BILEVEL LINEAR PROGRAMMING WITH FUZZY PARAMETERS

F. HAMIDI AND H. MISHMAST NEHI

Abstract. Bilevel linear programming is a decision making problem with a two-level decentralized organization. The "leader" is in the upper level and the "follower", in the lower. Making a decision at one level affects that at the other one. In this paper, bilevel linear programming with inexact parameters has been studied and a method is proposed to solve a fuzzy bilevel linear programming using interval bilevel linear programming.

1. Introduction

Bilevel linear programming (BLP), first introduced by Stackelberg [22], is a nested optimization problem including a leader and a follower problem. Making a decision at one level, affects the objective function and the decision space of the other and the order is from top to bottom. "Kth-best" [6], "Branch and bound" [4] and "Complementary pivot" [13] are among the most important methods presented for the solution of this problem. In real cases, the parameters of an optimization problem may not be exact, i.e. they may be interval or fuzzy; same is the case with BLP problems. It is worth mentioning that when the parameters of a Fuzzy Bilevel Linear Programming (FBLP) problem are fuzzy quantities, the objective functions are fuzzy quantities too. Recently, Calvete et al. [9] have presented two algorithms, similar to the Kth-best, for the calculation of the best and the worst optimal values of the Interval Bilevel Linear Programming (IBLP) objective function. They are based on the ordering of the extreme points. In the IBLP problem, investigated by the above researchers, only the coefficients of the leader and the follower objective functions are interval; the other coefficients are ordinary. These algorithms can be extended to the solution of the IBLP problems wherein all parameters are interval. In this paper, using "λ-cut", an IBLP is obtained from a fuzzy one. The best and the worst optimal values of the objective function will be found first, and then a linear piecewise trapezoidal approximate fuzzy number will be presented for the optimal value of the leader objective function related to the FBLP problem.

2. BLP Foundation

Definitions, important theorems and solution classification of BLP problems are given in this section.

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2.1. **Main Theorems and Definitions.** BLP is a nested optimization model including an upper level decision maker (the leader or the upper level player) and a lower level one (the follower or the lower level player). The leader has control over the $\mathbf{x} \in X \subset R^n$ vector and the follower controls the $\mathbf{y} \in Y \subset R^m$ vector. First, the leader chooses an $\mathbf{x}$ and, probably under some constraints, tries to minimize $F(\mathbf{x}, \mathbf{y})$. The follower, observing the decision made by the leader, chooses a $\mathbf{y}$ and, under some constraints for a specific value of $\mathbf{x}$, minimizes its own objective function, $f(\mathbf{x}, \mathbf{y})$. It is to be noted that the leader's decision affects the objective function and the decision space of the follower's. For $F : X \times Y \longrightarrow R$ and $f : X \times Y \longrightarrow R$, BLP can be written as follows:

$$
\begin{aligned}
\min_{\mathbf{x} \in X} \quad & F(\mathbf{x}, \mathbf{y}) = \mathbf{c}^1\mathbf{x} + \mathbf{d}^1\mathbf{y} \\
s.t. \quad & A^1\mathbf{x} + B^1\mathbf{y} \geq \mathbf{b}^1 \\
& \min_{\mathbf{y} \in Y} \quad f(\mathbf{x}, \mathbf{y}) = \mathbf{c}^2\mathbf{x} + \mathbf{d}^2\mathbf{y} \\
& s.t. \quad\;\; A^2\mathbf{x} + B^2\mathbf{y} \geq \mathbf{b}^2
\end{aligned} \tag{1}
$$

where $\mathbf{c}^1, \mathbf{c}^2 \in R^n$, $\mathbf{d}^1, \mathbf{d}^2 \in R^m$, $\mathbf{b}^1 \in R^p$, $\mathbf{b}^2 \in R^q$, $A^1 \in R^{p \times n}$, $B^1 \in R^{p \times m}$, $A^2 \in R^{q \times n}$, $B^2 \in R^{q \times m}$. For an $\mathbf{x}$, chosen by the leader, $\mathbf{c}^2\mathbf{x}$ in the follower's objective function is a constant value and can be eliminated.

Definition 2.1. [3] a) Constraint region of the BLP:

$$S = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, A^1\mathbf{x} + B^1\mathbf{y} \geq \mathbf{b}^1, A^2\mathbf{x} + B^2\mathbf{y} \geq \mathbf{b}^2\}.$$

b) Feasible set for the follower for each fixed $\mathbf{x} \in X$:

$$S(\mathbf{x}) = \{\mathbf{y} \in Y : B^2\mathbf{y} \geq \mathbf{b}^2 - A^2\mathbf{x}\}.$$

c) Projection of S onto the leader's decision space:

$$S(X) = \{\mathbf{x} \in X : \exists\ \mathbf{y} \in Y, A^1\mathbf{x} + B^1\mathbf{y} \geq \mathbf{b}^1, A^2\mathbf{x} + B^2\mathbf{y} \geq \mathbf{b}^2\}.$$

d) Follower's rational reaction set for $\mathbf{x} \in S(X)$:

$$P(\mathbf{x}) = \{\mathbf{y} \in Y : \mathbf{y} \in \operatorname{argmin}\{f(\mathbf{x}, \widehat{\mathbf{y}}) : \widehat{\mathbf{y}} \in S(\mathbf{x})\}\}.$$

e) Inducible region:

$$IR = \{(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in S, \mathbf{y} \in P(\mathbf{x})\}.$$

In fact, IR is a set over which the leader optimizes. So, a BLP problem can be summarized as follows:

$$\min\{F(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in IR\}. \tag{2}$$

X and Y are usually considered as follows:

$$X = \{\mathbf{x} \in R^n : \mathbf{x} \geq \mathbf{0}\} \quad , \quad Y = \{\mathbf{y} \in R^m : \mathbf{y} \geq \mathbf{0}\}.$$

In order for the BLP to be well-posed, it is assumed that:
1) $\emptyset \neq S$ is compact.
2) $\emptyset \neq P(\mathbf{x})$ is a point-to-point map.

The following theorem shows that IR is formed by faces of S.

Theorem 2.2. *The inducible region can be written equivalently as a piecewise linear equally constraint compromised of supporting hyperplanes of S that is*

$$IR = \{(\mathbf{x}, \mathbf{y}) \in S : Q(\mathbf{x}) - \mathbf{d}^2\mathbf{y} = \mathbf{0}\}$$

where

$$Q(\mathbf{x}) = \min\{\mathbf{d}^2\mathbf{y} : B^2\mathbf{y} \geq \mathbf{b}^2 - A^2\mathbf{x}, \mathbf{y} \in Y\}.$$

Proof. See [2]. □

The next two theorems play important roles in presenting the problem solution algorithms.

Theorem 2.3. *If $(\mathbf{x}, \mathbf{y})$ is an extreme point of IR, then it is an extreme point of S. Also a solution to the BLP occurs at a vertex of IR.*

Proof. See [2]. □

Theorem 2.4. *A necessary condition that $(\mathbf{x}^*, \mathbf{y}^*)$ solves the BLP (1) is that there exist vectors $\mathbf{u}^*$ and $\mathbf{v}^*$ such that $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{u}^*, \mathbf{v}^*)$ solves:*

$$\begin{aligned}
\min \quad & \mathbf{c}^1\mathbf{x} + \mathbf{d}^1\mathbf{y} \\
s.t. \quad & A^1\mathbf{x} + B^1\mathbf{y} \geq \mathbf{b}^1 \\
& \mathbf{u}B^2 + \mathbf{v} = \mathbf{d}^2 \\
& \mathbf{u}(A^2\mathbf{x} + B^2\mathbf{y} - \mathbf{b}^2) + \mathbf{v}\mathbf{y} = \mathbf{0} \\
& A^2\mathbf{x} + B^2\mathbf{y} \geq \mathbf{b}^2 \\
& \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0}.
\end{aligned} \tag{3}$$

Proof. See [2]. □

Remark 2.5. Some BLP characteristics are as follows:
a) The optimal solution to the BLP is not Pareto optimal.
b) The BLP is an NP-hard problem(See [5]).

