$m_{21} = 15, n_{21} = 15, \alpha_{21}^L = 3, \alpha_{21}^R = 0$
$m_{22} = 5, n_{22} = 20, \alpha_{22}^L = 2, \alpha_{22}^R = 5$

On the basis of the above result, it can be concluded that although the chosen system of fuzzy equations is feasible but no feasible solution can be obtained by applying existing method [15].

4.3. **Reason of Shortcoming.** On solving the system of ordinary differential equations, obtained in Step 3 of the existing method [15], the following shortcoming may occur:

4.3.1. **Shortcoming Due to Analytical Methods.** If the system of ordinary differential equations, obtained in Step 3 of the existing method [15], is solved by any analytical method then it is not possible to apply any restriction on the dependent variables y_1 and y_2. If there exist more than one fuzzy solution of a fuzzy differential equation then by using the existing method [15] that solution may be obtained which satisfies the condition $y_1 \leq y_2$ and on the basis of the obtained solution, one may conclude that the solution of the fuzzy differential equation does not exist whereas in actual there may possibly exist feasible solution of fuzzy differential equation.

4.3.2. **Shortcoming Due to Numerical Methods.** In practice, it is not always possible to find the exact solution of ordinary differential equations and in such cases the numerical methods are employed to solve ordinary differential equations. If the system of ordinary differential equations, obtained in Step 3 of the existing method [15], is solved by numerical methods then it is not possible to apply any restriction on the dependent variables y_1 and y_2. Therefore, the confusion regarding the feasibility of chosen fuzzy differential equation may take place.

5. **Shortcoming of Existing Method in Real Life Problems**

In the literature, several methods [34, 35, 36, 44, 49, 53, 55, 56, 57] are proposed to evaluate fuzzy reliability of industrial systems. One of these existing methods for evaluating the fuzzy reliability is by using fuzzy Markov modeling [12, 41, 43, 45, 54], wherein fuzzy Kolmogorov's differential equations are developed by fuzzy Markov model. Further, fuzzy reliability is evaluated by solving the attained fuzzy Kolmogorov's differential equations.

In this section, fuzzy Kolmogorov's differential equations, obtained by using fuzzy Markov model of piston manufacturing system, are solved by using an analytical method [15] and it is illustrated that the obtained solution does not define a fuzzy number. Therefore, the solution of fuzzy Kolmogorov's differential equations can not be used to evaluate fuzzy reliability of piston manufacturing system.

5.1. **Fuzzy Markov Modeling of Piston Manufacturing System.** Piston manufacturing system consists of two sub-systems namely R_1 and R_2 , which are connected in series. Further the sub-system R_1 consists of six sub-systems namely A, B, C, D, E and F and similarly, six sub-systems namely G, H, I, J, K and L constitute the sub-system R_2. Markov models for sub-system R_1 and sub-system R_2 are shown in Figure 1 and Figure 2 respectively.

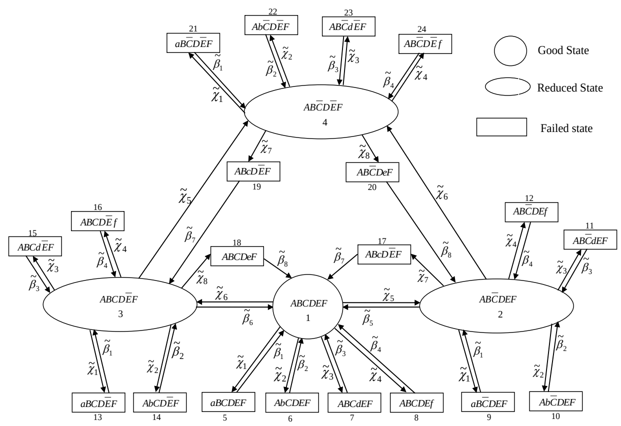

FIGURE 1. Fuzzy Markov Model of Sub-System R_1

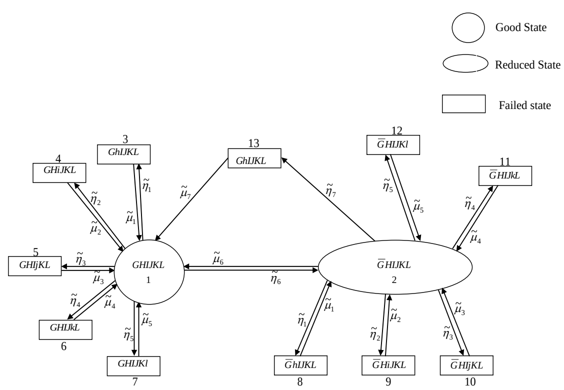

FIGURE 2. Fuzzy Markov Model of Sub-System R_2

The operations that are performed on these machines or sub-systems are as follows:

1. **Sub-system A (Fixture Seat Machine):** This machine is used to clamp the piston.
2. **Sub-system B (Rough Grooving and Turning Machine):** On this machine, rough grooves are made on piston. Turning operation is performed on this machine i.e., to bring the dia of piston to proper size.

3. **Sub-system C (Rough Pin Hole Boring Machine):** Pin hole boring operation is performed using this machine i.e., proper size is given to holes.
4. **Sub-system D (Oil Hole Drilling Machine):** On this machine, one hole is made on the piston to supply the oil. The oil is used to move piston in cylinder smoothly.
5. **Sub-system E (Finishing Grooving Machine):** On this machine, the finishing is given to rough grooves which are prepared using sub-system B.
6. **Sub-system F (Finish Profile Turning Machine):** Oval shape is given to piston using this machine.
7. **Sub-system G (Finish Pin Hole Boring Machine):** On this machine, finishing is given to the pin hole portion which is prepared using sub-system C.
8. **Sub-system H (Finish Crown and Cavity Machine):** On this machine, finishing operation is performed on the crown of piston.
9. **Sub-system I (Valve Milling Machine):** On this machine, valve recess is made on the piston.
10. **Sub-system J (Chamfering or Radius Machine):** This machine rounds off the corners of the piston so that it can run smoothly in the cylinder.
11. **Sub-system K (Circlip Grooving Machine):** On this machine, circlip grooves are made on the piston.
12. **Sub-system L (Piston Cleaning Machine):** This machine is used to clean the inside and outside portion of the piston.

5.2. **Notation.** In this section, notation that is used to evaluate the fuzzy reliability of piston manufacturing system are presented:

1. A, B, C, D, E, F and G, H, I, J, K, L denote good conditions of sub-systems of R_1 and R_2 respectively.
2. The symbols $a, b, c, d, e, f, g, h, i, j, k$ and l represent the failed state of the sub-systems $A, B, C, D, E, F, G, H, I, J, K$ and L respectively.
3. $\bar{C}, \bar{E}$ and $\bar{G}$ indicate that the sub-systems C, E and G are working in reduced state.
4. $\tilde{\chi}_i (i = 1 \text{ to } 8)$ represents the fuzzy failure rates of the relevant sub-systems, when the transition is from A to a, B to b, D to d, F to f, C to $\bar{C}$, E to $\bar{E}$, $\bar{C}$ to c and $\bar{E}$ to e respectively.
5. $\tilde{\beta}_i (i = 1 \text{ to } 8)$ represents the fuzzy repair rates of the relevant sub-systems, when the transition is from a to A, b to B, d to D, f to F, $\bar{C}$ to C, $\bar{E}$ to E, c to C and e to E respectively.
6. $\tilde{\eta}_i (i = 1 \text{ to } 7)$ represents the fuzzy failure rates of the relevant sub-systems, when the transition is from H to h, I to i, J to j, K to k, L to l, G to $\bar{G}$ and $\bar{G}$ to g respectively.
7. $\tilde{\mu}_i (i = 1 \text{ to } 7)$ represents the fuzzy repair rates of the relevant sub-systems, when the transition is from h to H, i to I, j to J, k to K, l to L, $\bar{G}$ to G and g to G respectively.
8. $\tilde{P}_j(t), j = 1, 2, ..., n$ represents the fuzzy probability that the system is in state S_j at time t, where n is number of states. $\tilde{P}'_j(t), j = 1, 2, ..., n$ represents derivative of $\tilde{P}_j(t)$ with respect to t.

9. $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ denote the fuzzy reliability of the sub-systems R_1 and R_2 respectively.
10. $\tilde{R}(t)$ represents the fuzzy reliability of the whole system.

5.3. **Assumptions.** In this section, the assumptions that are made to evaluate fuzzy reliability of piston manufacturing system are as follows:

(i) Fuzzy failure rates and fuzzy repair rates are independent with each other and their unit is per hour.
(ii) There are no simultaneous failures among the sub-systems.
(iii) Sub-systems C, E and G fails through reduced states only.

5.4. **Data.** On the basis of perception of experts, the appropriate failure rates and repair rates for different sub-systems of R_1 and R_2 are shown in Table 1 and Table 2 respectively.

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\chi}_1$=(0.00126,0.00154,0.00021,0.00021)$_{LR}$	$\tilde{\beta}_1$=(1.0584,1.1016,0.0324,0.0324)$_{LR}$
$\tilde{\chi}_2$=(0.00054,0.00066,0.00009,0.00009)$_{LR}$	$\tilde{\beta}_2$=(0.04214,0.04386,0.00129,0.00129)$_{LR}$
$\tilde{\chi}_3$=(0.00081,0.00099,0.000135,0.000135)$_{LR}$	$\tilde{\beta}_3$=(0.49,0.51,0.015,0.015)$_{LR}$
$\tilde{\chi}_4$=(0.00081,0.00099,0.000135,0.000135)$_{LR}$	$\tilde{\beta}_4$=(0.28028,0.29172,0.00858,0.00858)$_{LR}$
$\tilde{\chi}_5$=(0.01872,0.02288,0.00312,0.00312)$_{LR}$	$\tilde{\beta}_5$=(0.15092,0.15702,0.00462,0.00462)$_{LR}$
$\tilde{\chi}_6$=(0.01872,0.02288,0.00312,0.00312)$_{LR}$	$\tilde{\beta}_6$=(0.245,0.255,0.0078,0.0075)$_{LR}$
$\tilde{\chi}_7$=(0.00081,0.00099,0.000135,0.000135)$_{LR}$	$\tilde{\beta}_7$=(0.05782,0.06018,0.00177,0.00177)$_{LR}$
$\tilde{\chi}_8$=(0.00351,0.00429,0.000585,0.000585)$_{LR}$	$\tilde{\beta}_8$=(0.08526,0.08874,0.00261,0.00261)$_{LR}$

