

A NOTE ON INTUITIONISTIC FUZZY MAPPINGS

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ABSTRACT. In this paper, the concept of intuitionistic fuzzy mapping as a generalization of fuzzy mapping is presented, and its' relationship with intuitionistic fuzzy relations is derived. Moreover, some basic operations of intuitionistic fuzzy mappings are defined, hence we can conclude that all of intuitionistic fuzzy mappings constitute a soft algebra with respect to these operations. Afterwards, the Atanassov's operator is applied to intuitionistic fuzzy mappings and the corresponding properties are examined. Finally, the decomposition and representation theorems of intuitionistic fuzzy mappings are established.

1. Introduction

In 1981, Heilpern [10] first introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings. These results can be regarded as the extensions of the classical mappings and Banach contraction principle in a complete metric space. In recent years, further studies have been made on the fixed point theory of fuzzy mappings by many scholars [4, 13, 14, 18]. As we all know, the convexity and semi-continuity of real-valued functions play important roles in optimization theory and abstract analysis. Analogously, the convexity and semi-continuity of fuzzy mappings were also introduced and studied by various authors [5, 17, 20, 21, 22, 23, 24]. Obviously, these researches will be benefit to the developments of the fuzzy optimization theory.

The concept of intuitionistic fuzzy set was formally introduced by Atanassov in 1986 [1], which can be viewed as a generalization of fuzzy set. Because intuitionistic fuzzy set can not only characterize the degree of membership of an element, but also simultaneously express the degree of non-membership, it will be better to deal with the imprecise and uncertain information than fuzzy sets. At present, many researchers have devoted to the study of intuitionistic fuzzy sets, such as Atanassov [1, 2, 3], Burillo and Bustince [6, 7, 8, 9], Hosseini et al. [11], Hur et al. [12], Rafiand et al. [19], etc. However, as far as we know, there are few researches on the intuitionistic fuzzy mappings in previous work. In this paper, we will extend fuzzy mappings to intuitionistic fuzzy mappings and discuss some relevant properties.

The rest of the paper is organized into four parts. In section 2, we review some related concepts and introduce the concept of the truncation projection of intuitionistic fuzzy relations. Meantime, an important conclusion is given, which provides

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a basis for the next part. In section 3, the concept of intuitionistic fuzzy mapping is proposed and its' relationship with the intuitionistic fuzzy relations is discussed. Afterwards, some operations of intuitionistic fuzzy mappings are defined and the corresponding properties are discussed. Section 4 discusses two basic theorems of intuitionistic fuzzy mapping based on the decomposition and representation theorems of intuitionistic fuzzy sets. Finally, some conclusions are given in section 5.

2. Preliminaries

For completeness and clarity, in this section, some related concepts and results from [1, 2, 3, 8, 9, 15, 16, 25] are summarized below. Moreover, a new concept of the truncation projection of intuitionistic fuzzy relations is proposed.

Let X be the universe of discourse. An *intuitionistic fuzzy set* E on X is given by

$$E = \{ \langle x, \mu_E(x), \nu_E(x) \rangle \mid x \in X \}$$

with

$$\mu_E : X \rightarrow [0, 1] \quad \text{and} \quad \nu_E : X \rightarrow [0, 1]$$

such that $0 \leq \mu_E(x) + \nu_E(x) \leq 1$ for all $x \in X$.

Generally, $\mu_E(x)$ and $\nu_E(x)$ are called the *degree of membership* and the *degree of non-membership* of the element x in the set E , respectively. The *complement* of E is denoted by $\complement E = \{ \langle x, \nu_E(x), \mu_E(x) \rangle \mid x \in X \}$. $IFS(X)$ denotes the set of all intuitionistic fuzzy sets on X . Especially, if $\nu_E(x) = 1 - \mu_E(x)$, then the set E becomes a fuzzy set.

In fact, an ordinary set can be viewed as a special intuitionistic fuzzy set, and it can be expressed by the degree of membership and the degree of non-membership, i.e. if $A \subseteq X$, then $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$, where $\mu_A(x) = 1$, $\nu_A(x) = 0$ if $x \in A$, and $\mu_A(x) = 0$, $\nu_A(x) = 1$ if $x \notin A$.

For every $A, B \in IFS(X)$, the relations and operations between two intuitionistic fuzzy sets are defined as

- (i) $A \leq B \Leftrightarrow \forall x \in X, \mu_A(x) \leq \mu_B(x) \ \& \ \nu_A(x) \geq \nu_B(x)$;
- (ii) $A = B \Leftrightarrow \forall x \in X, \mu_A(x) = \mu_B(x) \ \& \ \nu_A(x) = \nu_B(x)$;
- (iii) $A \vee B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X \}$
 $= \{ \langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\} \rangle \mid x \in X \}$;
- (iv) $A \wedge B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X \}$
 $= \{ \langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\} \rangle \mid x \in X \}$.

An *intuitionistic fuzzy relation* R is an intuitionistic fuzzy subset of $X \times Y$, which is defined as

$$R = \{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \mid x \in X, y \in Y \},$$

where

$$\mu_R : X \times Y \rightarrow [0, 1] \quad \text{and} \quad \nu_R : X \times Y \rightarrow [0, 1]$$

satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for every $(x, y) \in X \times Y$. The set of all intuitionistic fuzzy relations of $X \times Y$ will be denoted by $IFR(X \times Y)$.

Let $R \in IFR(X \times Y)$, for every $x \in X$ and $y \in Y$, the value of intuitionistic fuzzy relation between x and y is abbreviated as $R(x, y) = \langle \mu_R(x, y), \nu_R(x, y) \rangle$.