2.2. **Solution Methods.** BLP solution methods can be generally classified as follows:
1) Vertex enumeration(e.g., See [6]).
2) Penalty function(e.g., See [1]).
3) KKT conditions(e.g., See [4]).
4) Heuristic(e.g., See [14]).

A well-known method, lying in the first class, is the Kth-best the working basis for which is Theorem 2.3 [6].

Let $(\mathbf{x}_{[1]}, \mathbf{y}_{[1]}), \ldots, (\mathbf{x}_{[N]}, \mathbf{y}_{[N]})$ denote the N ordered basic feasible solutions to the linear programming problem

$$\min\ \{\mathbf{c}^1\mathbf{x} + \mathbf{d}^1\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in S\} \tag{4}$$

such that

$$\mathbf{c}^1\mathbf{x}_{[i]} + \mathbf{d}^1\mathbf{y}_{[i]} \leq \mathbf{c}^1\mathbf{x}_{[i+1]} + \mathbf{d}^1\mathbf{y}_{[i+1]},\ i = 1, \ldots, N-1.$$

Then solving (1) is equivalent to finding the index

$$K^* = \min\{i \in \{1, \ldots, N\} : (\mathbf{x}_{[i]}, \mathbf{y}_{[i]}) \in IR\}$$

yielding the global optimum solution $(\mathbf{x}_{[K^*]}, \mathbf{y}_{[K^*]})$. This requires finding the (K^*)th best extreme point solution to problem (4) and may be accomplished with the following algorithm.

Step 1: Put $i \leftarrow 1$. Solve the linear programming problem (4) to obtain the optimal solution $(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})$. Let $W \leftarrow \{(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})\}$ and $T \leftarrow \emptyset$. Go to Step 2.

Step 2: Solve the follower's linear program below to see if the current point is in the rational reaction set $P(\mathbf{x}_{[i]})$:

$$\min \{\mathbf{d}^2\mathbf{y} : \mathbf{y} \in P(\mathbf{x}_{[i]})\}. \tag{5}$$

Let $\widetilde{\mathbf{y}}$ denote the optimal solution to (5). If $\widetilde{\mathbf{y}} = \mathbf{y}_{[i]}$, stop; $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ is global optimum to (1) with $K^* \leftarrow i$. Otherwise, go to Step 3.

Step 3: Let $W_{[i]}$ denote the set of adjacent extreme points of $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ such that $(\mathbf{x}, \mathbf{y}) \in W_{[i]}$ implies $\mathbf{c}^1\mathbf{x} + \mathbf{d}^1\mathbf{y} \geq \mathbf{c}^1\mathbf{x}_{[i]} + \mathbf{d}^1\mathbf{y}_{[i]}$. Let $T \leftarrow T \cup \{(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})\}$ and $W \leftarrow (W \cup W_{[i]}) - T$. Go to Step 4.

Step 4: Set $i \leftarrow i + 1$ and choose $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ so that

$$\mathbf{c}^1\mathbf{x}_{[i]} + \mathbf{d}^1\mathbf{y}_{[i]} \leftarrow \min\{\mathbf{c}^1\mathbf{x} + \mathbf{d}^1\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in W\}.$$

Go to Step 2.

3. Interval Linear Programming(ILP) and IBLP

Here, the ILP problem and the theorem related to the best and the worst values of the objective functions have been studied (other ILP cases may be seen in [18]). This theorem will be proved for the IBLP later.

3.1. **ILP Problem.** Consider the following classical linear programming(LP) problem

$$\begin{aligned} \min \quad & z = \mathbf{c}\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{6}$$

where $A \in R^{m\times n}$, $\mathbf{b} \in R^m$, $\mathbf{x}, \mathbf{c} \in R^n$. ILP is in fact an LP problem wherein all the coefficients are intervals:

$$\begin{aligned} \min \quad & z = [\underline{\mathbf{c}}\ , \overline{\mathbf{c}}]\mathbf{x} \\ s.t. \quad & [\underline{A}\ , \overline{A}]\mathbf{x} \geq [\underline{\mathbf{b}}\ , \overline{\mathbf{b}}] \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{7}$$

The following theorems are used to find the best and the worst optimal values of the ILP problem.

Theorem 3.1. *Consider the interval inequality* $[\underline{A}\ , \overline{A}]\mathbf{x} \geq [\underline{\mathbf{b}}\ , \overline{\mathbf{b}}]$. *Then* $\overline{A}\mathbf{x} \geq \underline{\mathbf{b}}$ *and* $\underline{A}\mathbf{x} \geq \overline{\mathbf{b}}$ *are the maximum and the minimum value range inequalities respectively.*

Proof. See [10]. □

Theorem 3.2. *Let* $z = \mathbf{c}\mathbf{x}$ *be an objective function. Then* $\overline{\mathbf{c}}\mathbf{x} \geq \underline{\mathbf{c}}\mathbf{x}$ *for any given* $\mathbf{x} \geq \mathbf{0}$.

Proof. It is trivial. □

Considering the above theorems, the best and the worst optimal values of the ILP problem are those of the objective functions of the following problems respectively:

$$\begin{aligned} \min \quad & \underline{z} = \underline{\mathbf{c}}\mathbf{x} \\ s.t. \quad & \overline{A}\mathbf{x} \geq \underline{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{8}$$

$$\begin{aligned} \min \quad & \overline{z} = \overline{\mathbf{c}}\mathbf{x} \\ s.t. \quad & \underline{A}\mathbf{x} \geq \overline{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{9}$$

and $\underline{z}^* \leq z^* \leq \overline{z}^*$ where z^* is the optimal solution of the ILP problem (7).

3.2. **IBLP Problem.** In the BLP problem, suppose the coefficients are intervals. Then it can be written in the following form:

$$\begin{aligned} \min_{\mathbf{x}\in X} \quad & F(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{c}}^1, \overline{\mathbf{c}}^1]\mathbf{x} + [\underline{\mathbf{d}}^1, \overline{\mathbf{d}}^1]\mathbf{y} \\ s.t. \quad & [\underline{A}^1, \overline{A}^1]\mathbf{x} + [\underline{B}^1, \overline{B}^1]\mathbf{y} \geq [\underline{\mathbf{b}}^1, \overline{\mathbf{b}}^1] \\ & \min_{\mathbf{y}\in Y} \quad f(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2, \overline{\mathbf{d}}^2]\mathbf{y} \\ & s.t. \quad [\underline{A}^2, \overline{A}^2]\mathbf{x} + [\underline{B}^2, \overline{B}^2]\mathbf{y} \geq [\underline{\mathbf{b}}^2, \overline{\mathbf{b}}^2]. \end{aligned} \tag{10}$$

To solve the IBLP problem where only the coefficients of the objective functions are interval, Calvete et al. [9] have presented two separate algorithms to find the best and the worst optimal values of the leader objective function. These algorithms can be easily extended for the solution of IBLP (10).

Generally, the following theorems are used for the solution of IBLP problem (10).