Table 1. Fuzzy Failure Rates and Fuzzy Repair Rates for the Different Sub-systems of R_1

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\eta}_1$=(0.00126,0.00154,0.00021,0.00021)$_{LR}$	$\tilde{\mu}_1$=(0.3234,0.3366,0.0099,0.0099)$_{LR}$
$\tilde{\eta}_2$=(0.00027,0.00033,0.00004,0.00004)$_{LR}$	$\tilde{\mu}_2$=(0.49,0.51,0.015,0.015)$_{LR}$
$\tilde{\eta}_3$=(0.00009,0.00011,0.00001,0.00002)$_{LR}$	$\tilde{\mu}_3$=(0.6566,0.6834,0.0201,0.0201)$_{LR}$
$\tilde{\eta}_4$=(0.00027,0.00033,0.00004,0.00004)$_{LR}$	$\tilde{\mu}_4$=(0.0343,0.0357,0.00105,0.00105)$_{LR}$
$\tilde{\eta}_5$=(0.00009,0.00011,0.00001,0.00002)$_{LR}$	$\tilde{\mu}_5$=(2.9694,3.0906,0.0909,0.0909)$_{LR}$
$\tilde{\eta}_6$=(0.01872,0.02288,0.00312,0.00312)$_{LR}$	$\tilde{\mu}_6$=(0.21756,0.22644,0.00666,0.00666)$_{LR}$
$\tilde{\eta}_7$=(0.0036,0.0044,0.0006,0.0006)$_{LR}$	$\tilde{\mu}_7$=(0.1225,0.1275,0.00375,0.00375)$_{LR}$

Table 2. Fuzzy Failure Rates and Fuzzy Repair Rates for the Different Sub-systems of R_2

where, $L(x) = R(x) = max\{0, 1 - x\}$

5.5. **Fuzzy Kolmogorov's Differential Equations.** In this section, fuzzy Kolmogorov's differential equations are developed by using Markov model of sub-system R_1 and sub-system R_2.

Fuzzy Kolmogorov's differential equations for sub-system R_1 associated with Markov model (Figure 1) are:

$\tilde{P}_1^{(1)}(t) \oplus \tilde{\lambda}_1\tilde{P}_1(t) = \tilde{\beta}_1\tilde{P}_5(t) \oplus \tilde{\beta}_2\tilde{P}_6(t) \oplus \tilde{\beta}_3\tilde{P}_7(t) \oplus \tilde{\beta}_4\tilde{P}_8(t) \oplus \tilde{\beta}_5\tilde{P}_2(t) \oplus \tilde{\beta}_6\tilde{P}_3(t) \oplus \tilde{\beta}_7\tilde{P}_{17}(t) \oplus \tilde{\beta}_8\tilde{P}_{18}(t)$

$\tilde{P}_2^{(1)}(t) \oplus \tilde{\lambda}_2\tilde{P}_2(t) = \tilde{\beta}_1\tilde{P}_9(t) \oplus \tilde{\beta}_2\tilde{P}_{10}(t) \oplus \tilde{\beta}_3\tilde{P}_{11}(t) \oplus \tilde{\beta}_4\tilde{P}_{12}(t) \oplus \tilde{\beta}_8\tilde{P}_{20}(t) \oplus \tilde{\chi}_5\tilde{P}_1(t)$
$\tilde{P}_3^{(1)}(t) \oplus \tilde{\lambda}_3\tilde{P}_3(t) = \tilde{\beta}_1\tilde{P}_{13}(t) \oplus \tilde{\beta}_2\tilde{P}_{14}(t) \oplus \tilde{\beta}_3\tilde{P}_{15}(t) \oplus \tilde{\beta}_4\tilde{P}_{16}(t) \oplus \tilde{\beta}_7\tilde{P}_{19}(t) \oplus \tilde{\chi}_6\tilde{P}_1(t)$
$\tilde{P}_4^{(1)}(t) \oplus \tilde{\lambda}_4\tilde{P}_4(t) = \tilde{\beta}_1\tilde{P}_{21}(t) \oplus \tilde{\beta}_2\tilde{P}_{22}(t) \oplus \tilde{\beta}_3\tilde{P}_{23}(t) \oplus \tilde{\beta}_4\tilde{P}_{24}(t) \oplus \tilde{\chi}_5\tilde{P}_3(t) \oplus \tilde{\chi}_6\tilde{P}_2(t)$
$\tilde{P}_{4+i}^{(1)}(t) \oplus \tilde{\beta}_i\tilde{P}_{4+i}(t) = \tilde{\chi}_i\tilde{P}_1(t), i = 1,2,3,4$
$\tilde{P}_{8+i}^{(1)}(t) \oplus \tilde{\beta}_i\tilde{P}_{8+i}(t) = \tilde{\chi}_i\tilde{P}_2(t), i = 1,2,3,4$
$\tilde{P}_{12+i}^{(1)}(t) \oplus \tilde{\beta}_i\tilde{P}_{12+i}(t) = \tilde{\chi}_i\tilde{P}_3(t), i = 1,2,3,4$
$\tilde{P}_{17}^{(1)}(t) \oplus \tilde{\beta}_7\tilde{P}_{17}(t) = \tilde{\chi}_7\tilde{P}_2(t)$
$\tilde{P}_{18}^{(1)}(t) \oplus \tilde{\beta}_8\tilde{P}_{18}(t) = \tilde{\chi}_8\tilde{P}_3(t)$
$\tilde{P}_{19}^{(1)}(t) \oplus \tilde{\beta}_7\tilde{P}_{19}(t) = \tilde{\chi}_7\tilde{P}_4(t)$
$\tilde{P}_{20}^{(1)}(t) \oplus \tilde{\beta}_8\tilde{P}_{20}(t) = \tilde{\chi}_8\tilde{P}_4(t)$
$\tilde{P}_{20+i}^{(1)}(t) \oplus \tilde{\beta}_i\tilde{P}_{20+i}(t) = \tilde{\chi}_i\tilde{P}_4(t), i = 1,2,3,4$
where,
$\tilde{P}_i^{(1)}(t) = \frac{d\tilde{P}_i}{dt}$ for i=1 to 24
$\tilde{\lambda}_1 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_5 \oplus \tilde{\chi}_6$
$\tilde{\lambda}_2 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_6 \oplus \tilde{\chi}_7 \oplus \tilde{\beta}_5$
$\tilde{\lambda}_3 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_5 \oplus \tilde{\chi}_8 \oplus \tilde{\beta}_6$
$\tilde{\lambda}_4 = \tilde{\chi}_1 \oplus \tilde{\chi}_2 \oplus \tilde{\chi}_3 \oplus \tilde{\chi}_4 \oplus \tilde{\chi}_7 \oplus \tilde{\chi}_8$
with initial conditions
$\tilde{P}_1(0)$=$(0.945,0.955,0.005,0.005)_{LR}$,
$\tilde{P}_2(0)$=$(0.0065,0.0075,0.0005,0.0005)_{LR}$,
$\tilde{P}_3(0)$=$(0.0045,0.0055,0.0005,0.0005)_{LR}$,
$\tilde{P}_4(0)$=$(0.0025,0.0035,0.0005,0.0005)_{LR}$ and
$\tilde{P}_j(0)$=$(0,0,0,0)_{LR}$, j=5 to 24.

Fuzzy Kolmogorov's differential equations for sub-system R_2 associated with Markov model (Figure 2) are:
$\tilde{P}_1^{(1)}(t) \oplus \tilde{\delta}_1\tilde{P}_1(t) = \tilde{\mu}_1\tilde{P}_3(t) \oplus \tilde{\mu}_2\tilde{P}_4(t) \oplus \tilde{\mu}_3\tilde{P}_5(t) \oplus \tilde{\mu}_4\tilde{P}_6(t) \oplus \tilde{\mu}_5\tilde{P}_7(t) \oplus \tilde{\mu}_6\tilde{P}_2(t) \oplus \tilde{\mu}_7\tilde{P}_{13}(t)$
$\tilde{P}_2^{(1)}(t) \oplus \tilde{\delta}_2\tilde{P}_2(t) = \tilde{\mu}_1\tilde{P}_8(t) \oplus \tilde{\mu}_2\tilde{P}_9(t) \oplus \tilde{\mu}_3\tilde{P}_{10}(t) \oplus \tilde{\mu}_4\tilde{P}_{11}(t) \oplus \tilde{\mu}_5\tilde{P}_{12}(t) \oplus \tilde{\eta}_6\tilde{P}_1(t)$
$\tilde{P}_{2+i}^{(1)}(t) \oplus \tilde{\mu}_i\tilde{P}_{2+i}(t) = \tilde{\eta}_i\tilde{P}_1(t), i = 1,2,3,4,5$
$\tilde{P}_{7+i}^{(1)}(t) \oplus \tilde{\mu}_i\tilde{P}_{7+i}(t) = \tilde{\eta}_i\tilde{P}_2(t), i = 1,2,3,4,5$
$\tilde{P}_{13}^{(1)}(t) \oplus \tilde{\mu}_7\tilde{P}_{13}(t) = \tilde{\eta}_7\tilde{P}_2(t)$
where,
$\tilde{P}_i^{(1)}(t) = \frac{d\tilde{P}_i}{dt}$ for i=1 to 13
$\tilde{\delta}_1 = \tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3 \oplus \tilde{\eta}_4 \oplus \tilde{\eta}_5 \oplus \tilde{\eta}_6$
$\tilde{\delta}_2 = \tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3 \oplus \tilde{\eta}_4 \oplus \tilde{\eta}_5 \oplus \tilde{\eta}_7 \oplus \tilde{\mu}_6$
with initial conditions
$\tilde{P}_1(0)$=$(0.955,0.965,0.005,0.005)_{LR}$,
$\tilde{P}_2(0)$=$(0.0045,0.0055,0.0005,0.0005)_{LR}$
and $\tilde{P}_j(0)$=$(0,0,0,0)_{LR}$, j=3 to 13.

5.6. **Solution of Fuzzy Kolmogorov's Differential Equations.** The solution of fuzzy Kolmogorov's differential equations of sub-system R_1 and sub-system R_2 is obtained by applying the existing method [15] for α = 0, 0.2, 0.4, 0.6, 0.8, 1 at $t = 360$ hours and the attained solution is illustrated in Table 3 and Table 4 respectively.