It should be noted that, if R is an ordinary binary relation on $X \times Y$, then it can be regarded as a special intuitionistic fuzzy relation. Moreover, the binary relation R can also be represented as the above form, where

$$\mu_R(x, y) = \begin{cases} 1, & (x, y) \in R, \\ 0, & (x, y) \notin R, \end{cases} \quad \nu_R(x, y) = \begin{cases} 0, & (x, y) \in R, \\ 1, & (x, y) \notin R. \end{cases}$$

For convenience, we use the notation I^2 to express the following set

$$I^2 = \{ \langle \alpha, \beta \rangle \mid 0 \leq \alpha + \beta \leq 1, \alpha, \beta \in I \},$$

where I denotes the unit interval, i.e., $I = [0, 1]$.

Additionally, for every $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, we introduce the following relations:

- (i) $\langle \alpha_1, \beta_1 \rangle = \langle \alpha_2, \beta_2 \rangle \Leftrightarrow \alpha_1 = \alpha_2 \ \& \ \beta_1 = \beta_2$;
- (ii) $\langle \alpha_1, \beta_1 \rangle \leq \langle \alpha_2, \beta_2 \rangle \Leftrightarrow \alpha_1 \leq \alpha_2 \ \& \ \beta_1 \geq \beta_2$;
- (iii) $\langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle \Leftrightarrow \langle \alpha_1, \beta_1 \rangle \leq \langle \alpha_2, \beta_2 \rangle \ \& \ \langle \alpha_1, \beta_1 \rangle \neq \langle \alpha_2, \beta_2 \rangle$.

Let $R \in IFR(X \times Y)$ be an intuitionistic fuzzy relation. For all $\langle \alpha, \beta \rangle \in I^2$, the $\langle \alpha, \beta \rangle$ -cut relation of R is defined as

$$R_{\langle \alpha, \beta \rangle} = \{ \langle (x, y), \alpha_{\mu_R(x, y)}, \beta_{\nu_R(x, y)} \rangle \mid (x, y) \in X \times Y \},$$

where

$$(\alpha_{\mu_R(x, y)}, \beta_{\nu_R(x, y)}) = \begin{cases} (1, 0), & \mu_R(x, y) \geq \alpha, \nu_R(x, y) \leq \beta, \\ (0, 1), & \text{otherwise.} \end{cases}$$

Obviously, the cut relation $R_{\langle \alpha, \beta \rangle}$ is an ordinary binary relation on $X \times Y$.

In [15], Liu extended the decomposition theorem of fuzzy sets to intuitionistic fuzzy sets. Now, we briefly recall the decomposition theorem which will be used later.

If A is an intuitionistic fuzzy set on X , then

$$A = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot A_{\langle \alpha, \beta \rangle},$$

where $\langle \alpha, \beta \rangle \cdot A_{\langle \alpha, \beta \rangle} := \{ \langle x, \alpha \wedge \mu_A(x), \beta \vee \nu_A(x) \rangle \mid x \in X \}$, $A_{\langle \alpha, \beta \rangle}$ denotes the $\langle \alpha, \beta \rangle$ -cut set of A , its definition is similar to the one of $\langle \alpha, \beta \rangle$ -cut relation.

Similarly, for all $R \in IFR(X \times Y)$, the intuitionistic fuzzy relation R can be decomposed as follows [25]:

$$R = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}$$

where $\mu_R(x, y) = \bigvee_{\langle \alpha, \beta \rangle \in I^2} (\alpha \wedge \mu_{R_{\langle \alpha, \beta \rangle}}(x, y))$, $\nu_R(x, y) = \bigwedge_{\langle \alpha, \beta \rangle \in I^2} (\beta \vee \nu_{R_{\langle \alpha, \beta \rangle}}(x, y))$.

Next, we will review the notion of the truncation projection of an ordinary binary relation.

If R is an ordinary binary relation on $X \times Y$, for a given $x \in X$, the set $R|_x := \{y \mid (x, y) \in R\} \in P(Y)$ is said to be the *truncation projection* of R at x . Conversely, for a given $y \in Y$, the set $R|_y := \{x \mid (x, y) \in R\} \in P(X)$ is said to be the *truncation projection* of R at y . $P(X)$ and $P(Y)$ denote the power set of X and Y , respectively.

Given x and y , it can easily be seen that $R|_x(y) = R|_y(x) = R(x, y)$, where $R|_x(y) = (\{x\} \times R|_x) \cap (\{y\} \times X)$, $R|_y(x) = (\{y\} \times R|_y) \cap (\{x\} \times Y)$.

Analogously, we will introduce the truncation projection of an intuitionistic fuzzy relation.

Definition 2.1. Let $R \in IFR(X \times Y)$ be an intuitionistic fuzzy relation, if $x \in X$ and $y \in Y$, the sets

$$R|_x := \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}|_x \in IFS(Y),$$

$$R|_y := \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}|_y \in IFS(X)$$

are called the *truncation projection* of R at x and *truncation projection* of R at y , respectively.

Theorem 2.2. Let $R \in IFR(X \times Y)$ be an intuitionistic fuzzy relation, if $x \in X$ and $y \in Y$, then

$$R|_x(y) = R|_y(x) = R(x, y).$$

$$\begin{aligned} \text{Proof. } R|_x(y) &= \left(\bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}|_x \right)(y) = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}|_x(y) \\ &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot R_{\langle \alpha, \beta \rangle}(x, y) = R(x, y). \end{aligned}$$

Similarly, it is easy to verify that $R|_y(x) = R(x, y)$. \square

3. Intuitionistic Fuzzy Mappings

We first introduce the concept of intuitionistic fuzzy mapping based on fuzzy mapping. Meantime, a relationship between an intuitionistic fuzzy mapping and an intuitionistic fuzzy relation is discussed. Furthermore, some operations of intuitionistic fuzzy mappings are defined in this section.