Theorem 3.3. *The best and the worst optimal values of the leader objective function of the IBLP are the optimal values of the objective functions of the following problems respectively:*

$$\begin{aligned} \min_{\mathbf{x}\in X} \quad & \underline{F}(\mathbf{x},\mathbf{y}) = \underline{\mathbf{c}}^1\mathbf{x} + \underline{\mathbf{d}}^1\mathbf{y} \\ s.t. \quad & \overline{A}^1\mathbf{x} + \overline{B}^1\mathbf{y} \geq \underline{\mathbf{b}}^1 \\ & \min_{\mathbf{y}\in Y} \quad f(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2, \overline{\mathbf{d}}^2]\mathbf{y} \\ & s.t. \quad \overline{A}^2\mathbf{x} + \overline{B}^2\mathbf{y} \geq \underline{\mathbf{b}}^2 \end{aligned} \tag{11}$$

$$\begin{aligned} \min_{\mathbf{x}\in X} \quad & \overline{F}(\mathbf{x},\mathbf{y}) = \overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} \\ s.t. \quad & \underline{A}^1\mathbf{x} + \underline{B}^1\mathbf{y} \geq \overline{\mathbf{b}}^1 \\ & \min_{\mathbf{y}\in Y} \quad f(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2, \overline{\mathbf{d}}^2]\mathbf{y} \\ & s.t. \quad \underline{A}^2\mathbf{x} + \underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2. \end{aligned} \tag{12}$$

Proof. Considering the definition in 2.1, for the $\mathbf{x}$ strategy chosen by the leader, the follower's feasible set is in the following interval form:

$$S_I(\mathbf{x}) = \{\mathbf{y} \in Y : [\underline{B}^2, \overline{B}^2]\mathbf{y} \geq [\underline{\mathbf{b}}^2, \overline{\mathbf{b}}^2] - [\underline{A}^2, \overline{A}^2]\mathbf{x}\}$$

and, according to Theorem 3.1, the maximum and minimum value range regions for the follower's interval problem are respectively:

$$\overline{S}(\mathbf{x}) = \{\mathbf{y} \in Y : \overline{B}^2\mathbf{y} \geq \underline{\mathbf{b}}^2 - \overline{A}^2\mathbf{x}\},$$

$$\underline{S}(\mathbf{x}) = \{\mathbf{y} \in Y : \underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2 - \underline{A}^2\mathbf{x}\}.$$

Moreover, considering the definition of the follower's rational reaction set, $P(\mathbf{x})$:

$$\underline{P}(\mathbf{x}) = \{\mathbf{y} \in Y : \mathbf{y} \in \operatorname{argmin}\{f(\mathbf{x}, \widehat{\mathbf{y}}) : \widehat{\mathbf{y}} \in \overline{S}(\mathbf{x})\}\},$$

$$\overline{P}(\mathbf{x}) = \{\mathbf{y} \in Y : \mathbf{y} \in \operatorname{argmin}\{f(\mathbf{x}, \widehat{\mathbf{y}}) : \widehat{\mathbf{y}} \in \underline{S}(\mathbf{x})\}\}$$

where $f(\mathbf{x}, \widehat{\mathbf{y}}) = [\underline{\mathbf{d}}^2, \overline{\mathbf{d}}^2]\widehat{\mathbf{y}}$. Therefore, the maximum and the minimum value range regions related to the inducible region, are respectively:

$$\overline{IR} = \{(\mathbf{x}, \mathbf{y}) \in \overline{S}, \mathbf{y} \in \underline{P}(\mathbf{x})\},$$

$$\underline{IR} = \{(\mathbf{x}, \mathbf{y}) \in \underline{S}, \mathbf{y} \in \overline{P}(\mathbf{x})\}$$

where

$$\overline{S} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, \overline{A}^1\mathbf{x} + \overline{B}^1\mathbf{y} \geq \underline{\mathbf{b}}^1, \overline{A}^2\mathbf{x} + \overline{B}^2\mathbf{y} \geq \underline{\mathbf{b}}^2\},$$

$$\underline{S} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, \underline{A}^1\mathbf{x} + \underline{B}^1\mathbf{y} \geq \overline{\mathbf{b}}^1, \underline{A}^2\mathbf{x} + \underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2\}.$$

In this case, assuming that $\overline{S}$ and $\underline{S}$ are both non-empty and polyhedron, the best and the worst optimal values of the objective functions of the leader's interval problem are respectively

$$\underline{F}^* = \min\{\underline{F}(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in \overline{IR}\},$$

$$\overline{F}^* = \min\{\overline{F}(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in \underline{IR}\}$$

such that $\underline{F}^* \leq \overline{F}^*$, and the proof is complete. □

Theorem 3.4. *Suppose $I = \{\mathbf{d}^2 \in R^m : \mathbf{d}_i^2 \in [\underline{\mathbf{d}}_i^2, \overline{\mathbf{d}}_i^2], i = 1, \cdots, m\}$ and I_E represents the set of extreme points of? I. If $\overline{IR}$ does not change for any $\mathbf{d}^2 \in I_E$, then the best optimal value of the leader objective function for the IBLP problem will be the optimal value of the objective function of the following problem:*

$$\begin{aligned} \min_{\mathbf{x} \in X} \quad & \underline{F}(\mathbf{x}, \mathbf{y}) = \underline{\mathbf{c}}^1\mathbf{x} + \underline{\mathbf{d}}^1\mathbf{y} \\ s.t. \quad & \overline{A}^1\mathbf{x} + \overline{B}^1\mathbf{y} \geq \underline{\mathbf{b}}^1 \\ & \min_{\mathbf{y} \in Y} \quad \underline{f}(\mathbf{x}, \mathbf{y}) = \underline{\mathbf{d}}^2\mathbf{y} \\ & s.t. \quad \overline{A}^2\mathbf{x} + \overline{B}^2\mathbf{y} \geq \underline{\mathbf{b}}^2. \end{aligned} \tag{13}$$

Similarly, if $\underline{IR}$ does not change for any $\mathbf{d}^2 \in I_E$, then the worst optimal value of the leader objective function for the IBLP problem will be the optimal value of the objective function of the following problem:

$$\begin{aligned} \min_{\mathbf{x} \in X} \quad & \overline{F}(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} \\ s.t. \quad & \underline{A}^1\mathbf{x} + \underline{B}^1\mathbf{y} \geq \overline{\mathbf{b}}^1 \\ & \min_{\mathbf{y} \in Y} \quad \overline{f}(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{d}}^2\mathbf{y} \\ & s.t. \quad \underline{A}^2\mathbf{x} + \underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2. \end{aligned} \tag{14}$$

Proof. Considering theorem 3.3 and theorem 9 in [9], it is obvious. □

Example 3.5. Consider the IBLP below

$$
\begin{aligned}
\min_{x_1 \geq 0} \quad & F(x_1,x_2) = [-1,-0.5]x_2 \\
s.t. \quad & \min_{x_2 \geq 0} \quad f(x_1,x_2) = [1,2]x_2 \\
& s.t. \quad [0.5,1]x_1 + [1.9,2]x_2 \geq [10,10.5] \\
& \qquad\; [-2,-1]x_1 + [1,2]x_2 \geq [-6,-5] \\
& \qquad\; [-3,-2]x_1 + [0.5,1]x_2 \geq [-21,20] \\
& \qquad\; [-2,-1]x_1 + [-3,-2]x_2 \geq [-38,-37] \\
& \qquad\; [0.5,1]x_1 + [-3,-2]x_2 \geq [-18,-17].
\end{aligned} \tag{15}
$$