	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
j	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$
1	0.525421	0.396102	0.515146	0.401844	0.504872	0.407586	0.494597	0.413328	0.484323	0.419071	0.474049	0.424813
2	0.055996	0.065514	0.05667	0.065057	0.057344	0.064601	0.058018	0.064144	0.058692	0.063687	0.059366	0.063231
3	0.032869	0.036752	0.033158	0.036576	0.033447	0.036401	0.033737	0.036226	0.034026	0.036051	0.034316	0.035876
4	0.306812	0.424067	0.315888	0.419049	0.324964	0.414032	0.33404	0.409015	0.343116	0.403998	0.352192	0.398981
5	0.000537	0.000611	0.000542	0.000607	0.000547	0.000604	0.000553	0.000601	0.000558	0.000597	0.000564	0.000594
6	0.005867	0.006627	0.005923	0.006592	0.00598	0.006557	0.006036	0.006522	0.006093	0.006487	0.00615	0.006453
7	0.000747	0.000849	0.000754	0.000844	0.000761	0.000839	0.000769	0.000834	0.000776	0.000829	0.000784	0.000825
8	0.001307	0.001485	0.00132	0.001476	0.001333	0.001468	0.001346	0.001459	0.001359	0.001451	0.001372	0.001443
9	0.000057	0.000101	0.000059	0.000098	0.000062	0.000095	0.000065	0.000093	0.000068	0.000091	0.000071	0.000088
10	0.000623	0.001093	0.000651	0.001065	0.000681	0.001038	0.000709	0.001011	0.000738	0.000984	0.000767	0.000957
11	0.000079	0.00014	0.000082	0.000136	0.000086	0.000133	0.00009	0.000129	0.000094	0.000126	0.000098	0.000122
12	0.000139	0.000245	0.000145	0.000238	0.000151	0.000232	0.000158	0.000226	0.000164	0.00022	0.000171	0.000214
13	0.000033	0.000056	0.000034	0.000054	0.000036	0.000053	0.000037	0.000052	0.000039	0.000051	0.000041	0.00005
14	0.000366	0.000614	0.000381	0.0006	0.000397	0.000586	0.000412	0.000572	0.000428	0.000558	0.000444	0.000544
15	0.000046	0.000078	0.000048	0.000076	0.00005	0.000074	0.000052	0.000072	0.000054	0.000071	0.000056	0.000069
16	0.000081	0.000137	0.000084	0.000134	0.000088	0.000131	0.000091	0.000127	0.000095	0.000124	0.000099	0.000121
17	0.000679	0.001193	0.00071	0.001163	0.000741	0.001133	0.000773	0.001103	0.000804	0.001073	0.000836	0.001044
18	0.00117	0.001967	0.00122	0.001921	0.00127	0.001876	0.00132	0.001831	0.00137	0.001786	0.00142	0.001741
19	0.003628	0.007661	0.003876	0.007431	0.004125	0.007201	0.004373	0.006972	0.004622	0.006742	0.004871	0.006513
20	0.010731	0.022557	0.01146	0.021884	0.01219	0.021211	0.012919	0.020538	0.013649	0.019865	0.014379	0.019193
21	0.000313	0.000654	0.000334	0.000634	0.000355	0.000615	0.000376	0.000595	0.000397	0.000576	0.000419	0.000557
22	0.003293	0.006991	0.00352	0.00678	0.003748	0.006569	0.003975	0.006358	0.004203	0.006147	0.004431	0.005936
23	0.000435	0.000908	0.000464	0.000881	0.000493	0.000853	0.000522	0.000826	0.000551	0.000799	0.000581	0.000772
24	0.000759	0.001587	0.00081	0.00154	0.000861	0.001493	0.000912	0.001446	0.000963	0.001399	0.001015	0.001352

Table 3. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-System R_1 Obtained by Applying the Existing Method [15]

It is clear from Table 3 and Table 4 that for $j = 1$, $p_{j1}(t, \alpha) > p_{j2}(t, \alpha)$, i.e., $[p_{j1}(t, \alpha), p_{j2}(t, \alpha)]$ does not define α-cut of a fuzzy number $\tilde{p}_j(t)$ for $j = 1$. Thus, the solution obtained by using the existing method [15] can not be used to evaluate fuzzy reliability of piston manufacturing system.

Remark 5.1. In section 5.6, it is shown that the attained solution by applying the existing method [15] can not be used to evaluate fuzzy reliability of piston manufacturing system. Similarly, numerous real life problems can exist wherein the solution obtained with the help of existing method [15] is not applicable.

6. Proposed Method with *JMD* *LR* Flat Fuzzy Numbers

Kumar et al. [42] proposed *JMD LR* flat fuzzy number and proved that it is better to use the proposed representation of *LR* flat fuzzy numbers instead of existing representation of *LR* flat fuzzy numbers for finding the fuzzy optimal solution of fully fuzzy project crashing problems.

	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
j	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$	$p_{j1}(t,\alpha)$	$p_{j2}(t,\alpha)$
1	0.87807	0.862585	0.876803	0.86353	0.875536	0.864525	0.87427	0.86552	0.873003	0.866515	0.871737	0.867511
2	0.064038	0.094192	0.065987	0.092551	0.067937	0.09091	0.069887	0.08927	0.071837	0.087629	0.073787	0.085989
3	0.002941	0.004356	0.003032	0.004278	0.003123	0.004201	0.003214	0.004123	0.003305	0.004046	0.003396	0.003969
4	0.000425	0.000624	0.000436	0.000611	0.000447	0.000598	0.000458	0.000586	0.000469	0.000573	0.00048	0.000561
5	0.00011	0.000159	0.000111	0.000155	0.000113	0.000151	0.000115	0.000147	0.000117	0.000143	0.000119	0.000139
6	0.006073	0.008919	0.00623	0.008739	0.006388	0.008559	0.006546	0.008379	0.006704	0.008199	0.006862	0.008019
7	0.000024	0.000035	0.000024	0.000034	0.000024	0.000033	0.000025	0.000032	0.000025	0.000031	0.000026	0.000031
8	0.000228	0.000475	0.000229	0.000459	0.000243	0.000442	0.000257	0.000425	0.000272	0.000409	0.000287	0.000393
9	0.000031	0.000068	0.000033	0.000065	0.000035	0.000062	0.000037	0.00006	0.000039	0.000057	0.000041	0.000055
10	0.000008	0.000017	0.000008	0.000016	0.000009	0.000015	0.000009	0.000014	0.00001	0.000013	0.00001	0.000013
11	0.000442	0.000973	0.000469	0.000937	0.000497	0.000901	0.000525	0.000865	0.000553	0.000829	0.000581	0.000794
12	0.000001	0.000004	0.000001	0.000003	0.000001	0.000003	0.000002	0.000003	0.000002	0.000003	0.000002	0.000003
13	0.001617	0.003588	0.001727	0.003463	0.001837	0.003339	0.001947	0.003215	0.002057	0.003091	0.002168	0.002967

TABLE 4. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-system R_2 Obtained by Applying the Existing Method [15]

In this section, to overcome the drawback of existing method [15], a new analytical method is proposed to find the solution of nth-order fuzzy linear differential equations with the help of *JMD LR* flat fuzzy numbers and the advantage of proposed method over existing method [15] is also discussed. Furthermore, it is illustrated that *JMD LR* flat fuzzy numbers are better as compared to existing *LR* flat fuzzy numbers to find the solution of fuzzy differential equations.

6.1. ***JMD*** **Representation of** LR **Flat Fuzzy Number.** In this section, definitions of JMD LR flat fuzzy number, zero JMD LR flat fuzzy number and equality of JMD LR flat fuzzy numbers are presented [42].

Definition 6.1. Let $(m, n, \alpha^L, \alpha^R)_{LR}$ be an LR flat fuzzy number then its JMD representation is $(x, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$, where $x = m - \alpha^L$, $\alpha^M = n - m$.

Definition 6.2. A JMD LR flat fuzzy number $\tilde{A} = (x, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$ is said to be zero JMD LR flat fuzzy number if and only if $x = 0, \alpha^L = 0, \alpha^M = 0, \alpha^R = 0$.

Definition 6.3. A JMD LR flat fuzzy number $\widetilde{A} = (x, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$ is said to be non-negative JMD LR flat fuzzy number iff $x \geq 0$.

Definition 6.4. Two JMD LR flat fuzzy numbers $\tilde{A} = (x_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD}$ and $\tilde{B} = (x_2, \alpha_2^L, \alpha_2^M, \alpha_2^R)_{LR}^{JMD}$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $x_1 = x_2$, $\alpha_1^L = \alpha_2^L$, $\alpha_1^M = \alpha_1^M$ and $\alpha_1^R = \alpha_2^R$.

6.2. **Arithmetic Operations Between** JMD LR **Flat Fuzzy Numbers.** In this section, arithmetic operations between JMD LR flat fuzzy numbers are presented [42].

Let $\widetilde{A}_1 = (x_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD}$ and $\tilde{A}_2 = (x_2, \alpha_2^L, \alpha_2^M, \alpha_2^R)_{LR}^{JMD}$ be any JMD LR flat fuzzy numbers and $\tilde{A}_3 = (x_3, \alpha_3^L, \alpha_3^M, \alpha_3^R)_{RL}^{JMD}$ be any JMD RL flat fuzzy number. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, \alpha_1^L + \alpha_2^L, \alpha_1^M + \alpha_2^M, \alpha_1^R + \alpha_2^R)_{LR}^{JMD}$

(ii) $\tilde{A}_1 \ominus \tilde{A}_3 = (x_1 - x_3 - \alpha_3^L - \alpha_3^M - \alpha_3^R, \alpha_1^L + \alpha_3^R, \alpha_1^M + \alpha_3^M, \alpha_1^R + \alpha_3^L)_{LR}^{JMD}$

(iii) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-negative, then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (x_1x_2, x_1\alpha_2^L + \alpha_1^L\alpha_2^L + x_2\alpha_1^L, x_1\alpha_2^M + x_2\alpha_1^M + \alpha_1^L\alpha_2^M + \alpha_2^L\alpha_1^M + \alpha_1^M\alpha_2^M, x_1\alpha_2^R + \alpha_1^L\alpha_2^R + \alpha_1^M\alpha_2^R + x_2\alpha_1^R + \alpha_2^L\alpha_1^R + \alpha_2^M\alpha_1^R + \alpha_1^R\alpha_2^R)_{LR}^{JMD}$