3.1. The Concept of Intuitionistic Fuzzy Mapping.

Definition 3.1. Let X, Y be two non-empty sets. The mapping $\tilde{f} : X \rightarrow IFS(Y)$ is called an *intuitionistic fuzzy mapping* from X to Y , namely,

$$x \mapsto \tilde{f}(x) = \{ \langle y, (\triangleleft \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) \rangle \mid y \in Y \}$$

where, for a $x \in X$, $(\triangleleft \tilde{f})_x : Y \rightarrow [0, 1]$, $(\tilde{f} \triangleright)_x : Y \rightarrow [0, 1]$, and satisfying $0 \leq (\triangleleft \tilde{f})_x(y) + (\tilde{f} \triangleright)_x(y) \leq 1$ for all $y \in Y$. The set of all intuitionistic fuzzy mappings from X to Y will be represented as $IFS(Y)^X$.

According to Definition 3.1, it is easy to see that the image or value of an intuitionistic fuzzy mapping is an intuitionistic fuzzy set on Y for every $x \in X$. Given $x \in X$ and $y \in Y$, its value can be denoted by

$$\tilde{f}(x)(y) = \langle (\triangleleft \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) \rangle.$$

In general, let $\tilde{f}, \tilde{g} \in IFS(Y)^X$, $\tilde{f} \leq \tilde{g} \iff \tilde{f}(x) \leq \tilde{g}(x)$ for all $x \in X$.

Similar to the fuzzy mappings, there exists a relationship between an intuitionistic fuzzy mapping and an intuitionistic fuzzy relation.

Theorem 3.2. *Given an intuitionistic fuzzy mapping $\tilde{f} \in IFS(Y)^X$, it can uniquely establish an intuitionistic fuzzy relation $R_{\tilde{f}} \in IFR(X \times Y)$, satisfying $R_{\tilde{f}}|_x = \tilde{f}(x)$. Conversely, if given an intuitionistic fuzzy relation $R \in IFR(X \times Y)$, it can uniquely determine an intuitionistic fuzzy mapping \tilde{f}_R , such that $\tilde{f}_R(x) = R|_x$.*

Proof. Suppose that $\tilde{f} : X \rightarrow IFS(Y)$, $x \mapsto \tilde{f}(x) \in IFS(Y)$. Let

$$R_{\tilde{f}} \in IFR(X \times Y) \text{ and } R_{\tilde{f}}(x, y) := \tilde{f}(x)(y).$$

Then, we can obtain

$$R_{\tilde{f}}|_x = \tilde{f}(x) \text{ and } R_{\tilde{f}}|_x(y) = R_{\tilde{f}}(x, y) = \tilde{f}(x)(y).$$

Moreover, if an intuitionistic fuzzy relation R satisfies $R|_x = \tilde{f}(x)$, then we have

$$R(x, y) = R|_x(y) = \tilde{f}(x)(y) = R_{\tilde{f}}|_x(y).$$

Therefore, the established relation $R_{\tilde{f}}$ is unique.

On the other hand, if $R \in IFR(X \times Y)$, then we let $\tilde{f}_R(x)(y) := R(x, y)$.

Thus, for all $y \in Y$, $\tilde{f}_R(x) = R|_x$, we can assume that

$$\tilde{f}_R : X \rightarrow IFS(Y), x \mapsto \tilde{f}_R(x) = R|_x \in IFS(Y).$$

It is obvious that \tilde{f}_R satisfying $R|_x = \tilde{f}(x)$ is an intuitionistic fuzzy mapping, and $\tilde{f}(x)$ is uniquely determined by $R|_x$. Therefore, \tilde{f}_R is an intuitionistic fuzzy mapping, which is uniquely determined by the relation R .

This completes the proof. \square

Next, an example is given to illustrate Theorem 3.1.

Example 3.3. Let $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_m\}$ be two finite sets.

(i) Given an intuitionistic fuzzy mapping $\tilde{f} \in IFS(Y)^X$, namely,

$$x_i \mapsto \tilde{f}(x_i) = \{ \langle u_{i1}, v_{i1} \rangle, \langle u_{i2}, v_{i2} \rangle, \dots, \langle u_{im}, v_{im} \rangle \} \in IFS(Y),$$

it can uniquely establish an intuitionistic fuzzy relation $R_{\tilde{f}}$, i.e.,

$$R_{\tilde{f}} = \begin{pmatrix} \langle u_{11}, v_{11} \rangle & \langle u_{12}, v_{12} \rangle & \cdots & \langle u_{1m}, v_{1m} \rangle \\ \langle u_{21}, v_{21} \rangle & \langle u_{22}, v_{22} \rangle & \cdots & \langle u_{2m}, v_{2m} \rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle u_{n1}, v_{n1} \rangle & \langle u_{n2}, v_{n2} \rangle & \cdots & \langle u_{nm}, v_{nm} \rangle \end{pmatrix}.$$

Obviously, $R_{\tilde{f}}(x_i, y_j) = \langle u_{ij}, v_{ij} \rangle = \tilde{f}(x_i)(y_j)$, and satisfy $0 \leq u_{ij} + v_{ij} \leq 1$.