According to Theorem 3.4, the best and the worst optimal values of the objective function of the IBLP (15), are the optimal values of the objective functions of problems (16) and (17) respectively:

$$
\begin{aligned}
\min_{x_1 \geq 0} \quad & \underline{F}(x_1,x_2) = -x_2 \\
s.t. \quad & \min_{x_2 \geq 0} \quad \underline{f}(x_1,x_2) = x_2 \\
& s.t. \quad x_1 + 2x_2 \geq 10 \\
& \qquad\; -x_1 + 2x_2 \geq -6 \\
& \qquad\; -2x_1 + x_2 \geq -21 \\
& \qquad\; -x_1 - 2x_2 \geq -38 \\
& \qquad\; x_1 - 2x_2 \geq -18
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
\min_{x_1 \geq 0} \quad & \overline{F}(x_1,x_2) = -0.5x_2 \\
s.t. \quad & \min_{x_2 \geq 0} \quad \overline{f}(x_1,x_2) = 2x_2 \\
& s.t. \quad 0.5x_1 + 1.9x_2 \geq 10.5 \\
& \qquad\; -2x_1 + x_2 \geq -5 \\
& \qquad\; -3x_1 + 0.5x_2 \geq -20 \\
& \qquad\; -2x_1 - 3x_2 \geq -37 \\
& \qquad\; 0.5x_1 - 3x_2 \geq -17.
\end{aligned} \tag{17}
$$

The constraint and the inducible regions of the above BLP problems are shown in Figure 1; it is also clear that

$$-11 = \underline{F}^* \leq F^* \leq \overline{F}^* = -3.22.$$

Now, the algorithms presented by Calvete et al. [9], can be extended to the calculation of the best and the worst optimal values of the IBLP objective function wherein all parameters are interval.

3.2.1. **BEST Algorithm for Finding the Best Optimal Value of the IBLP Objective Function.** In this section, the KBB algorithm presented by Calvete et al. is extended for IBLP (10).

Step 1: Put $i \leftarrow 1$. Solve the linear programming problem (18) to obtain the optimal solution $(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})$.

$$\begin{array}{ll} \min & \underline{\mathbf{c}}^1\mathbf{x} + \underline{\mathbf{d}}^1\mathbf{y} \\ s.t. & (\mathbf{x},\mathbf{y}) \in \overline{S} \end{array} \tag{18}$$

where $\overline{S} = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, \overline{A}^1\mathbf{x} + \overline{B}^1\mathbf{y} \geq \underline{\mathbf{b}}^1, \overline{A}^2\mathbf{x} + \overline{B}^2\mathbf{y} \geq \underline{\mathbf{b}}^2\}$.
Let $W \leftarrow \{(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})\}$ and $T \leftarrow \emptyset$. Go to Step 2.

Step 2: Set $(\mathbf{x}_*, \mathbf{y}_*) \leftarrow (\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ and check system (19).

$$\begin{aligned} &\mathbf{u}\overline{B}^2 \leq \mathbf{d}^2 \\ &\mathbf{d}^2\mathbf{y}_* = (\underline{\mathbf{b}}^2 - \overline{A}^2\mathbf{x}_*)\mathbf{u} \\ &\mathbf{d}^2 \in [\underline{\mathbf{d}}^2, \overline{\mathbf{d}}^2],\ \mathbf{u} \geq \mathbf{0}. \end{aligned} \tag{19}$$

If the system is feasible, stop; $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ is the best optimal solution to (10).Otherwise, go to Step 3.

Step 3: Let $W_{[i]}$ denote the set of adjacent extreme points of $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ such that $(\mathbf{x},\mathbf{y}) \in W_{[i]}$ implies $\underline{\mathbf{c}}^1\mathbf{x} + \underline{\mathbf{d}}^1\mathbf{y} \geq \underline{\mathbf{c}}^1\mathbf{x}_{[i]} + \underline{\mathbf{d}}^1\mathbf{y}_{[i]}$. Let $T \leftarrow T \cup \{(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})\}$ and $W \leftarrow (W \cup W_{[i]}) - T$. Go to Step 4.

Step 4: Set $i \leftarrow i + 1$ and choose $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ so that

$$\underline{\mathbf{c}}^1\mathbf{x}_{[i]} + \underline{\mathbf{d}}^1\mathbf{y}_{[i]} \leftarrow \min\{\underline{\mathbf{c}}^1\mathbf{x} + \underline{\mathbf{d}}^1\mathbf{y} : (\mathbf{x},\mathbf{y}) \in W\}.$$

Go to Step 2.

3.2.2. **WORST Algorithm for Finding the Worst Optimal Value of the IBLP Objective Function.** In this section, the KBW algorithm presented by Calvete et al. is extended for IBLP (10).

In the following algorithm, let $I_{(\mathbf{x}_*,\mathbf{y}_*)} = \{\mathbf{d}^2 \in I : \mathbf{v}D = \mathbf{d}^2, \mathbf{v} \geq 0\}$ where $D\mathbf{y} = \mathbf{t}$ is the set of inequalities from $\underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2 - \underline{A}^2\mathbf{x}$ that are binding at $(\mathbf{x}_*, \mathbf{y}_*)$, and $W_{[i]}$ denotes the set of adjacent extreme points of $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ such that $(\mathbf{x},\mathbf{y}) \in W_{[i]}$ implies $\overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} \leq \overline{\mathbf{c}}^1\mathbf{x}_{[i]} + \overline{\mathbf{d}}^1\mathbf{y}_{[i]}$.

Step 1: Put $i \leftarrow 1$. Solve the linear programming problem (20) to obtain the optimal solution $(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})$.

$$\begin{array}{ll} \max & \overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} \\ s.t. & (\mathbf{x},\mathbf{y}) \in \underline{S} \end{array} \tag{20}$$

where $\underline{S} = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, \underline{A}^1\mathbf{x} + \underline{B}^1\mathbf{y} \geq \overline{\mathbf{b}}^1, \underline{A}^2\mathbf{x} + \underline{B}^2\mathbf{y} \geq \overline{\mathbf{b}}^2\}$.
Let $W \leftarrow \{(\mathbf{x}_{[1]}, \mathbf{y}_{[1]})\}$, $W_e \leftarrow \emptyset$, $W_p \leftarrow \emptyset$ and $T \leftarrow \emptyset$. Go to Step 2.

Step 2: Calculate $I_{(\mathbf{x}_{[i]},\mathbf{y}_{[i]})}$. If $I_{(\mathbf{x}_{[i]},\mathbf{y}_{[i]})} = \emptyset$, set $W_e \leftarrow W_e \cup \{(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})\}$. Go to Step 8. Otherwise, set $W_p \leftarrow W_{[i]} - (T \cup W_e)$. If $W_p = \emptyset$, stop; $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ is the worst optimal solution. Otherwise, set $I_* \leftarrow I_{(\mathbf{x}_{[i]},\mathbf{y}_{[i]})}$, go to Step 3.

Step 3: If $W_p = \emptyset$, go to Step 8. Otherwise go to Step 4.

Step 4: Choose $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ so that

$$\overline{\mathbf{c}}^1\widehat{\mathbf{x}} + \overline{\mathbf{d}}^1\widehat{\mathbf{y}} \leftarrow \max\{\overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in W_p\}.$$

Calculate $I_{(\widehat{\mathbf{x}},\widehat{\mathbf{y}})}$. If $I_{(\widehat{\mathbf{x}},\widehat{\mathbf{y}})} = \emptyset$, set $W_e \leftarrow W_e \cup \{(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})\}$, $W_p \leftarrow W_p - \{(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})\}$. Go to Step 3.

Step 5: If $I_* \cap I_{(\widehat{\mathbf{x}},\widehat{\mathbf{y}})} = \emptyset$, set $W_p \leftarrow W_p - \{(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})\}$, go to Step 3. Otherwise, go to Step 6.