(iv) If $\tilde{A}_1$ is non-positive and $\tilde{A}_2$ is non-negative, then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (x_1x_2 + x_1\alpha_2^L + x_1\alpha_2^M + x_1\alpha_2^R, x_2\alpha_1^L + \alpha_1^L\alpha_2^L + \alpha_1^L\alpha_2^M - x_1\alpha_2^R, x_2\alpha_1^M + \alpha_2^L\alpha_1^M - x_1\alpha_2^M - \alpha_1^L\alpha_2^M, x_2\alpha_1^R - x_1\alpha_2^L - \alpha_1^L\alpha_2^L - \alpha_2^L\alpha_1^M)_{LR}^{JMD}$

(v) If $\tilde{A}_1$ is non-negative and $\tilde{A}_2$ is non-positive, then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (x_1x_2 + x_2\alpha_1^L + x_2\alpha_1^M + x_2\alpha_1^R, x_1\alpha_2^L + \alpha_1^L\alpha_2^L + \alpha_2^L\alpha_1^M - x_2\alpha_1^R, x_1\alpha_2^M + \alpha_1^L\alpha_2^M - x_2\alpha_1^M - \alpha_2^L\alpha_1^M, x_1\alpha_2^R - x_2\alpha_1^L - \alpha_1^L\alpha_2^L - \alpha_1^L\alpha_2^M)_{LR}^{JMD}$

(vi) If $\tilde{A}_1$ and $\tilde{A}_2$ both are non-positive, then
$\tilde{A}_1 \odot \tilde{A}_2 \simeq (x_1x_2 + x_1\alpha_2^L + x_1\alpha_2^M + x_2\alpha_1^L + \alpha_1^L\alpha_2^L + \alpha_1^L\alpha_2^M + x_2\alpha_1^M + \alpha_2^L\alpha_1^M + \alpha_1^M\alpha_2^M + x_1\alpha_2^R + \alpha_1^L\alpha_2^R + \alpha_1^M\alpha_2^R + x_2\alpha_1^R + \alpha_1^R\alpha_2^L + \alpha_1^R\alpha_2^M + \alpha_1^R\alpha_2^R, -x_1\alpha_2^R - \alpha_1^L\alpha_2^R - \alpha_1^M\alpha_2^R - x_2\alpha_1^R - \alpha_1^R\alpha_2^L - \alpha_1^R\alpha_2^M - \alpha_1^R\alpha_2^R, -x_1\alpha_2^M - \alpha_1^L\alpha_2^M - x_2\alpha_1^M - \alpha_1^M\alpha_2^L - \alpha_1^M\alpha_2^M, -x_1\alpha_2^L - \alpha_1^L\alpha_2^L - x_2\alpha_1^L)_{LR}^{JMD}$

(vii) $\lambda\tilde{A}_1 = \begin{cases} (\lambda x_1, \lambda\alpha_1^L, \lambda\alpha_1^M, \lambda\alpha_1^R)_{LR}^{JMD} & \lambda \geq 0 \\ (\lambda x_1 + \lambda\alpha_1^L + \lambda\alpha_1^M + \lambda\alpha_1^R, -\lambda\alpha_1^R, -\lambda\alpha_1^M, -\lambda\alpha_1^L)_{RL}^{JMD} & \lambda < 0 \end{cases}$

6.3. **Proposed Method.** In this section, a new analytical method is proposed to solve nth-order fuzzy linear differential equations.

The solution of nth-order fuzzy linear differential equation (1), where $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0$ are LR flat fuzzy numbers, can be obtained by using the following steps:

Step 1: Convert all the parameters of fuzzy differential equation, represented by LR flat fuzzy number $(m, n, \alpha^L, \alpha^R)_{LR}$, into JMD LR flat fuzzy number $(x, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$, where, $x = m - \alpha^L$, $\alpha^M = n - m$. Assuming $\tilde{a}_n = (a_n, \beta_n^L, \beta_n^M, \beta_n^R)_{LR}^{JMD}$, $\tilde{a}_{n-1} = (a_{n-1}, \beta_{n-1}^L, \beta_{n-1}^R, \beta_{n-1}^R)_{LR}^{JMD}$, $\tilde{a}_{n-2} = (a_{n-2}, \beta_{n-2}^L, \beta_{n-2}^M, \beta_{n-2}^R)_{LR}^{JMD}, ...,$ $\tilde{a}_1 = (a_1, \beta_1^L, \beta_1^M, \beta_1^R)_{LR}^{JMD}$, $\tilde{a}_0 = (a_0, \beta_0^L, \beta_0^M, \beta_0^R)_{LR}^{JMD}$, $\tilde{y}^{(n)} = (y_1^{(n)}, (\alpha_1^L)^{(n)}, (\alpha_1^M)^{(n)}, (\alpha_1^R)^{(n)})_{LR}^{JMD}$, $\tilde{y}^{(n-1)} = (y_1^{(n-1)}, (\alpha_1^L)^{(n-1)}, (\alpha_1^M)^{(n-1)}, (\alpha_1^R)^{(n-1)})_{LR}^{JMD}, ..., \tilde{y}^{(1)} = (y_1^{(1)}, (\alpha_1^L)^{(1)}, (\alpha_1^M)^{(1)}, (\alpha_1^R)^{(1)})_{LR}^{JMD}$, $\tilde{y} = (y_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD}$ and $\tilde{\gamma}_0 = (\gamma_0, \zeta_0^L, \zeta_0^M, \zeta_{0(1)}^R)_{LR}^{JMD}$, $\tilde{\gamma}_1 = (\gamma_1, \zeta_1^L, \zeta_1^M, \zeta_1^R)_{LR}^{JMD}, ..., \tilde{\gamma}_{n-1} = (\gamma_{n-1}, \zeta_{n-1}^L, \zeta_{n-1}^M, \zeta_{n-1}^R)_{LR}^{JMD}$ the nth order fuzzy linear differential equation (1), can be written as:

$$(a_n, \beta_n^L, \beta_n^M, \beta_n^R)_{LR}^{JMD} \otimes (y_1^{(n)}, (\alpha_1^L)^{(n)}, (\alpha_1^M)^{(n)}, (\alpha_1^R)^{(n)})_{LR}^{JMD} \oplus (a_{n-1}, \beta_{n-1}^L, \beta_{n-1}^M, \beta_{n-1}^R)_{LR}^{JMD} \otimes (y_1^{(n-1)}, (\alpha_1^L)^{(n-1)}, (\alpha_1^M)^{(n-1)}, (\alpha_1^R)^{(n-1)})_{LR}^{JMD} \oplus \ldots \oplus (a_1, \beta_1^L, \beta_1^M, \beta_1^R)_{LR}^{JMD} \otimes (y_1^{(1)}, (\alpha_1^L)^{(1)}, (\alpha_1^M)^{(1)}, (\alpha_1^R)^{(1)})_{LR}^{JMD} \oplus (a_0, \beta_0^L, \beta_0^M, \beta_0^R)_{LR}^{JMD} \otimes (y_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD} = (g, 0, 0, 0)_{LR}^{JMD} \tag{7}$$

with initial conditions

$(y_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD} = (\gamma_0, \zeta_0^L, \zeta_0^M, \zeta_0^R)_{LR}^{JMD}, (y_1^{(1)}, (\alpha_1^L)^{(1)}, (\alpha_1^M)^{(1)}, (\alpha_1^R)^{(1)})_{LR}^{JMD} = (\gamma_1, \zeta_1^L, \zeta_1^M, \zeta_1^R)_{LR}^{JMD}, ..., (y_1^{(n-1)}, (\alpha_1^L)^{(n-1)}, (\alpha_1^M)^{(n-1)}, (\alpha_1^R)^{(n-1)})_{LR}^{JMD} = (\gamma_{n-1}, \zeta_{n-1}^L, \zeta_{n-1}^M, \zeta_{n-1}^R)_{LR}^{JMD}.$

Step 2: Using Definition 6.4 and section 6.2, equation (7) can be split into four ordinary differential equations.

Step 3: Solve the ordinary differential equations to find the values of y_1, α_1^L, α_1^M and α_1^R.

Step 4: Put the values of y_1, α_1^L, α_1^M and α_1^R in $\tilde{y} = (y_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD}$ to find the solution of nth order fuzzy linear differential equation (1).

Step 5: Convert $\tilde{y} = (y_1, \alpha_1^L, \alpha_1^M, \alpha_1^R)_{LR}^{JMD}$ into $\tilde{y} = (m, n, \alpha_1^L, \alpha_1^R)_{LR}$, where, $m = y_1 + \alpha_1^L$ and $n = y_1 + \alpha_1^L + \alpha_1^M$.

6.4. **Advantage of Proposed Method.** The solution of nth-order fuzzy linear differential equation, obtained by using the existing method [15], does not define α-cut of a fuzzy number whereas the solution of same equation, obtained by using proposed method, always represent α-cut of a fuzzy number.

In this section, the advantage of proposed method over existing method is discussed. By applying the existing method [15], the attained solution of fuzzy Kolmogorov's differential equations is not appropriate whereas the solution of same equations obtained by using proposed method is shown in Table 5 and Table 6.

From Table 5 and Table 6, It is clear that $(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$ defines LR flat fuzzy number $\tilde{p}_j(t)$ for $j = 1$ to 24 and $j = 1$ to 13 respectively.

6.5. **Advantage of *JMD* LR Flat Fuzzy Number Over Existing LR Flat Fuzzy Number.** If the existing LR flat fuzzy number $(y_1, y_2, \alpha^L, \alpha^R)_{LR}$ is used to represent the dependent variable $\tilde{y}$ of fuzzy differential equations then to obtain the feasible solution it is essential to solve ordinary differential equations, obtained in Step 2 of the proposed method, with the restrictions $y_1 \leq y_2, \alpha^L \geq 0$ and $\alpha^R \geq 0$. However, the restrictions $y_1 \leq y_2, \alpha^L \geq 0$ and $\alpha^R \geq 0$ are not possible to apply as discussed in section 4.3. Therefore, the obtained solution is not necessarily a fuzzy number.

If the dependent variable $\tilde{y}$ is represented by JMD LR flat fuzzy number i.e., $\tilde{y} = (y, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$ then the dependent variables of ordinary differential equations, obtained in Step 2 of the proposed method, will be y, α^L, α^M and α^R. Further, if on solving the ordinary differential equations, any of the variables α^L, α^M and α^R is negative then the nth-order fuzzy linear differential equation (1) will be infeasible otherwise the solution of nth-order fuzzy linear differential equation will always represent a fuzzy number $(y, \alpha^L, \alpha^M, \alpha^R)_{LR}^{JMD}$ or $(y + \alpha^L, y + \alpha^L + \alpha^M, \alpha^L, \alpha^R)_{LR}$.