(ii) Given an intuitionistic fuzzy relation

$$R = \begin{pmatrix} \langle u_{11}, v_{11} \rangle & \langle u_{12}, v_{12} \rangle & \cdots & \langle u_{1m}, v_{1m} \rangle \\ \langle u_{21}, v_{21} \rangle & \langle u_{22}, v_{22} \rangle & \cdots & \langle u_{2m}, v_{2m} \rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle u_{n1}, v_{n1} \rangle & \langle u_{n2}, v_{n2} \rangle & \cdots & \langle u_{nm}, v_{nm} \rangle \end{pmatrix},$$

it can uniquely determine an intuitionistic fuzzy mapping

$$\tilde{f}_R : X \rightarrow IFS(Y),$$

$$x_i \mapsto \tilde{f}_R(x_i) = \{ \langle u_{i1}, v_{i1} \rangle, \langle u_{i2}, v_{i2} \rangle, \dots, \langle u_{im}, v_{im} \rangle \} \in IFS(Y),$$

and fulfill $\tilde{f}_R(x_i)(y_j) = \langle u_{ij}, v_{ij} \rangle = R(x_i, y_j)$.

3.2. Some Operations of Intuitionistic Fuzzy Mappings.

For all $\tilde{f}, \tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(\gamma)} \in IFS(Y)^X$ ($\gamma \in \Gamma$), Γ denotes index set, we can define the following operations for intuitionistic fuzzy mappings.

(a) Union:

$$\tilde{f}^{(1)} \cup \tilde{f}^{(2)} : X \rightarrow IFS(Y), x \mapsto (\tilde{f}^{(1)} \cup \tilde{f}^{(2)})(x) := \tilde{f}^{(1)}(x) \vee \tilde{f}^{(2)}(x);$$

(b) Intersection:

$$\tilde{f}^{(1)} \cap \tilde{f}^{(2)} : X \rightarrow IFS(Y), x \mapsto (\tilde{f}^{(1)} \cap \tilde{f}^{(2)})(x) := \tilde{f}^{(1)}(x) \wedge \tilde{f}^{(2)}(x);$$

(c) Complement:

$$\mathfrak{C}\tilde{f} : X \rightarrow IFS(Y), x \mapsto \mathfrak{C}\tilde{f}(x) := \mathfrak{C}(\tilde{f}(x));$$

(d) Arbitrary union:

$$\bigcup_{\gamma \in \Gamma} \tilde{f}^{(\gamma)} : X \rightarrow IFS(Y), x \mapsto \left(\bigcup_{\gamma \in \Gamma} \tilde{f}^{(\gamma)} \right)(x) := \bigvee_{\gamma \in \Gamma} \tilde{f}^{(\gamma)}(x);$$

(e) Arbitrary intersection:

$$\bigcap_{\gamma \in \Gamma} \tilde{f}^{(\gamma)} : X \rightarrow IFS(Y), x \mapsto \left(\bigcap_{\gamma \in \Gamma} \tilde{f}^{(\gamma)} \right)(x) := \bigwedge_{\gamma \in \Gamma} \tilde{f}^{(\gamma)}(x).$$

Notice that the intuitionistic fuzzy mapping is defined by every element of X , and the value of each mapping is an intuitionistic fuzzy set for every element. Therefore, the intuitionistic fuzzy mappings have the same operation rules as the intuitionistic fuzzy sets. If we define the *minimum element* $\tilde{0}$ and *maximum element* $\tilde{1}$ in $IFS(Y)^X$, respectively,

$$\tilde{0} : X \rightarrow IFS(Y), x \mapsto \tilde{0}(x) := \{ \langle y, 0, 1 \rangle \mid y \in Y \},$$

$$\tilde{1} : X \rightarrow IFS(Y), x \mapsto \tilde{1}(x) := \{ \langle y, 1, 0 \rangle \mid y \in Y \},$$

then it is easy to obtain the following Theorem.

Theorem 3.4. *The algebraic system $(IFS(Y)^X, \cup, \cap, \mathfrak{C})$ is a soft algebra.*

It should be pointed out that the system $(IFS(Y)^X, \cup, \cap, \mathfrak{C})$ is not a Boolean algebra, because, in general, the complementary law cannot be established.

In 1989, Atanassov [2] defined an operator which can change an intuitionistic fuzzy set into a fuzzy set. Later, Burillo and Bustince [6, 7, 8, 9] called this operator the *Atanassov's operator* and studied some properties of this operator.

Next, we will define the same operator on $IFS(Y)^X$ and discuss some basic properties of this operator.

Let $\alpha \in [0, 1]$, for the intuitionistic fuzzy mapping $\tilde{f} \in IFS(Y)^X$, the operator $D_\alpha(\tilde{f})$ can be defined as follows:

$$D_\alpha(\tilde{f})(x) = \{ \langle y, (\triangleleft \tilde{f})_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) + (1 - \alpha) \cdot (\diamond \tilde{f})_x(y) \rangle \mid y \in Y \},$$

where $(\diamond \tilde{f})_x(y) = 1 - (\triangleleft \tilde{f})_x(y) - (\tilde{f} \triangleright)_x(y)$.

Obviously, for every $x \in X$, $D_\alpha(\tilde{f})(x)$ is a fuzzy set of Y , because

$$\begin{aligned} & (\triangleleft \tilde{f})_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y) + (\tilde{f} \triangleright)_x(y) + (1 - \alpha) \cdot (\diamond \tilde{f})_x(y) \\ &= (\triangleleft \tilde{f})_x(y) + (\tilde{f} \triangleright)_x(y) + (\diamond \tilde{f})_x(y) = 1 \text{ for all } y \in Y. \end{aligned}$$

Based on the above analysis, we can see that $D_\alpha(\tilde{f})$ is a fuzzy mapping from X to Y , which was defined in [13, 18].

The following theorem shows some important properties of the operator $D_\alpha(\tilde{f})$.