Step 6: Set $I_* \leftarrow I_* - I_{(\widehat{\mathbf{x}},\widehat{\mathbf{y}})}$. If $I_* = \emptyset$, set $T \leftarrow T \cup \{(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})\}$, go to Step 8. Otherwise, go to Step7.

Step 7: Set $W_p \leftarrow W_p - \{(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})\}$. If $W_p = \emptyset$, stop; $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ is the worst optimal solution. Otherwise, go to Step 4.

Step 8: Set $W \leftarrow (W \cup W_{[i]}) - (T \cup W_e)$. Set $i \leftarrow i + 1$ and choose $(\mathbf{x}_{[i]}, \mathbf{y}_{[i]})$ so that

$$\overline{\mathbf{c}}^1\mathbf{x}_{[i]} + \overline{\mathbf{d}}^1\mathbf{y}_{[i]} \leftarrow \max\{\overline{\mathbf{c}}^1\mathbf{x} + \overline{\mathbf{d}}^1\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in W\}.$$

Go to Step 2.

Note that since the conditions of Theorem 3.4 are true for the IBLP examples discussed in this paper, it is easier to use this theorem instead of using BEST and WORST algorithms.

4. FBLP Problem

In this section, first a method is proposed for the solution of the fuzzy linear programming (FLP) problem using the ILP. Then, the method is used to solve the FBLP problem. Finally, the drawbacks of Zhang's method [23, 24, 25, 26] for the solution of the FBLP problem are discussed.

4.1. Solving FLP Using ILP. Consider the FLP problem below

$$\begin{aligned} \min \quad & \widetilde{z} = \widetilde{\mathbf{c}}\mathbf{x} \\ s.t. \quad & \widetilde{A}\mathbf{x} \succsim \widetilde{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{21}$$

where $\widetilde{\mathbf{c}} \in \mathscr{F}(R^n)$, $\widetilde{\mathbf{b}} \in \mathscr{F}(R^m)$, $\widetilde{A} = (\widetilde{a}_{ij})_{m\times n}$, $\widetilde{a}_{ij} \in \mathscr{F}(R)$ and the symbol $\succsim$, a fuzzy version of $\geq$, is called "fuzzy greater than or equal to". It is worth mentioning that in the model (21), $\succsim$ does not mean the vagueness in the constraints, as there is not vagueness in the objective function, but it means the comparison between the fuzzy quantities, in fact the decision maker does not allow any violation in the fulfilment of the constraints. The comparison between the fuzzy quantities has studied in the various articles by the many researchers [7, 8, 11, 15, 19].

The several approaches have been used for solving the FLP by the different researchers [8, 12, 15, 16, 17, 21, 27]. In the approaches, depending on the model consists of the vagueness or ambiguity or both of them, the FLP can be transformed to a classical LP problem. One of the most prevalent approaches is based on the Representation theorem and "α-cut" [21].

In our method which is proposed for solving the FLP problem and the following in section 4.2 for the FBLP, using "α-cut", the original problem is transformed to an interval programming.

Now, suppose $\lambda_k \in [0,1]$ is arbitrary; then, cutting it from the objective function as well as the constraints of the FLP problem, we will have the following ILP problem related to λ_k:

$$\begin{aligned} \min \quad & z_{\lambda_k} = [\underline{\mathbf{c}}_{\lambda_k}, \overline{\mathbf{c}}_{\lambda_k}]\mathbf{x} \\ s.t. \quad & [\underline{A}_{\lambda_k}, \overline{A}_{\lambda_k}]\mathbf{x} \geq [\underline{\mathbf{b}}_{\lambda_k}, \overline{\mathbf{b}}_{\lambda_K}] \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{22}$$

and, according to the features of the ILP problem, we will obtain the following two problems:

$$\begin{aligned} \min \quad & \underline{z}_{\lambda_k} = \underline{\mathbf{c}}_{\lambda_k}\mathbf{x} \\ s.t. \quad & \overline{A}_{\lambda_k}\mathbf{x} \geq \underline{\mathbf{b}}_{\lambda_k} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{23}$$

$$\begin{aligned} \min \quad & \overline{z}_{\lambda_k} = \overline{\mathbf{c}}_{\lambda_k}\mathbf{x} \\ s.t. \quad & \underline{A}_{\lambda_k}\mathbf{x} \geq \overline{\mathbf{b}}_{\lambda_K} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{24}$$

and $\underline{z}^*_{\lambda_k} \leq z^*_{\lambda_k} \leq \overline{z}^*_{\lambda_k}$ wherein $z^*_{\lambda_K}$, $\underline{z}^*_{\lambda_k}$ and $\overline{z}^*_{\lambda_k}$ are the optimal values of the objective functions of problems (22), (23) and (24) respectively. Also, according to the decomposition principle,

$$\widetilde{z}^* = \bigcup_{\lambda_k \in [0,1]} \lambda_k \widetilde{z}^*_{\lambda_k} = \bigcup_{\lambda_k \in [0,1]} \lambda_k [\underline{z}^*_{\lambda_k}, \overline{z}^*_{\lambda_k}].$$

Therefore, using $\lambda_1, \ldots, \lambda_t$ cuts, the linear piecewise trapezoidal approximate fuzzy number for $\widetilde{z}^*$ will be found as follows:

$$\mu_{\widetilde{z}^*}(x) \approx \begin{cases} 0 & x < \underline{z}^*_{\lambda_0} \\ \frac{\lambda_1 - \lambda_0}{\underline{z}^*_{\lambda_1} - \underline{z}^*_{\lambda_0}}(x - \underline{z}^*_{\lambda_0}) + \lambda_0 & \underline{z}^*_{\lambda_0} \leq x < \underline{z}^*_{\lambda_1} \\ \frac{\lambda_2 - \lambda_1}{\underline{z}^*_{\lambda_2} - \underline{z}^*_{\lambda_1}}(x - \underline{z}^*_{\lambda_1}) + \lambda_1 & \underline{z}^*_{\lambda_1} \leq x < \underline{z}^*_{\lambda_2} \\ \vdots & \vdots \\ \lambda_t & \underline{z}^*_{\lambda_t} \leq x < \overline{z}^*_{\lambda_t} \\ \vdots & \vdots \\ \frac{\lambda_1 - \lambda_0}{\overline{z}^*_{\lambda_1} - \overline{z}^*_{\lambda_0}}(x - \overline{z}^*_{\lambda_0}) + \lambda_0 & \overline{z}^*_{\lambda_1} \leq x < \overline{z}^*_{\lambda_0} \\ 0 & \overline{z}^*_{\lambda_0} \leq x. \end{cases} \tag{25}$$

It is to be noted that the more is the number of λ_k-cuts, the better will be the approximation for $\widetilde{z}^*$.