To show the advantage of *JMD* LR flat fuzzy numbers over existing LR flat fuzzy number, the fuzzy Kolmogorov's differential equations are solved by using proposed method with existing LR flat fuzzy number and the attained solution is shown in Table 7 and Table 8.

It is obvious from Table 7 and Table 8 that $p_{j1}(t) > p_{j2}(t)$, $\alpha_j^L(t) < 0$ and $\alpha_j^R(t) < 0$ for $j = 1$ i.e., $(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$ does not define LR flat fuzzy number for $j = 1$.

7. Fuzzy Reliability Evaluation of Piston Manufacturing System

section 5.6 illustrates that the solution of fuzzy Kolmogorov's differential equations, obtained by using the existing method, does not define α-cut of a fuzzy

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t) \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$
1	$(0.724095,0.731181, 0.003816,0.003807)_{LR}$	$(0.655343,0.66104, 0.003178,0.003163)_{LR}$	$(0.606389,0.611232, 0.002773,0.002756)_{LR}$	$(0.571173,0.5754, 0.002482,0.002463)_{LR}$	$(0.54602,0.550247, 0.002482,0.002463)_{LR}$	$(0.527484,0.530833, 0.002063,0.002044)_{LR}$
2	$(0.072092,0.074107, 0.000939,0.000938)_{LR}$	$(0.066938,0.069151, 0.001092,0.001137)_{LR}$	$(0.063193,0.065337, 0.00112,0.001116)_{LR}$	$(0.060453,0.06345, 0.001094,0.000089)_{LR}$	$(0.058498,0.060495, 0.001094,0.001089)_{LR}$	$(0.056988,0.058685, 0.000992,0.000986)_{LR}$
3	$(0.045198,0.046916, 0.000817,0.000825)_{LR}$	$(0.041218,0.042925 0.000866,0.00088)_{LR}$	$(0.038314,0.039853, 0.000828,0.000511)_{LR}$	$(0.036189,0.037551, 0.001959,0.000782)_{LR}$	$(0.034704,0.036066, 0.000766,0.000782)_{LR}$	$(0.033514,0.034591, 0.000645,0.000659)_{LR}$
4	$(0.096893,0.098549, 0.000731,0.000732)_{LR}$	$(0.172501,0.174963, 0.003178,0.001035)_{LR}$	$(0.223285,0.22648, 0.001334,0.001338)_{LR}$	$(0.261463,0.265271, 0.001612,0.001618)_{LR}$	$(0.28731,0.291118, 0.000161,0.001618)_{LR}$	$(0.308887,0.313591, 0.002075,0.002084)_{LR}$
5	$(0.000761,0.000807, 0.000023,0.00023)_{LR}$	$(0.00069,0.000729, 0.000002,0.000021)_{LR}$	$(0.000637,0.00067, 0.000019,0.000019)_{LR}$	$(0.000599,0.000627 0.000017,0.000016)_{LR}$	$(0.000573,0.000601, 0.000017,0.000016)_{LR}$	$(0.000551,0.000573, 0.000014,0.000013)_{LR}$
6	$(0.007585,0.007641, 0.000022,0.000022)_{LR}$	$(0.007452,0.007546, 0.000038,0.000038)_{LR}$	$(0.006907,0.007028, 0.000051,0.000051)_{LR}$	$(0.006478,0.006618, 0.000061,0.000061)_{LR}$	$(0.006158,0.006298, 0.000061,0.000061)_{LR}$	$(0.005941,0.006103, 0.000074,0.000074)_{LR}$
7	$(0.001049,0.0011, 0.000022,0.000022)_{LR}$	$(0.000956,0.00101, 0.000027,0.000027)_{LR}$	$(0.000885,0.000933, 0.000026,0.000026)_{LR}$	$(0.000833,0.000875, 0.000024,0.000024)_{LR}$	$(0.000797,0.000839, 0.000024,0.000024)_{LR}$	$(0.000768,0.000801, 0.000021,0.00002)_{LR}$
8	$(0.001829,0.001915, 0.000027,0.000027)_{LR}$	$(0.001667,0.001813, 0.000038,0.000037)_{LR}$	$(0.001546,0.001737, 0.000041,0.000041)_{LR}$	$(0.001458,0.001684, 0.000041,0.000041)_{LR}$	$(0.001394,0.00162, 0.000041,0.000041)_{LR}$	$(0.001344,0.001616, 0.000037,0.000036)_{LR}$
9	$(0.000076,0.000087, 0.000004,0.000004)_{LR}$	$(0.000073,0.000086, 0.000006,0.000006)_{LR}$	$(0.00007,0.000084, 0.000007,0.000007)_{LR}$	$(0.000068,0.000081, 0.000007,0.000007)_{LR}$	$(0.000065,0.000078, 0.000007,0.000007)_{LR}$	$(0.000063,0.000074 0.000006,0.000006)_{LR}$
10	$(0.000722,0.000733, 0.000003,0.000003)_{LR}$	$(0.000749,0.000773, 0.000009,0.000008)_{LR}$	$(0.000713,0.00075, 0.000014,0.000014)_{LR}$	$(0.000683,0.00073, 0.000018,0.000018)_{LR}$	$(0.000659,0.000706, 0.000018,0.000018)_{LR}$	$(0.000649,0.000711, 0.000026,0.000026)_{LR}$
11	$(0.000105,0.000115, 0.000004,0.000004)_{LR}$	$(0.0001,0.000116, 0.000007,0.000007)_{LR}$	$(0.000096,0.000114, 0.000008,0.000008)_{LR}$	$(0.000093,0.000111, 0.000007,0.000009)_{LR}$	$(0.00009,0.000108, 0.000009,0.000009)_{LR}$	$(0.000088,0.000104, 0.000009,0.000005)_{LR}$
12	$(0.000181,0.000197, 0.000004,0.000004)_{LR}$	$(0.000173,0.000211 0.000009,0.000009)_{LR}$	$(0.000166,0.000224, 0.000012,0.000012)_{LR}$	$(0.000161,0.000237 0.000014,0.000014)_{LR}$	$(0.000156,0.000232 0.000014,0.000014)_{LR}$	$(0.000154,0.000257, 0.000015,0.000015)_{LR}$
13	$(0.000049,0.000058, 0.000004,0.000004)_{LR}$	$(0.000046,0.000057, 0.000005,0.000005)_{LR}$	$(0.000043,0.000053, 0.000005,0.000005)_{LR}$	$(0.000041,0.00005, 0.000005,0.000005)_{LR}$	$(0.000039,0.000048, 0.000005,0.000005)_{LR}$	$(0.000037,0.000044, 0.000004,0.000004)_{LR}$
14	$(0.00046,0.00047, 0.000003,0.000003)_{LR}$	$(0.000464,0.000485, 0.000007,0.000007)_{LR}$	$(0.000436,0.000466, 0.000011,0.000011)_{LR}$	$(0.000413,0.00045, 0.000014,0.000015)_{LR}$	$(0.000394,0.000431, 0.000014,0.000015)_{LR}$	$(0.000385,0.00043, 0.000019,0.00002)_{LR}$
15	$(0.000066,0.000075, 0.000003,0.000003)_{LR}$	$(0.000063,0.000076, 0.000006,0.000006)_{LR}$	$(0.00006,0.000074, 0.000007,0.000007)_{LR}$	$(0.000057,0.00007, 0.000007,0.000007)_{LR}$	$(0.000055,0.000068 0.000007,0.000007)_{LR}$	$(0.000052,0.000062, 0.000006,0.000006)_{LR}$
16	$(0.000115,0.00013, 0.000004,0.000004)_{LR}$	$(0.000109,0.000141, 0.000008,0.000008)_{LR}$	$(0.000103,0.00015, 0.00001,0.00001)_{LR}$	$(0.000099,0.000158, 0.000011,0.000011)_{LR}$	$(0.000095,0.000154, 0.000011,0.000013)_{LR}$	$(0.000092,0.000168, 0.000011,0.000011)_{LR}$
17	$(0.00084,0.000856, 0.000005,0.000005)_{LR}$	$(0.000822,0.000857, 0.000013,0.000013)_{LR}$	$(0.00078,0.000833, 0.00002,0.00002)_{LR}$	$(0.000749,0.000816 0.000026,0.000026)_{LR}$	$(0.000705,0.000772, 0.000026,0.000026)_{LR}$	$(0.000715,0.0008, 0.000036,0.000036)_{LR}$
18	$(0.001613,0.001675, 0.000022,0.000022)_{LR}$	$(0.0015,0.001624, 0.000047,0.000047)_{LR}$	$(0.001412,0.001581, 0.000067,0.000068)_{LR}$	$(0.00135,0.001548, 0.000084,0.000085)_{LR}$	$(0.001295,0.001493, 0.000081,0.000085)_{LR}$	$(0.001273,0.001495, 0.000103,0.000105)_{LR}$
19	$(0.000839,0.000852, 0.000004,0.000013)_{LR}$	$(0.001804,0.001836, 0.00001,0.00001)_{LR}$	$(0.002514,0.00257, 0.000018,0.000019)_{LR}$	$(0.00303,0.003114, 0.000028,0.000028)_{LR}$	$(0.003394,0.003478, 0.000028,0.000028)_{LR}$	$(0.003678,0.003822, 0.00005,0.00005)_{LR}$
20	$(0.002772,0.002827, 0.000019,0.000019)_{LR}$	$(0.00597,0.00729, 0.00045,0.00045)_{LR}$	$(0.007592,0.007819, 0.000077,0.000077)_{LR}$	$(0.009065,0.009398, 0.000114,0.000114)_{LR}$	$(0.0101,0.010433, 0.000114,0.000114)_{LR}$	$(0.010927,0.011477, 0.000196,0.000196)_{LR}$
21	$(0.000099,0.000107, 0.000003,0.000003)_{LR}$	$(0.000177,0.00019, 0.000005,0.000005)_{LR}$	$(0.000233,0.000251, 0.000007,0.000007)_{LR}$	$(0.000274,0.000297, 0.000009,0.000009)_{LR}$	$(0.000302,0.000325, 0.000009,0.000009)_{LR}$	$(0.000325,0.000354, 0.000012,0.000012)_{LR}$
22	$(0.000675,0.000683, 0.000003,0.000003)_{LR}$	$(0.001556,0.001578, 0.000007,0.000007)_{LR}$	$(0.002227,0.002266, 0.000013,0.000013)_{LR}$	$(0.002945,0.003004, 0.000249,0.000019)_{LR}$	$(0.003062,0.003121, 0.000019,0.000019)_{LR}$	$(0.003328,0.003432, 0.000035,0.000035)_{LR}$
23	$(0.000135,0.000143, 0.000003,0.000003)_{LR}$	$(0.000243,0.000259, 0.000006,0.000006)_{LR}$	$(0.000321,0.000344, 0.000008,0.000008)_{LR}$	$(0.000378,0.000407, 0.000011,0.000011)_{LR}$	$(0.000417,0.000446, 0.000011,0.000011)_{LR}$	$(0.000451,0.00049, 0.000016,0.000016)_{LR}$
24	$(0.000228,0.000241, 0.000003,0.000003)_{LR}$	$(0.000418,0.000448, 0.000007,0.000007)_{LR}$	$(0.000556,0.000617, 0.000012,0.000012)_{LR}$	$(0.000657,0.000751, 0.000016,0.000016)_{LR}$	$(0.000725,0.000819, 0.000016,0.000016)_{LR}$	$(0.000784,0.000952, 0.000025,0.000025)_{LR}$