Theorem 3.5. *Let $\tilde{f}, \tilde{g} \in IFS(Y)^X$ be two intuitionistic fuzzy mappings, for every $\alpha, \beta \in [0, 1]$, the following conclusions hold.*

- (a) if $\alpha \leq \beta$, then $D_\alpha(\tilde{f}) \leq D_\beta(\tilde{f})$; (b) if $\tilde{f} \leq \tilde{g}$, then $D_\alpha(\tilde{f}) \leq D_\alpha(\tilde{g})$;
- (c) $D_\alpha(\tilde{f} \cup \tilde{g}) \geq D_\alpha(\tilde{f}) \cup D_\alpha(\tilde{g})$; (d) $D_\alpha(\tilde{f} \cap \tilde{g}) \leq D_\alpha(\tilde{f}) \cap D_\alpha(\tilde{g})$;
- (e) $D_\alpha(D_\beta(\tilde{f})) = D_\beta(\tilde{f})$; (f) $\mathbb{C}(D_\alpha(\mathbb{C}\tilde{f})) = D_{1-\alpha}(\tilde{f})$.

Proof. (a) It follows from the previous definition of the operator.

(b) If $\tilde{f} \leq \tilde{g}$, then for every $x \in X$, we have

$$(\triangleleft \tilde{f})_x(y) \leq (\triangleleft \tilde{g})_x(y) \text{ and } (\tilde{f} \triangleright)_x(y) \geq (\tilde{g} \triangleright)_x(y) \text{ for all } y \in Y.$$

Since

$$\begin{aligned} & (\triangleleft \tilde{f})_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y) - [(\triangleleft \tilde{g})_x(y) + \alpha \cdot (\diamond \tilde{g})_x(y)] \\ &= (\triangleleft \tilde{f})_x(y) + \alpha \cdot [1 - (\triangleleft \tilde{f})_x(y) - (\tilde{f} \triangleright)_x(y)] - [(\triangleleft \tilde{g})_x(y) + \alpha \cdot [1 - (\triangleleft \tilde{g})_x(y) - (\tilde{g} \triangleright)_x(y)]] \\ &= (1 - \alpha) \cdot [(\triangleleft \tilde{f})_x(y) - (\triangleleft \tilde{g})_x(y)] + \alpha \cdot [(\tilde{g} \triangleright)_x(y) - (\tilde{f} \triangleright)_x(y)] \leq 0. \end{aligned}$$

Hence, we can obtain that $(\triangleleft \tilde{f})_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y) \leq (\triangleleft \tilde{g})_x(y) + \alpha \cdot (\diamond \tilde{g})_x(y)$.

Similarly, we can prove that $(\tilde{f} \triangleright)_x(y) + (1 - \alpha) \cdot (\diamond \tilde{f})_x(y) \geq (\tilde{g} \triangleright)_x(y) + (1 - \alpha) \cdot (\diamond \tilde{g})_x(y)$.

Therefore, for every $x \in X$, $D_\alpha(\tilde{f})(x) \leq D_\alpha(\tilde{g})(x)$, i.e., $D_\alpha(\tilde{f}) \leq D_\alpha(\tilde{g})$.

(c) Since $\tilde{f} \leq \tilde{f} \cup \tilde{g}$ and $\tilde{g} \leq \tilde{f} \cup \tilde{g}$, according to the conclusion (b), we have

$$D_\alpha(\tilde{f}) \leq D_\alpha(\tilde{f} \cup \tilde{g}) \text{ and } D_\alpha(\tilde{g}) \leq D_\alpha(\tilde{f} \cup \tilde{g}).$$

Hence, we obtain $D_\alpha(\tilde{f} \cup \tilde{g}) \geq D_\alpha(\tilde{f}) \cup D_\alpha(\tilde{g})$.

(d) It can be proved in a similar way as the previous one.

(e) For every $x \in X$,

$$\begin{aligned}
& D_\alpha(D_\beta(\tilde{f})(x)) \\
&= D_\alpha(\{\langle y, (\triangleleft \tilde{f})_x(y) + \beta \cdot (\diamond \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) + (1 - \beta) \cdot (\diamond \tilde{f})_x(y) \rangle \mid y \in Y\}) \\
&= \{\langle y, (\triangleleft \tilde{f})_x(y) + \beta \cdot (\diamond \tilde{f})_x(y) + \alpha \cdot [1 - (\triangleleft \tilde{f})_x(y) - \beta \cdot (\diamond \tilde{f})_x(y) - (\tilde{f} \triangleright)_x(y) - (1 - \beta) \cdot (\diamond \tilde{f})_x(y)], \\
&\quad (\tilde{f} \triangleright)_x(y) + (1 - \beta) \cdot (\diamond \tilde{f})_x(y) + (1 - \alpha) \cdot [1 - (\triangleleft \tilde{f})_x(y) - \beta \cdot (\diamond \tilde{f})_x(y) - (\tilde{f} \triangleright)_x(y) - \\
&\quad (1 - \beta) \cdot (\diamond \tilde{f})_x(y)] \rangle \mid y \in Y\} \\
&= \{\langle y, (\triangleleft \tilde{f})_x(y) + \beta \cdot (\diamond \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) + (1 - \beta) \cdot (\diamond \tilde{f})_x(y) \rangle \mid y \in Y\} \\
&= D_\beta(\tilde{f})(x).
\end{aligned}$$

(f) For every $x \in X$,

$$\begin{aligned}
& \mathbb{C}(D_\alpha(\mathbb{C}\tilde{f})(x)) \\
&= \mathbb{C}(\{\langle y, (\tilde{f} \triangleright)_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y), (\triangleleft \tilde{f})_x(y) + (1 - \alpha) \cdot (\diamond \tilde{f})_x(y) \rangle \mid y \in Y\}) \\
&= \{\langle y, (\triangleleft \tilde{f})_x(y) + (1 - \alpha) \cdot (\diamond \tilde{f})_x(y), (\tilde{f} \triangleright)_x(y) + \alpha \cdot (\diamond \tilde{f})_x(y) \rangle \mid y \in Y\} \\
&= D_{1-\alpha}(\tilde{f})(x). \quad \square
\end{aligned}$$

4. Truncation Mappings and Two Basic Theorems of Intuitionistic Fuzzy Mappings

In this section, based on the cut sets (relations) of an intuitionistic fuzzy set (relation) proposed by Liu and Zhou in [15, 25], we will introduce the concept of truncation mapping of an intuitionistic fuzzy mapping, and then present some properties of the truncation mapping. Finally, the decomposition theorem and representation theorem of an intuitionistic fuzzy mapping are established, respectively.