Example 4.1. Consider the following FLP problem in which the coefficients are triangular fuzzy numbers:

$$\begin{aligned} \min \quad & \widetilde{z} = (19, 20, 21)x_1 + (29, 30, 31)x_2 \\ s.t. \quad & (4.5, 5, 5.5)x_1 + (2.5, 3, 4)x_2 \gtrsim (194, 200, 206) \\ & (3, 4, 5)x_1 + (6.5, 7, 7.5)x_2 \gtrsim (230, 240, 250) \\ & x_1, x_2 \geq 0. \end{aligned} \tag{26}$$

λ_k	$[\underline{z}^*_{\lambda_k}, \overline{z}^*_{\lambda_k}]$
0.1	[896.2204,1373.09]
0.2	[920.8,1336.058]
0.3	[941.9282,1300.771]
0.4	[963.9206,1267.11]
0.5	[986.8312,1234.964]
0.6	[1010.718,1204.235]
0.7	[1035.646,1174.83]
0.8	[1061.6820,1146.666]
0.9	[1088.903,1119.667]

Table 1. Optimal Values of the ILP Problems

By a λ_k-cut from the FLP problem, an ILP problem will be obtained; and, considering section 3.1, the problems related to the best and the worst optimal values of the objective function of the ILP problem will be respectively as follows:

$$\begin{aligned} \min \quad & \underline{z}_{\lambda_k} = (\lambda_k + 19)x_1 + (\lambda_k + 29)x_2 \\ s.t. \quad & (5.5 - \lambda_k)x_1 + (4 - \lambda_k)x_2 \geq (\lambda_k + 194) \\ & (5 - \lambda_k)x_1 + (7.5 - \lambda_k)x_2 \geq (\lambda_k + 230) \\ & x_1, x_2 \geq 0 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \min \quad & \overline{z}_{\lambda_k} = (21 - \lambda_k)x_1 + (31 - \lambda_k)x_2 \\ s.t. \quad & (4.5 + \lambda_k)x_1 + (2.5 + \lambda_k)x_2 \geq (206 - \lambda_k) \\ & (3 + \lambda_k)x_1 + (6.5 + \lambda_k)x_2 \geq (250 - \lambda_k) \\ & x_1, x_2 \geq 0. \end{aligned} \tag{28}$$

Associated with the λ_k-cuts, the optimal values of problems (27) and (28) are shown in Table 1, where $\lambda_k = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. Then according to (25), the linear piecewise trapezoidal approximate fuzzy number for $\widetilde{z}^*$ can be written as:

$$\mu_{\widetilde{z}^*}(x) \approx \begin{cases} 0 & x < 896.2204 \\ 0.00407x - 3.54620 & 896.2204 \leq x < 920.8 \\ 0.00473x - 4.15816 & 920.8 \leq x < 941.9282 \\ 0.00455x - 3.98297 & 941.9282 \leq x < 963.9206 \\ 0.00436x - 3.80731 & 963.9206 \leq x < 986.8312 \\ 0.00419x - 3.63128 & 986.8312 \leq x < 1010.718 \\ 0.00401x - 3.45455 & 1010.718 \leq x < 1035.646 \\ 0.00384x - 3.27775 & 1035.646 \leq x < 1061.682 \\ 0.00367x - 3.10023 & 1061.682 \leq x < 1088.903 \\ 0.9 & 1088.903 \leq x < 1119.667 \\ -0.00370x + 5.04707 & 1119.667 \leq x < 1146.666 \\ -0.00355x + 4.87139 & 1146.666 \leq x < 1174.83 \\ -0.00340x + 4.69534 & 1174.83 \leq x < 1204.235 \\ -0.00325x + 4.51889 & 1204.235 \leq x < 1234.964 \\ -0.00311x + 4.34173 & 1234.964 \leq x < 1267.11 \\ -0.00297x + 4.16433 & 1267.11 \leq x < 1300.771 \\ -0.00283x + 3.98626 & 1300.771 \leq x < 1336.058 \\ -0.00270x + 3.80785 & 1336.058 \leq x < 1373.09 \\ 0 & 1373.09 \leq x \end{cases} \tag{29}$$

Figure 2 shows the approximate fuzzy number $\widetilde{z}^*$.

4.2. **Solving FBLP Using IBLP.** Consider an FBLP problem in the following form:

$$\begin{aligned} \min_{\mathbf{x} \in X} \quad & \widetilde{F}(\mathbf{x}, \mathbf{y}) = \widetilde{\mathbf{c}}^1 \mathbf{x} + \widetilde{\mathbf{d}}^1 \mathbf{y} \\ s.t. \quad & \widetilde{A}^1 \mathbf{x} + \widetilde{B}^1 \mathbf{y} \gtrsim \widetilde{\mathbf{b}}^1 \\ & \min_{\mathbf{y} \in Y} \quad \widetilde{f}(\mathbf{x}, \mathbf{y}) = \widetilde{\mathbf{d}}^2 \mathbf{y} \\ & s.t. \quad \widetilde{A}^2 \mathbf{x} + \widetilde{B}^2 \mathbf{y} \gtrsim \widetilde{\mathbf{b}}^2 \end{aligned} \tag{30}$$

where $X \subset R^n$, $Y \subset R^m$, $F, f : X \times Y \to \mathscr{F}(R)$, $\widetilde{\mathbf{c}}^1, \widetilde{\mathbf{c}}^2 \in \mathscr{F}(R^n)$, $\widetilde{\mathbf{d}}^1, \widetilde{\mathbf{d}}^2 \in \mathscr{F}(R^m)$, $\widetilde{\mathbf{b}}^1 \in \mathscr{F}(R^p)$, $\widetilde{\mathbf{b}}^2 \in \mathscr{F}(R^q)$, $\widetilde{A}^1 \in \mathscr{F}(R^{p\times n})$, $\widetilde{B}^1 \in \mathscr{F}(R^{p\times m})$, $\widetilde{A}^2 \in \mathscr{F}(R^{q\times n})$, $\widetilde{B}^2 \in \mathscr{F}(R^{q\times m})$. Now, for an arbitrary λ_k, we will have the following IBLP problem:

$$\begin{aligned}
&\min_{\mathbf{x}\in X} \quad F_{\lambda_k}(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{c}}^1_{\lambda_k}, \overline{\mathbf{c}}^1_{\lambda_k}]\mathbf{x} + [\underline{\mathbf{d}}^1_{\lambda_k}, \overline{\mathbf{d}}^1_{\lambda_k}]\mathbf{y} \\
&s.t. \quad [\underline{A}^1_{\lambda_k}, \overline{A}^1_{\lambda_k}]\mathbf{x} + [\underline{B}^1_{\lambda_k}, \overline{B}^1_{\lambda_k}]\mathbf{y} \geq [\underline{\mathbf{b}}^1_{\lambda_k}, \overline{\mathbf{b}}^1_{\lambda_k}] \\
&\quad \min_{\mathbf{y}\in Y} \quad f_{\lambda_k}(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2_{\lambda_k}, \overline{\mathbf{d}}^2_{\lambda_k}]\mathbf{y} \\
&\quad s.t. \quad [\underline{A}^2_{\lambda_k}, \overline{A}^2_{\lambda_k}]\mathbf{x} + [\underline{B}^2_{\lambda_k}, \overline{B}^2_{\lambda_k}]\mathbf{y} \geq [\underline{\mathbf{b}}^2_{\lambda_k}, \overline{\mathbf{b}}^2_{\lambda_k}].
\end{aligned} \tag{31}$$