TABLE 5. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-System R_1 Obtained by Using Proposed Method

number. Also, the solution of same equations, obtained by using proposed method with existing LR flat fuzzy number does not define a fuzzy number. Consequently, the obtained solution cannot be used to evaluate fuzzy reliability of piston manufacturing system. However, Table 5 and Table 6 prove that if the same equations are solved by using proposed method with JMD LR flat fuzzy numbers then the obtained solution always represent fuzzy number.

In this section, the solution of fuzzy Kolmogorov's differential equations, obtained by using proposed method, is utilized to evaluate fuzzy reliability of piston manufacturing system.

Using the fuzzy probabilities for sub-system R_1 and sub-system R_2, shown in Table 5 and Table 6, the corresponding fuzzy reliability $\tilde{R}_1(t) = \sum_{j=1}^{4} \tilde{p}_j(t)$,$\tilde{R}_2(t) = \sum_{j=1}^{2} \tilde{p}_j(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \odot \tilde{R}_2(t)$ i.e., $(R_{11}(t), R_{12}(t), \alpha_1^L(t), \alpha_1^R(t))_{LR} = (\sum_{j=1}^{4} p_{j1}(t), \sum_{j=1}^{4} p_{j2}(t),$ $\sum_{j=1}^{4} \alpha_j^L(t), \sum_{j=1}^{4} \alpha_j^R(t))$, $(R_{21}(t), R_{22}(t), \alpha_2^L(t), \alpha_2^R(t))_{LR} = (\sum_{j=1}^{2} p_{j1}(t), \sum_{j=1}^{2} p_{j2}(t), \sum_{j=1}^{2} \alpha_j^L(t),$

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t) \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$
1	(0.883263,0.891744, 0.004381,0.004374)$_{LR}$	(0.882218,0.890005, 0.004039,0.004031)$_{LR}$	(0.881929,0.889381, 0.003845,0.003835)$_{LR}$	(0.881804,0.889083, 0.003732,0.00372)$_{LR}$	(0.881734,0.888916, 0.003663,0.003649)$_{LR}$	(0.881689,0.888812, 0.003619,0.003604)$_{LR}$
2	(0.065113,0.067364, 0.001017,0.001016)$_{LR}$	(0.065329,0.068091, 0.001283,0.001281)$_{LR}$	(0.065457,0.068421, 0.001418,0.001414)$_{LR}$	(0.065522,0.068559, 0.001484,0.001479)$_{LR}$	(0.065554,0.068611, 0.001516,0.00151)$_{LR}$	(0.065567,0.068624, 0.001529,0.001522)$_{LR}$
3	(0.002986,0.00309, 0.000043,0.000043)$_{LR}$	(0.003004,0.003145, 0.000063,0.000063)$_{LR}$	(0.003013,0.003164, 0.000072,0.000072)$_{LR}$	(0.003016,0.003169, 0.000075,0.000075)$_{LR}$	(0.003018,0.003171, 0.000077,0.000076)$_{LR}$	(0.003018,0.00317, 0.000077,0.000076)$_{LR}$
4	(0.000432,0.00045, 0.000007,0.000009)$_{LR}$	(0.000434,0.000456, 0.000009,0.000011)$_{LR}$	(0.000435,0.000457, 0.00001,0.000012)$_{LR}$	(0.000435,0.000457, 0.00001,0.000012)$_{LR}$	(0.000434,0.000455, 0.000009,0.000012)$_{LR}$	(0.000434,0.000455, 0.000009,0.000012)$_{LR}$
5	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$	(0.000111,0.000116, 0.000001,0.000003)$_{LR}$
6	(0.00528,0.005311, 0.000011,0.000013)$_{LR}$	(0.005985,0.006041, 0.000019,0.000024)$_{LR}$	(0.006086,0.006164, 0.000027,0.000034)$_{LR}$	(0.006105,0.006202, 0.000034,0.000004)$_{LR}$	(0.006114,0.006228, 0.000041,0.000051)$_{LR}$	(0.00612,0.00625, 0.000047,0.000059)$_{LR}$
7	(0.000025,0.000026, 0.000001,0.000001)$_{LR}$	(0.0000244,0.0000254, 0.0000004,0.0000006)$_{LR}$	(0.0000244,0.0000254, 0.0000004,0.0000008)$_{LR}$	(0.0000244,0.0000254, 0.0000004,0.0000008)$_{LR}$	(0.0000244,0.0000254, 0.0000004,0.0000006)$_{LR}$	(0.0000243,0.000025, 0.0000003,0.000001)$_{LR}$
8	(0.000221,0.000242 0.000007,0.000007)$_{LR}$	(0.000229,0.000268, 0.000015,0.000015)$_{LR}$	(0.000235,0.000286, 0.000021,0.000021)$_{LR}$	(0.000239,0.000297, 0.000025,0.000025)$_{LR}$	(0.000242,0.000303, 0.000028,0.000028)$_{LR}$	(0.000244,0.000307, 0.00003,0.00003)$_{LR}$
9	(0.000032,0.000035, 0.000001,0.000001)$_{LR}$	(0.000033,0.000039, 0.000002,0.000003)$_{LR}$	(0.000034,0.000042, 0.000003,0.000003)$_{LR}$	(0.000034,0.000042, 0.000003,0.000004)$_{LR}$	(0.000034,0.000043, 0.000003,0.000004)$_{LR}$	(0.000034,0.000043, 0.000003,0.000003)$_{LR}$
10	(0.0000082,0.0000093, 0.0000002,0.0000005)$_{LR}$	(0.0000085,0.00001, 0.0000005,0.000001)$_{LR}$	(0.0000086,0.0000106, 0.0000006,0.000001)$_{LR}$	(0.0000086,0.0000106, 0.0000006,0.000001)$_{LR}$	(0.000009,0.000011, 0.000001,0.000001)$_{LR}$	(0.000009,0.00003, 0.000001,0.000002)$_{LR}$
11	(0.000374,0.000379, 0.000001,0.000002)$_{LR}$	(0.000437,0.000451, 0.000004,0.000005)$_{LR}$	(0.000448,0.000471, 0.000007,0.000009)$_{LR}$	(0.000452,0.000483, 0.00001,0.000012)$_{LR}$	(0.000455,0.000494, 0.000013,0.000016)$_{LR}$	(0.000457,0.000503, 0.000015,0.000019)$_{LR}$
12	(0.0000021,0.0000024, 0.0000001,0.0000002)$_{LR}$	(0.0000011,0.0000015, 0.0000001,0.0000002)$_{LR}$	(0.0000011,0.0000015, 0.0000001,0.0000003)$_{LR}$	(0.0000011,0.0000016, 0.0000001,0.0000003)$_{LR}$	(0.0000011,0.0000016, 0.0000001,0.0000003)$_{LR}$	(0.0000011,0.0000016, 0.0000001,0.0000003)$_{LR}$
13	(0.001642,0.001714, 0.000025,0.000025)$_{LR}$	(0.001676,0.001835, 0.000058,0.000116)$_{LR}$	(0.001707,0.001944, 0.00009,0.00009)$_{LR}$	(0.001736,0.002037, 0.000119,0.000118)$_{LR}$	(0.00176,0.002109, 0.000143,0.000143)$_{LR}$	(0.00178,0.002165, 0.000163,0.000163)$_{LR}$

Table 6. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-System R_2 Obtained by Using Proposed Method