Definition 4.1. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, the following mappings,

$$\begin{aligned}
\tilde{f}_{\langle \alpha, \beta \rangle} &: X \rightarrow P(Y), x \rightarrow \tilde{f}_{\langle \alpha, \beta \rangle}(x) := (\tilde{f}(x))_{\langle \alpha, \beta \rangle} \\
&= \{y \in Y \mid (\triangleleft \tilde{f})_x(y) \geq \alpha, (\tilde{f} \triangleright)_x(y) \leq \beta\}, \\
\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} &: X \rightarrow P(Y), x \rightarrow \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle}(x) := (\tilde{f}(x))_{\langle \underline{\alpha}, \underline{\beta} \rangle} \\
&= \{y \in Y \mid (\triangleleft \tilde{f})_x(y) > \alpha, (\tilde{f} \triangleright)_x(y) < \beta\}, \\
\tilde{f}_{\langle \underline{\alpha}, \beta \rangle} &: X \rightarrow P(Y), x \rightarrow \tilde{f}_{\langle \underline{\alpha}, \beta \rangle}(x) := (\tilde{f}(x))_{\langle \underline{\alpha}, \beta \rangle} \\
&= \{y \in Y \mid (\triangleleft \tilde{f})_x(y) > \alpha, (\tilde{f} \triangleright)_x(y) \leq \beta\}, \\
\tilde{f}_{\langle \alpha, \underline{\beta} \rangle} &: X \rightarrow P(Y), x \rightarrow \tilde{f}_{\langle \alpha, \underline{\beta} \rangle}(x) := (\tilde{f}(x))_{\langle \alpha, \underline{\beta} \rangle} \\
&= \{y \in Y \mid (\triangleleft \tilde{f})_x(y) \geq \alpha, (\tilde{f} \triangleright)_x(y) < \beta\}
\end{aligned}$$

be called the $\langle \alpha, \beta \rangle$ -truncation mapping, $\langle \alpha, \beta \rangle$ -strong truncation mapping, $\langle \underline{\alpha}, \underline{\beta} \rangle$ -truncation mapping and $\langle \alpha, \underline{\beta} \rangle$ -truncation mapping of \tilde{f} , respectively, where $P(Y)$ denotes the power set of Y .

From Definition 4.1, it is easy to see that each truncation mapping of an intuitionistic fuzzy mapping is a set-valued mapping from X to Y . Similar to the properties of the cut sets of intuitionistic fuzzy sets, given by Liu [15], we can obtain the following properties of the truncation mappings.

Theorem 4.2. Let $\tilde{f}, \tilde{g} \in IFS(Y)^X$ be two intuitionistic fuzzy mappings, for all $\langle \alpha, \beta \rangle \in I^2$, the following expressions hold.

- (i) $\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \subseteq \tilde{f}_{\langle \alpha, \beta \rangle} \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$; (ii) $\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \subseteq \tilde{f}_{\langle \alpha, \underline{\beta} \rangle} \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$;
- (iii) $\tilde{f} \subseteq \tilde{g} \Rightarrow \tilde{f}_{\langle \alpha, \beta \rangle} \subseteq \tilde{g}_{\langle \alpha, \beta \rangle}$; (iv) $(\tilde{f} \cap \tilde{g})_{\langle \alpha, \beta \rangle} = \tilde{f}_{\langle \alpha, \beta \rangle} \cap \tilde{g}_{\langle \alpha, \beta \rangle}$;
- (v) $(\tilde{f} \cup \tilde{g})_{\langle \alpha, \beta \rangle} \supseteq \tilde{f}_{\langle \alpha, \beta \rangle} \cup \tilde{g}_{\langle \alpha, \beta \rangle}$; (vi) $\tilde{f}_{\langle 0, 1 \rangle} = Y, \tilde{f}_{\langle 1, 0 \rangle} = \emptyset$.

Theorem 4.3. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, if $\langle \alpha_1, \beta_1 \rangle \leq \langle \alpha_2, \beta_2 \rangle \in I^2$, then

- (i) $\tilde{f}_{\langle \alpha_2, \beta_2 \rangle} \subseteq \tilde{f}_{\langle \alpha_1, \beta_1 \rangle}$; (ii) $\tilde{f}_{\langle \underline{\alpha}_2, \underline{\beta}_2 \rangle} \subseteq \tilde{f}_{\langle \underline{\alpha}_1, \underline{\beta}_1 \rangle}$; (iii) $\tilde{f}_{\langle \alpha_2, \beta_2 \rangle} \subseteq \tilde{f}_{\langle \underline{\alpha}_1, \underline{\beta}_1 \rangle}$;
- (iv) $\tilde{f}_{\langle \underline{\alpha}_2, \underline{\beta}_2 \rangle} \subseteq \tilde{f}_{\langle \underline{\alpha}_1, \underline{\beta}_1 \rangle}$; (v) $\tilde{f}_{\langle \alpha_2, \underline{\beta}_2 \rangle} \subseteq \tilde{f}_{\langle \alpha_1, \underline{\beta}_1 \rangle}$.