And, according to what was said in section 3.2, the following two BLP problems can be written from the above problem:

$$\begin{aligned}
&\min_{\mathbf{x}\in X} \quad \underline{F}_{\lambda_k}(\mathbf{x},\mathbf{y}) = \underline{\mathbf{c}}^1_{\lambda_k}\mathbf{x} + \underline{\mathbf{d}}^1_{\lambda_k}\mathbf{y} \\
&s.t. \quad \overline{A}^1_{\lambda_k}\mathbf{x} + \overline{B}^1_{\lambda_k}\mathbf{y} \geq \underline{\mathbf{b}}^1_{\lambda_k} \\
&\quad \min_{\mathbf{y}\in Y} \quad f_{\lambda_k}(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2_{\lambda_k}, \overline{\mathbf{d}}^2_{\lambda_k}]\mathbf{y} \\
&\quad s.t. \quad \overline{A}^2_{\lambda_k}\mathbf{x} + \overline{B}^2_{\lambda_k}\mathbf{y} \geq \underline{\mathbf{b}}^2_{\lambda_k}
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\min_{\mathbf{x}\in X} \quad \overline{F}_{\lambda_k}(\mathbf{x},\mathbf{y}) = \overline{\mathbf{c}}^1_{\lambda_k}\mathbf{x} + \overline{\mathbf{d}}^1_{\lambda_k}\mathbf{y} \\
&s.t. \quad \underline{A}^1_{\lambda_k}\mathbf{x} + \underline{B}^1_{\lambda_k}\mathbf{y} \geq \overline{\mathbf{b}}^1_{\lambda_k} \\
&\quad \min_{\mathbf{y}\in Y} \quad f_{\lambda_k}(\mathbf{x},\mathbf{y}) = [\underline{\mathbf{d}}^2_{\lambda_k}, \overline{\mathbf{d}}^2_{\lambda_k}]\mathbf{y} \\
&\quad s.t. \quad \underline{A}^2_{\lambda_k}\mathbf{x} + \underline{B}^2_{\lambda_k}\mathbf{y} \geq \overline{\mathbf{b}}^2_{\lambda_k}.
\end{aligned} \tag{33}$$

Supposing that $F^*_{\lambda_k}$, $\underline{F}^*_{\lambda_k}$ and $\overline{F}^*_{\lambda_k}$ are the optimal values of the objective functions of problems (31), (32) and (33) respectively, then, according to section 4.1, $\underline{F}^*_{\lambda_k} \leq F^*_{\lambda_k} \leq \overline{F}^*_{\lambda_k}$,

$$\widetilde{F}^* = \bigcup_{\lambda_k\in[0,1]} \lambda_k \widetilde{F}^*_{\lambda_k} = \bigcup_{\lambda_k\in[0,1]} \lambda_k [\underline{F}^*_{\lambda_k}, \overline{F}^*_{\lambda_k}].$$

Using $\lambda_1, \ldots, \lambda_t$ cuts, a linear piecewise trapezoidal approximate fuzzy number, similar to (25), will be obtained for $\widetilde{F}^*$. It is to be noted that the same algorithm can be followed for the follower's objective function.

4.2.1. **Solution Algorithm for the FBLP.** Solution algorithm for the FBLP is as follows

Step 1: Choose t values for λ_k between 0 and 1, i.e. $\lambda_1, \ldots, \lambda_t$ so that $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_t \leq 1$.

Step 2: For each λ_k-cut, find the best and the worst optimal values of the objective function of the leader for the obtained IBLP problem.

Step 3: Form the membership function of the approximate fuzzy number related to the optimal values of the objective function of the leader according to (25).

Example 4.2. Consider the following FBLP problem:

$$\begin{aligned}
&\min_{x\geq 0} \quad \widetilde{F}(x,y) = \widetilde{1}x - \widetilde{4}y \\
&s.t. \quad \min_{y\geq 0} \quad \widetilde{f}(x,y) = \widetilde{1}y \\
&\qquad\quad s.t. \quad \widetilde{2}x - \widetilde{1}y \gtrsim \widetilde{0} \\
&\qquad\qquad\quad -\widetilde{2}x - \widetilde{1}y \gtrsim -\widetilde{12} \\
&\qquad\qquad\quad \widetilde{3}x - \widetilde{2}y \gtrsim \widetilde{4}
\end{aligned} \tag{34}$$

where

$$\mu_{\widetilde{1}}(x) = \begin{cases} 0 & x<0, x\geq 2 \\ x & 0\leq x<1 \\ 2-x & 1\leq x<2 \end{cases} \quad \mu_{\widetilde{2}}(x) = \begin{cases} 0 & x<1, x\geq 3 \\ x-1 & 1\leq x<2 \\ 3-x & 2\leq x<3 \end{cases}$$

$$\mu_{\widetilde{3}}(x) = \begin{cases} 0 & x<2, x\geq 4 \\ x-2 & 2\leq x<3 \\ 4-x & 3\leq x<4 \end{cases} \quad \mu_{\widetilde{4}}(x) = \begin{cases} 0 & x<3, x\geq 5 \\ x-3 & 3\leq x<4 \\ 5-x & 4\leq x<5 \end{cases}$$

$$\mu_{\widetilde{12}}(x) = \begin{cases} 0 & x<11, x\geq 13 \\ x-11 & 11\leq x<12 \\ 13-x & 12\leq x<13 \end{cases} \quad \mu_{\widetilde{0}}(x) = \begin{cases} 0 & x<-1, x\geq 1 \\ x+1 & -1\leq x<0 \\ 1-x & 0\leq x<1. \end{cases}$$

By a λ_k-cut from the FBLP problem, an IBLP problem will be obtained; and, considering section 4.2, the problems related to the best and the worst optimal values of the objective functions of the IBLP problem will be respectively:

$$\begin{aligned}
&\min_{x\geq 0} \quad \underline{F}(x,y) = \lambda_k x + (\lambda_k - 5)y \\
&s.t. \quad \min_{y\geq 0} \quad \underline{f}(x,y) = \lambda_k y \\
&\qquad\quad s.t. \quad (3-\lambda_k)x - \lambda_k y \geq \lambda_k - 1 \\
&\qquad\qquad\quad (-1-\lambda_k)x - \lambda_k y \geq \lambda_k - 13 \\
&\qquad\qquad\quad (4-\lambda_k)x - (-1-\lambda_k)y \geq \lambda_k + 3
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\min_{x\geq 0} \quad \overline{F}(x,y) = (2-\lambda_k)x + (-3-\lambda_k)y \\
&s.t. \quad \min_{y\geq 0} \quad \overline{f}(x,y) = (2-\lambda_k)y \\
&\qquad\quad s.t. \quad (\lambda_k+1)x + (\lambda_k-2)y \geq 1-\lambda_k \\
&\qquad\qquad\quad (\lambda_k-3)x + (\lambda_k-2)y \geq -11-\lambda_k \\
&\qquad\qquad\quad (\lambda_k+2)x + (\lambda_k-3)y \geq 5-\lambda_k.
\end{aligned} \tag{36}$$

The optimal values of the above problems for λ_k are equal to 0.1,0.2,0.3,0.4,0.5,0.6, 0.7,0.8 and 0.9 are shown in Table 2. Figure 3 shows the approximate fuzzy number $\widetilde{F}^*$.

λ_k	$[\underline{F}^*_{\lambda_k}, \overline{F}^*_{\lambda_k}]$
0.1	[0.079,4.427]
0.2	[0.168,3.924]
0.3	[0.267,3.468]
0.4	[0.376,3.072]
0.5	[0.50,2.70]
0.6	[0.636,2.366]
0.7	[0.784,2.067]
0.8	[0.952,1.80]
0.9	[1.134,1.551]

TABLE 2. Optimal Values of the IBLP Problems

The linear piecewise trapezoidal approximate fuzzy number for $\widetilde{F}^*$ will be found as follows:

$$\mu_{\widetilde{F}^*}(x) \approx \begin{cases} 0 & x < 0.079 \\ 1.124x + 0.0112 & 0.079 \le x < 0.168 \\ 1.0101x + 0.03030 & 0.168 \le x < 0.267 \\ 0.91743x + 0.0550 & 0.267 \le x < 0.376 \\ 0.8065x + 0.0968 & 0.376 \le x < 0.50 \\ 0.7353x + 0.1324 & 0.50 \le x < 0.636 \\ 0.6757x + 0.1703 & 0.636 \le x < 0.784 \\ 0.5952x + 0.2333 & 0.784 \le x < 0.952 \\ 0.5495x + 0.2769 & 0.952 \le x < 1.134 \\ 0.9 & 1.134 \le x < 1.551 \\ -0.4016x + 1.5229 & 1.551 \le x < 1.80 \\ -0.3745x + 1.4742 & 1.80 \le x < 2.067 \\ -0.3344x + 1.3913 & 2.067 \le x < 2.366 \\ -0.2994x + 1.3084 & 2.366 \le x < 2.70 \\ -0.2688x + 1.2258 & 2.70 \le x < 3.072 \\ -0.2525x + 1.1758 & 3.072 \le x < 3.468 \\ -0.2193x + 1.0605 & 3.468 \le x < 3.924 \\ -0.1988x + 0.9801 & 3.924 \le x < 4.427 \\ 0 & 4.427 \le x. \end{cases} \tag{37}$$

4.3. **Solving FBLP using Multi Objective Linear Programming.** For the solution of an FBLP problem, Zhang et al. [23, 24, 25, 26] have made use of two issues that can not be generally true.