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t) \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$
1	(0.681444,0.635114, −0.038835,−0.034395)$_{LR}$	(0.600728,0.541703, −0.051437,−0.040379)$_{LR}$	(0.547756,0.487123, −0.05586,−0.038837)$_{LR}$	(0.512688,0.454965, −0.056003,−0.035215)$_{LR}$	(0.48945,0.436, −0.054088,−0.031619)$_{LR}$	(0.474049,0.424813, −0.051372,−0.028711)$_{LR}$
2	(0.077829,0.084789, 0.006676,0.003555)$_{LR}$	(0.070633,0.075201, 0.004787,0.002042)$_{LR}$	(0.065921,0.069611, 0.003848,0.00172)$_{LR}$	(0.062802,0.066318, 0.003443,0.001832)$_{LR}$	(0.060736,0.064376, 0.003332,0.002061)$_{LR}$	(0.059366,0.063231, 0.00337,0.002283)$_{LR}$
3	(0.048441,0.052512, 0.00406,0.001976)$_{LR}$	(0.042938,0.045117, 0.002586,0.000749)$_{LR}$	(0.039332,0.040802, 0.001846,0.00047)$_{LR}$	(0.036946,0.038259, 0.001523,0.000542)$_{LR}$	(0.035364,0.036761, 0.001426,0.000708)$_{LR}$	(0.034316,0.035876, 0.001447,0.000876)$_{LR}$
4	(0.125721,0.166747, 0.029559,0.029782)$_{LR}$	(0.213389,0.26945, 0.044066,0.037187)$_{LR}$	(0.271397,0.329903, 0.049446,0.035899)$_{LR}$	(0.309835,0.365552, 0.049984,0.032111)$_{LR}$	(0.335309,0.386578, 0.04816,0.028246)$_{LR}$	(0.352192,0.398981, 0.04538,0.025086)$_{LR}$
5	(0.000813,0.00089, 0.000075,0.000004)$_{LR}$	(0.000716,0.000758, 0.000028,0.000017)$_{LR}$	(0.000652,0.000681, 0.000034,0.000011)$_{LR}$	(0.000611,0.000636, 0.000029,0.000012)$_{LR}$	(0.000583,0.000609, 0.000027,0.000015)$_{LR}$	(0.000564,0.000594, 0.000027,0.000017)$_{LR}$
6	(0.008481,0.009508, 0.000918,0.000593)$_{LR}$	(0.008024,0.008591, 0.00061,0.000242)$_{LR}$	(0.007273,0.007621, 0.000417,0.000126)$_{LR}$	(0.006741,0.007021, 0.000324,0.000115)$_{LR}$	(0.006385,0.006663, 0.000288,0.000142)$_{LR}$	(0.00615,0.006453, 0.000283,0.000174)$_{LR}$
7	(0.001132,0.001241, 0.000105,0.000055)$_{LR}$	(0.000996,0.001056, 0.000067,0.000023)$_{LR}$	(0.000907,0.000948, 0.000048,0.000015)$_{LR}$	(0.000849,0.000884, 0.00004,0.000017)$_{LR}$	(0.00081,0.000847, 0.000037,0.00002)$_{LR}$	(0.000784,0.000825, 0.000037,0.000024)$_{LR}$
8	(0.001987,0.002179, 0.000185,0.0001)$_{LR}$	(0.001747,0.001852, 0.000118,0.000041)$_{LR}$	(0.001591,0.001661, 0.000086,0.000026)$_{LR}$	(0.001486,0.001548, 0.000069,0.000028)$_{LR}$	(0.001417,0.001482, 0.000064,0.000035)$_{LR}$	(0.001372,0.001443, 0.000065,0.000042)$_{LR}$
9	(0.000092,0.000118, 0.00002,0.000018)$_{LR}$	(0.000084,0.000105, 0.000017,0.000014)$_{LR}$	(0.000078,0.000097, 0.000015,0.000013)$_{LR}$	(0.000074,0.000092, 0.000013,0.000013)$_{LR}$	(0.000072,0.00009, 0.000014,0.000012)$_{LR}$	(0.000071,0.000088, 0.000014,0.000013)$_{LR}$
10	(0.000932,0.001221, 0.000213,0.000205)$_{LR}$	(0.000931,0.001173, 0.000191,0.000162)$_{LR}$	(0.000867,0.001077, 0.000168,0.000141)$_{LR}$	(0.000819,0.001015, 0.000154,0.000135)$_{LR}$	(0.000788,0.000979, 0.000147,0.000134)$_{LR}$	(0.000767,0.000957, 0.000144,0.000136)$_{LR}$
11	(0.000129,0.0001651, 0.000028,0.000024)$_{LR}$	(0.000117,0.000146, 0.000024,0.00002)$_{LR}$	(0.000109,0.000135, 0.000021,0.000018)$_{LR}$	(0.000103,0.000128, 0.000019,0.000018)$_{LR}$	(0.0001,0.000125, 0.000019,0.000017)$_{LR}$	(0.000098,0.000122, 0.000019,0.000018)$_{LR}$
12	(0.000226,0.00029, 0.000049,0.000044)$_{LR}$	(0.000205,0.000256, 0.000041,0.000035)$_{LR}$	(0.000191,0.000237, 0.000037,0.000031)$_{LR}$	(0.000181,0.000225, 0.000034,0.00003)$_{LR}$	(0.000175,0.000218, 0.000033,0.000031)$_{LR}$	(0.000171,0.000214, 0.000032,0.000031)$_{LR}$
13	(0.000057,0.000073, 0.000012,0.000011)$_{LR}$	(0.000051,0.000063, 0.00001,0.000008)$_{LR}$	(0.000046,0.000057, 0.000008,0.000006)$_{LR}$	(0.000044,0.000053, 0.000008,0.000006)$_{LR}$	(0.000042,0.000051, 0.000008,0.000006)$_{LR}$	(0.000041,0.00005, 0.000008,0.000006)$_{LR}$
14	(0.000592,0.000772, 0.000135,0.000129)$_{LR}$	(0.000571,0.000712, 0.000114,0.000092)$_{LR}$	(0.000521,0.000636, 0.000096,0.000075)$_{LR}$	(0.000485,0.000589, 0.000086,0.000069)$_{LR}$	(0.000461,0.000561, 0.000081,0.000069)$_{LR}$	(0.000444,0.000544, 0.000078,0.00007)$_{LR}$
15	(0.00008,0.000102, 0.000017,0.000015)$_{LR}$	(0.000071,0.000087, 0.000014,0.000011)$_{LR}$	(0.000065,0.000079, 0.000012,0.000009)$_{LR}$	(0.000061,0.000074, 0.000011,0.000009)$_{LR}$	(0.000058,0.000071, 0.00001,0.000009)$_{LR}$	(0.000056,0.000069, 0.00001,0.000009)$_{LR}$
16	(0.000141,0.00018, 0.00003,0.000026)$_{LR}$	(0.000124,0.000154, 0.000023,0.000019)$_{LR}$	(0.000114,0.000139, 0.000021,0.000016)$_{LR}$	(0.000107,0.00013, 0.000019,0.000015)$_{LR}$	(0.000102,0.000124, 0.000018,0.000017)$_{LR}$	(0.000099,0.000121, 0.000018,0.000016)$_{LR}$
17	(0.001076,0.001398, 0.000241,0.000228)$_{LR}$	(0.001014,0.001274, 0.000205,0.000173)$_{LR}$	(0.000941,0.001167, 0.000181,0.000153)$_{LR}$	(0.000891,0.001104, 0.000168,0.000147)$_{LR}$	(0.000858,0.001066, 0.000161,0.000147)$_{LR}$	(0.000836,0.001044, 0.000157,0.000149)$_{LR}$
18	(0.002036,0.002617, 0.000445,0.0004)$_{LR}$	(0.001806,0.002237, 0.000353,0.000278)$_{LR}$	(0.001644,0.002005, 0.000299,0.000233)$_{LR}$	(0.001537,0.001869, 0.000271,0.000221)$_{LR}$	(0.001467,0.001788, 0.000256,0.000222)$_{LR}$	(0.00142,0.001741, 0.00025,0.000226)$_{LR}$
19	(0.001296,0.002071, 0.000461,0.00067)$_{LR}$	(0.002665,0.004017, 0.000871,0.001086)$_{LR}$	(0.003587,0.005182, 0.000641,0.00121)$_{LR}$	(0.004198,0.005869, 0.001196,0.001217)$_{LR}$	(0.004603,0.006274, 0.001237,0.001187)$_{LR}$	(0.004871,0.006513, 0.001243,0.001148)$_{LR}$
20	(0.004246,0.006735, 0.001493,0.002136)$_{LR}$	(0.008165,0.012241, 0.002645,0.003251)$_{LR}$	(0.010762,0.015486, 0.003247,0.003566)$_{LR}$	(0.012483,0.017399, 0.003532,0.00357)$_{LR}$	(0.013623,0.018527, 0.003637,0.003474)$_{LR}$	(0.014379,0.019193, 0.003648,0.003364)$_{LR}$
21	(0.000147,0.00023, 0.000051,0.000069)$_{LR}$	(0.000252,0.000375, 0.00008,0.000096)$_{LR}$	(0.000322,0.00046, 0.000096,0.000103)$_{LR}$	(0.000368,0.00051, 0.000103,0.000103)$_{LR}$	(0.000398,0.00054, 0.000105,0.000099)$_{LR}$	(0.000419,0.000557, 0.000106,0.000097)$_{LR}$
22	(0.001047,0.001684, 0.000375,0.000556)$_{LR}$	(0.002312,0.003506, 0.000763,0.000966)$_{LR}$	(0.003194,0.004636, 0.00098,0.001101)$_{LR}$	(0.003782,0.005307, 0.001086,0.001115)$_{LR}$	(0.004172,0.005703, 0.001129,0.001089)$_{LR}$	(0.004431,0.005936, 0.001138,0.001055)$_{LR}$
23	(0.000201,0.000315, 0.000069,0.000096)$_{LR}$	(0.000348,0.000518, 0.000111,0.000133)$_{LR}$	(0.000445,0.000637, 0.000132,0.000144)$_{LR}$	(0.00051,0.000707, 0.000143,0.000144)$_{LR}$	(0.000553,0.000749, 0.000147,0.000139)$_{LR}$	(0.000581,0.000773, 0.000146,0.000135)$_{LR}$
24	(0.000344,0.000539, 0.000119,0.000165)$_{LR}$	(0.000604,0.000898, 0.000193,0.000413)$_{LR}$	(0.000776,0.00111, 0.000232,0.000251)$_{LR}$	(0.000889,0.001235, 0.000248,0.000249)$_{LR}$	(0.000965,0.001308, 0.000256,0.000243)$_{LR}$	(0.001015,0.001352, 0.000256,0.000235)$_{LR}$

Table 7. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-System R_1 Obtained by Using Proposed Method with Existing Representation of LR Flat Fuzzy Numbers