Theorem 4.4. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, we have

- (i) $\tilde{f}_{\langle \alpha, \beta \rangle} = \bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} \tilde{f}_{\langle \lambda, \mu \rangle} \quad (\langle \alpha, \beta \rangle \neq \langle 0, 1 \rangle)$;
- (ii) $\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} = \bigcup_{\substack{\lambda > \alpha \\ \mu < \beta}} \tilde{f}_{\langle \lambda, \mu \rangle} \quad (\langle \alpha, \beta \rangle \neq \langle 1, 0 \rangle)$.

Next, we will define a special operation between I^2 and $IFS(Y)^X$, which is similar to the one between a number $\lambda \in I$ and a fuzzy mapping \tilde{f} in the fuzzy set theory [16].

Definition 4.5. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for every $\langle \alpha, \beta \rangle \in I^2$, the operation $\langle \alpha, \beta \rangle \cdot \tilde{f}$ is defined as follows

$$\langle \alpha, \beta \rangle \cdot \tilde{f} : X \rightarrow IFS(Y), x \rightarrow (\langle \alpha, \beta \rangle \cdot \tilde{f})(x) := \langle \alpha, \beta \rangle \cdot \tilde{f}(x),$$

where $\langle \alpha, \beta \rangle \cdot \tilde{f}(x) = \{ \langle y, \alpha \wedge (\triangleleft \tilde{f})_x(y), \beta \vee (\tilde{f} \triangleright)_x(y) \rangle \mid y \in Y \}$.

Especially, if $\tilde{f}(x) \in P(Y)$, we have

$$\langle \alpha, \beta \rangle \cdot \tilde{f}(x) = \begin{cases} \langle y, \alpha, \beta \rangle, & y \in \tilde{f}(x), \\ \langle y, 0, 1 \rangle, & y \notin \tilde{f}(x). \end{cases}$$

Based on Definition 4.2, the following two Theorems can be easily obtained. It should be noted that we omit the proof of these two Theorems, since these conclusions are similar with those in [15].

Theorem 4.6. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, we have

$$\begin{aligned} \text{(i)} \quad \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \alpha, \beta \rangle}; & \text{(ii)} \quad \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle}; \\ \text{(iii)} \quad \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle}; & \text{(iv)} \quad \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \alpha, \underline{\beta} \rangle}. \end{aligned}$$

Theorem 4.7. Let $\tilde{f}, \tilde{g} \in IFS(Y)^X$ be two intuitionistic fuzzy mappings, for all $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, the following conclusions hold.

$$\begin{aligned} \text{(i)} \quad \langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle &\Rightarrow \langle \alpha_1, \beta_1 \rangle \cdot \tilde{f} \leq \langle \alpha_2, \beta_2 \rangle \cdot \tilde{f}; \\ \text{(ii)} \quad \tilde{f} \leq \tilde{g} &\Rightarrow \langle \alpha_1, \beta_1 \rangle \cdot \tilde{f} \leq \langle \alpha_1, \beta_1 \rangle \cdot \tilde{g}. \end{aligned}$$

Definition 4.8. Suppose that

$$\begin{aligned} H &: I^2 \times X \rightarrow P(Y), \\ \langle \alpha, \beta \rangle \times x &\mapsto H(\alpha, \beta)(x), \end{aligned}$$

for all $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, if $\langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle$, and the mapping H satisfies

$$H(\alpha_1, \beta_1)(x) \supseteq H(\alpha_2, \beta_2)(x), \quad \forall x \in X \quad (\text{Abbreviated as } H(\alpha_1, \beta_1) \supseteq H(\alpha_2, \beta_2)),$$

we then call H the *binary set-valued nested mapping* from X to Y . The set of all binary set-valued nested mappings from X to Y is denoted by $BSNM(X, Y)$.

Theorem 4.9. (*Decomposition Theorem*) Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, $H \in BSNM(X, Y)$, for all $\langle \alpha, \beta \rangle \in I^2$, if $\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$, then

$$\begin{aligned} \text{(i)} \quad \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta); \\ \text{(ii)} \quad \langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle &\Rightarrow H(\alpha_1, \beta_1) \supseteq H(\alpha_2, \beta_2); \\ \text{(iii)} \quad \tilde{f}_{\langle \alpha, \beta \rangle} &= \bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} H(\lambda, \mu) \quad (\langle \alpha, \beta \rangle \neq \langle 0, 1 \rangle); \\ \text{(iv)} \quad \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} &= \bigcup_{\substack{\lambda > \alpha \\ \mu < \beta}} H(\lambda, \mu) \quad (\langle \alpha, \beta \rangle \neq \langle 1, 0 \rangle). \end{aligned}$$

Proof. (i) $\tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle} \Rightarrow \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \leq \langle \alpha, \beta \rangle \cdot H(\alpha, \beta) \leq \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \alpha, \beta \rangle}$

$$\begin{aligned} \Rightarrow \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \underline{\alpha}, \underline{\beta} \rangle} \leq \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta) \leq \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot \tilde{f}_{\langle \alpha, \beta \rangle} \\ &= \tilde{f} \\ \Rightarrow \tilde{f} &= \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta). \end{aligned}$$

$$\text{(ii)} \quad \langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle \Rightarrow H(\alpha_1, \beta_1) \supseteq \tilde{f}_{\langle \underline{\alpha}_1, \underline{\beta}_1 \rangle} \supseteq \tilde{f}_{\langle \underline{\alpha}_2, \underline{\beta}_2 \rangle} \supseteq H(\alpha_2, \beta_2).$$

(iii) Firstly, for all $\langle \lambda, \mu \rangle < \langle \alpha, \beta \rangle$, we have $H(\lambda, \mu) \supseteq \tilde{f}_{\langle \underline{\lambda}, \underline{\mu} \rangle} \supseteq \tilde{f}_{\langle \alpha, \beta \rangle} < \langle \alpha, \beta \rangle \neq \langle 0, 1 \rangle$.