First, they have made use of fuzzy numbers ranking in the following form:

Definition 4.3. For any n-dimensional fuzzy numbers $\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}$:

(1) $\widetilde{\mathbf{a}} \gtrapprox \widetilde{\mathbf{b}} \iff \underline{\mathbf{a}}_\lambda \geqq \underline{\mathbf{b}}_\lambda$ and $\overline{\mathbf{a}}_\lambda \geqq \overline{\mathbf{b}}_\lambda$, $\lambda \in (0, 1]$.
(2) $\widetilde{\mathbf{a}} \succsim \widetilde{\mathbf{b}} \iff \underline{\mathbf{a}}_\lambda \geqslant \underline{\mathbf{b}}_\lambda$ and $\overline{\mathbf{a}}_\lambda \geqslant \overline{\mathbf{b}}_\lambda$, $\lambda \in (0, 1]$.
(3) $\widetilde{\mathbf{a}} \succ \widetilde{\mathbf{b}} \iff \underline{\mathbf{a}}_\lambda > \underline{\mathbf{b}}_\lambda$ and $\overline{\mathbf{a}}_\lambda > \overline{\mathbf{b}}_\lambda$, $\lambda \in (0, 1]$.

This definition is not generally true for the ranking of fuzzy numbers. For example, it is not true for the case when the left and the right functions of the fuzzy numbers intersect each other like Figure 4; for $\lambda \in [0, \lambda_0]$, $\widetilde{\mathbf{a}} \succsim \widetilde{\mathbf{b}}$ and $\lambda \in [\lambda_0, 1]$, $\widetilde{\mathbf{a}} \precsim \widetilde{\mathbf{b}}$.

Second, they have found the following problem by using λ-cut on the FBLP problem:

$$\begin{aligned} &\min_{\mathbf{x} \in X} \quad \underline{F}_\lambda(\mathbf{x}, \mathbf{y}) = \underline{\mathbf{c}}^1_\lambda \mathbf{x} + \underline{\mathbf{d}}^1_\lambda \mathbf{x} \\ &\min_{x \in X} \quad \overline{F}_\lambda(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{c}}^1_\lambda \mathbf{x} + \overline{\mathbf{d}}^1_\lambda \mathbf{x} \\ &s.t. \quad \underline{A}^1_\lambda \mathbf{x} + \underline{B}^1_\lambda \mathbf{y} \ge \underline{\mathbf{b}}^1_\lambda \\ &\qquad \overline{A}^1_\lambda \mathbf{x} + \overline{B}^1_\lambda \mathbf{y} \ge \overline{\mathbf{b}}^1_\lambda \\ &\qquad \min_{y \in Y} \quad \underline{f}_\lambda(\mathbf{x}, \mathbf{y}) = \underline{\mathbf{c}}^2_\lambda \mathbf{x} + \underline{\mathbf{d}}^2_\lambda \mathbf{x} \\ &\qquad \min_{y \in Y} \quad \overline{f}_\lambda(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{c}}^2_\lambda \mathbf{x} + \overline{\mathbf{d}}^2_\lambda \mathbf{x} \\ &\qquad s.t. \quad \underline{A}^2_\lambda \mathbf{x} + \underline{B}^2_\lambda \mathbf{y} \ge \underline{\mathbf{b}}^2_\lambda \\ &\qquad\qquad \overline{A}^2_\lambda \mathbf{x} + \overline{B}^2_\lambda \mathbf{y} \ge \overline{\mathbf{b}}^2_\lambda \end{aligned} \tag{38}$$

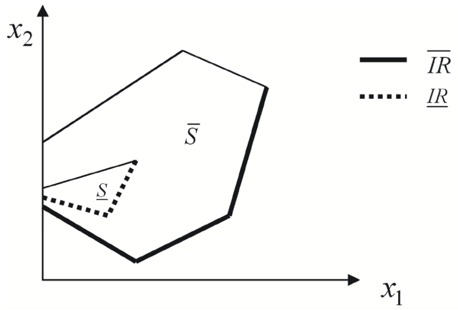

FIGURE 1. $\underline{S}, \underline{IR}$ and $\overline{S}, \overline{IR}$ Related to Problems (16) and (17) Respectively

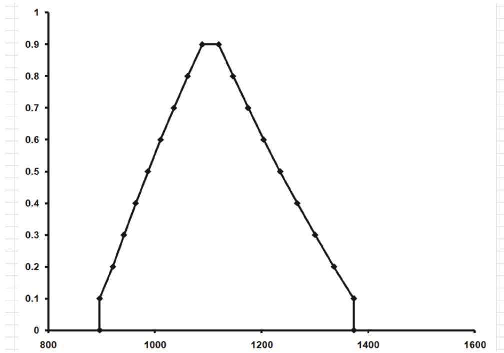

FIGURE 2. Approximate Fuzzy Number $\widetilde{z}^*$

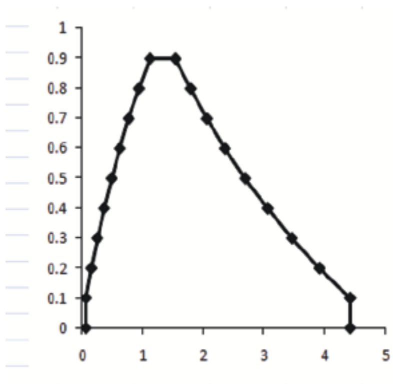

FIGURE 3. Approximate Fuzzy Number $\widetilde{F}^*$

and considered it as a multi objective BLP. Then, using weighting method, they have changed the multi objective BLP to an ordinary BLP. Here, two points are worth mentioning:

(1) For a λ-cut, an FBLP is changed to an IBLP and not a multi objective BLP.
(2) Solving a multi objective problem adds difficulties to the solution of a BLP problem.

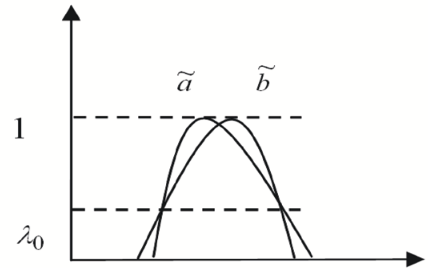

FIGURE 4. Comparison of Two Fuzzy Numbers

5. Conclusions

Bilevel optimization models can be widely used in the modeling of real world problems that are in the form of decentralized decision making problems at two levels, especially when the parameters are inexact. In this paper, a method has been proposed to find the approximate fuzzy number related to the objective function of the leader in the FBLP problem.

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Farhad Hamidi, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail address: f_hamidi@math.usb.ac.ir

Hassan Mishmast Nehi*, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

E-mail address: hmnehi@hamoon.usb.ac.ir

*Corresponding author