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t) \alpha_j^L(t), \alpha_j^R(t))_{LR}$	$(p_{j1}(t), p_{j2}(t), \alpha_j^L(t), \alpha_j^R(t))_{LR}$
1	$(0.872595,0.868478,-0.006287,-0.004941)_{LR}$	$(0.871845,0.867666,-0.006333,-0.004979)_{LR}$	$(0.871751,0.867573,-0.006333,-0.004977)_{LR}$	$(0.871739,0.867563,-0.006333,-0.004977)_{LR}$	$(0.871737,0.867561,-0.006334,-0.004976)_{LR}$	$(0.871737,0.867561,-0.006334,-0.004976)_{LR}$
2	$(0.073858,0.086077,0.009762,0.008216)_{LR}$	$(0.073796,0.085999,0.00975,0.008204)_{LR}$	$(0.073788,0.085991,0.009749,0.008202)_{LR}$	$(0.073788,0.085989,0.00975,0.008203)_{LR}$	$(0.073787,0.085989,0.009749,0.008023)_{LR}$	$(0.073787,0.085989,0.009749,0.008023)_{LR}$
3	$(0.0034,0.003973,0.000457,0.000388)_{LR}$	$(0.003396,0.003969,0.000455,0.000388)_{LR}$	$(0.003396,0.003969,0.000455,0.000387)_{LR}$	$(0.003396,0.003969,0.000455,0.000387)_{LR}$	$(0.003396,0.003969,0.000455,0.000387)_{LR}$	$(0.003396,0.003969,0.000455,0.000387)_{LR}$
4	$(0.000481,0.000562,0.000056,0.000063)_{LR}$	$(0.00048,0.000561,0.000055,0.000063)_{LR}$	$(0.00048,0.000561,0.000055,0.000063)_{LR}$	$(0.00048,0.000561,0.000055,0.000063)_{LR}$	$(0.00048,0.000561,0.000055,0.000063)_{LR}$	$(0.00048,0.000561,0.000055,0.000063)_{LR}$
5	$(0.000119,0.000139,0.000009,0.00002)_{LR}$	$(0.000119,0.000139,0.000009,0.00002)_{LR}$	$(0.000119,0.000139,0.000009,0.00002)_{LR}$	$(0.000119,0.000139,0.000009,0.00002)_{LR}$	$(0.000119,0.000139,0.000009,0.00002)_{LR}$	$(0.000119,0.000139,0.000009,0.00002)_{LR}$
6	$(0.006013,0.007112,0.000644,0.000865)_{LR}$	$(0.006755,0.007915,0.000789,0.000902)_{LR}$	$(0.006848,0.008007,0.000789,0.000901)_{LR}$	$(0.00686,0.008018,0.000789,0.0009)_{LR}$	$(0.006861,0.008019,0.000788,0.0009)_{LR}$	$(0.006862,0.008019,0.000789,0.0009)_{LR}$
7	$(0.000026,0.000031,0.000002,0.000004)_{LR}$	$(0.000026,0.000031,0.000002,0.000004)_{LR}$	$(0.000026,0.000031,0.000002,0.000004)_{LR}$	$(0.000026,0.000031,0.000002,0.000004)_{LR}$	$(0.000026,0.000031,0.000002,0.000004)_{LR}$	$(0.000026,0.000031,0.000002,0.000004)_{LR}$
8	$(0.000287,0.000393,0.000073,0.000083)_{LR}$	$(0.000287,0.000393,0.000073,0.000082)_{LR}$	$(0.000287,0.000393,0.000073,0.000082)_{LR}$	$(0.000287,0.000393,0.000073,0.000082)_{LR}$	$(0.000287,0.000393,0.000073,0.000082)_{LR}$	$(0.000287,0.000393,0.000073,0.000082)_{LR}$
9	$(0.000041,0.000055,0.00001,0.000013)_{LR}$	$(0.000041,0.000055,0.00001,0.000013)_{LR}$	$(0.000041,0.000055,0.00001,0.000013)_{LR}$	$(0.000041,0.000055,0.00001,0.000013)_{LR}$	$(0.000041,0.000055,0.00001,0.000013)_{LR}$	$(0.000041,0.000055,0.00001,0.000013)_{LR}$
10	$(0.00001,0.000013,0.000002,0.000004)_{LR}$	$(0.00001,0.000013,0.000002,0.000004)_{LR}$	$(0.00001,0.000013,0.000002,0.000004)_{LR}$	$(0.00001,0.000013,0.000002,0.000004)_{LR}$	$(0.00001,0.000013,0.000002,0.000004)_{LR}$	$(0.00001,0.000013,0.000002,0.000004)_{LR}$
11	$(0.000496,0.000689,0.000123,0.000163)_{LR}$	$(0.00057,0.000782,0.000137,0.000179)_{LR}$	$(0.000579,0.000793,0.000138,0.000179)_{LR}$	$(0.000581,0.000794,0.000139,0.000179)_{LR}$	$(0.000581,0.000794,0.000139,0.000179)_{LR}$	$(0.000581,0.000794,0.000139,0.000179)_{LR}$
12	$(0.000002,0.000003,0.000001,0.000001)_{LR}$	$(0.000002,0.000003,0.000001,0.000001)_{LR}$	$(0.000002,0.000003,0.000001,0.000001)_{LR}$	$(0.000002,0.000003,0.000001,0.000001)_{LR}$	$(0.000002,0.000003,0.000001,0.000001)_{LR}$	$(0.000002,0.000003,0.000001,0.000001)_{LR}$
13	$(0.002168,0.002969,0.000551,0.000622)_{LR}$	$(0.002168,0.002967,0.00055,0.000621)_{LR}$	$(0.002168,0.002967,0.00055,0.000621)_{LR}$	$(0.002168,0.002967,0.00055,0.000621)_{LR}$	$(0.002168,0.002967,0.00055,0.000621)_{LR}$	$(0.002168,0.002967,0.00055,0.000621)_{LR}$

Table 8. Solution of Fuzzy Kolmogorov's Differential Equations for Sub-System R_2 Obtained by Using Proposed Method with Existing Representation of LR Flat Fuzzy Numbers

$\sum_{j=1}^{2} \alpha_j^R(t))$ and $(R_1(t), R_2(t), \alpha^L(t), \alpha^R(t))_{LR} = (R_{11}(t), R_{12}(t), \alpha_1^L(t), \alpha_1^R(t))_{LR} \odot (R_{21}(t),$ $R_{22}(t), \alpha_2^L(t), \alpha_2^R(t))_{LR}$, of sub-system R_1, sub-system R_2 and whole system are computed respectively and are shown in Table 9.

Fuzzy Reliability $\longrightarrow$	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	$\tilde{R}(t)$
Time $(t) \downarrow$	$(R_{11}(t), R_{12}(t), \alpha_1^L(t), \alpha_1^R(t))_{LR}$	$(R_{21}(t), R_{22}(t), \alpha_2^L(t), \alpha_2^R(t))_{LR}$	$(R_1(t), R_2(t), \alpha^L(t), \alpha^R(t))_{LR}$
60	$(0.938278,0.950753,0.006303,0.006302)_{LR}$	$(0.948376,0.959108,0.005398,0.00539)_{LR}$	$(0.88984,0.911874,0.011009,0.011203)_{LR}$
120	$(0.936,0.948079,0.008314,0.006167)_{LR}$	$(0.947547,0.958096,0.005323,0.005312)_{LR}$	$(0.886903,0.908351,0.012815,0.010977)_{LR}$
180	$(0.931181,0.942902,0.006055,0.006054)_{LR}$	$(0.947386,0.957802,0.005263,0.005149)_{LR}$	$(0.882187,0.903113,0.010605,0.010685)_{LR}$
240	$(0.929278,0.940672,0.005894,0.005952)_{LR}$	$(0.947326,0.957642,0.005216,0.005199)_{LR}$	$(0.880329,0.900827,0.0104,0.010621)_{LR}$
300	$(0.926532,0.937926,0.004503,0.005952)_{LR}$	$(0.947288,0.957527,0.005179,0.005159)_{LR}$	$(0.877692,0.898089,0.009041,0.010569)_{LR}$
360	$(0.926873,0.9377,0.005775,0.005673)_{LR}$	$(0.947256,0.957436,0.005148,0.005126)_{LR}$	$(0.877986,0.897787,0.010213,0.010268)_{LR}$

Table 9. Fuzzy Reliability of Sub-System R_1, Sub-System R_2 and Whole System Obtained by Using Proposed Method

The variation in reliability of sub-system R_1, sub-system R_2 and the whole system corresponding to variation in time is depicted in Figure 3 to Figure 5 respectively.

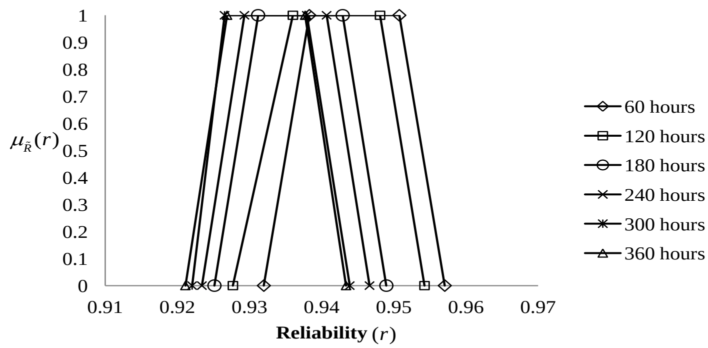

FIGURE 3. *LR* Flat Fuzzy Number Representing Fuzzy Reliability of Sub-System R_1

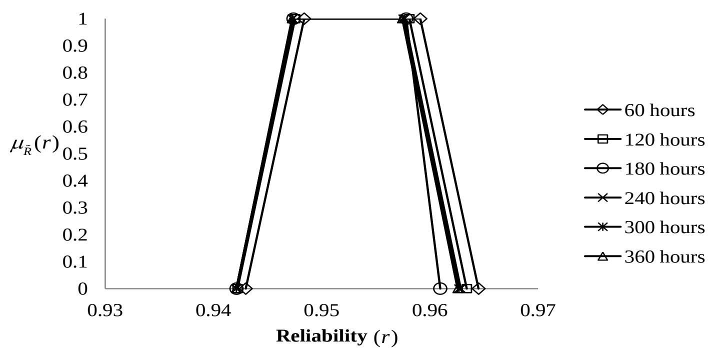

FIGURE 4. *LR* Flat Fuzzy Number Representing Fuzzy Reliability of Sub-System R_2

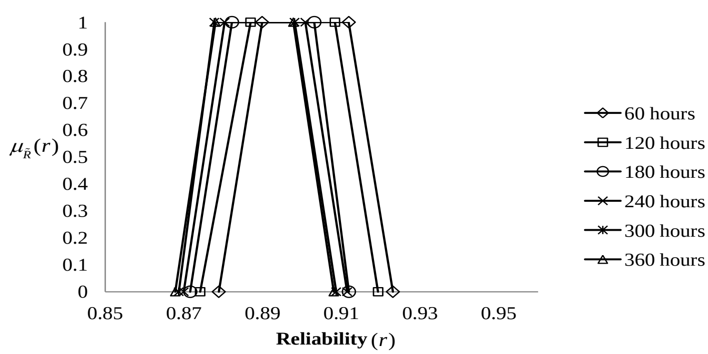

FIGURE 5. *LR* Flat Fuzzy Number Representing Fuzzy Reliability of Whole System

8. **Conclusion**

On the basis of the proposed study, it can be concluded that it is better to use the proposed method instead of existing method [15] for solving nth-order fuzzy linear differential equations and it is better to use JMD LR flat fuzzy number as compared to existing LR flat fuzzy number to solve nth-order fuzzy linear differential equations.

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