Thus, $\bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} H(\lambda, \mu) \supseteq \tilde{f}_{\langle \alpha, \beta \rangle}$. On the other hand, $\bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} H(\lambda, \mu) \subseteq \bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} \tilde{f}_{\langle \lambda, \mu \rangle} = \tilde{f}_{\langle \alpha, \beta \rangle}$ (\langle

$\alpha, \beta \rangle \neq \langle 0, 1 \rangle$).

Hence, we obtain that $\tilde{f}_{\langle \alpha, \beta \rangle} = \bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} H(\lambda, \mu)$.

(iv) This conclusion can be easily verified by the same method used in (iii). \square

Based on Theorems 4.1, 4.4 and 4.6, we easily obtain the following corollaries.

Corollary 4.10. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, $H \in BSNM(X, Y)$, for all $\langle \alpha, \beta \rangle \in I^2$, if $\tilde{f}_{\langle \alpha, \beta \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$, then $\tilde{f} = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta)$.

Corollary 4.11. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, $H \in BSNM(X, Y)$, if $\tilde{f}_{\langle \alpha, \underline{\beta} \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$, then $\tilde{f} = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta)$.

Corollary 4.12. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, $H \in BSNM(X, Y)$, if $\tilde{f}_{\langle \underline{\alpha}, \beta \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$, then $\tilde{f} = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta)$.

Corollary 4.13. Let $\tilde{f} \in IFS(Y)^X$ be an intuitionistic fuzzy mapping, for all $\langle \alpha, \beta \rangle \in I^2$, $H \in BSNM(X, Y)$, if $\tilde{f}_{\langle \alpha, \underline{\beta} \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \underline{\beta} \rangle}$, then $\tilde{f} = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta)$.

Theorem 4.14. (Representation Theorem) Suppose that $H \in BSNM(X, Y)$, the binary set-valued nested mapping H can uniquely establish an intuitionistic fuzzy mapping \tilde{f} from X to Y , which satisfies

- (i) $\tilde{f}_{\langle \alpha, \beta \rangle} \subseteq H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$;
- (ii) $\tilde{f}_{\langle \alpha, \beta \rangle} = \bigcap_{\substack{\lambda < \alpha \\ \mu > \beta}} H(\lambda, \mu)$ ($\langle \alpha, \beta \rangle \neq \langle 0, 1 \rangle$);
- (iii) $\tilde{f}_{\langle \alpha, \beta \rangle} = \bigcup_{\substack{\lambda > \alpha \\ \mu < \beta}} H(\lambda, \mu)$ ($\langle \alpha, \beta \rangle \neq \langle 1, 0 \rangle$).

Proof. For all $H \in BSNM(X, Y)$, if we take $\tilde{f} = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle \cdot H(\alpha, \beta)$, then $\tilde{f} \in IFS(Y)^X$, namely, \tilde{f} is an intuitionistic fuzzy mapping from X to Y .

(i) According to Definition 4.3, for every $x \in X$, $\langle \alpha, \beta \rangle \in I^2$, we know that $H(\alpha, \beta)(x) \subseteq Y$. Therefore, $H(\alpha, \beta)$ can also be viewed as an intuitionistic fuzzy

set and can be expressed as $H(\alpha, \beta)(x) = \{ \langle y, \mu_{H(\alpha, \beta)(x)}(y), \nu_{H(\alpha, \beta)(x)}(y) \rangle \mid y \in Y \}$. Obviously, we can obtain the following equivalence relations

$$\begin{aligned} y \in H(\alpha, \beta)(x) &\Leftrightarrow \mu_{H(\alpha, \beta)(x)}(y) = 1 \ \& \ \nu_{H(\alpha, \beta)(x)}(y) = 0, \\ y \notin H(\alpha, \beta)(x) &\Leftrightarrow \mu_{H(\alpha, \beta)(x)}(y) = 0 \ \& \ \nu_{H(\alpha, \beta)(x)}(y) = 1. \end{aligned}$$

Next, we will prove (i). First of all, for every $x \in X$,

$$\begin{aligned} \forall y \in H(\alpha, \beta)(x) &\Rightarrow \mu_{H(\alpha, \beta)(x)}(y) = 1 \ \& \ \nu_{H(\alpha, \beta)(x)}(y) = 0 \\ \Rightarrow \tilde{f}(x) &= \bigvee_{\langle \lambda, \mu \rangle \in I^2} \langle \lambda, \mu \rangle \cdot H(\lambda, \mu)(x) \\ &= \bigvee_{\langle \lambda, \mu \rangle \in I^2} \{ \langle y, \lambda \wedge \mu_{H(\lambda, \mu)(x)}(y), \mu \vee \nu_{H(\lambda, \mu)(x)}(y) \rangle \mid y \in Y \} \\ &\geq \{ \langle y, \alpha \wedge \mu_{H(\alpha, \beta)(x)}(y), \beta \vee \nu_{H(\alpha, \beta)(x)}(y) \rangle \mid y \in Y \} \\ &= \{ \langle y, \alpha, \beta \rangle \mid y \in Y \} \Rightarrow y \in \tilde{f}_{\langle \alpha, \beta \rangle}(x). \end{aligned}$$

Thus, $H(\alpha, \beta) \subseteq \tilde{f}_{\langle \alpha, \beta \rangle}$.

$$\begin{aligned} \text{On the other hand, if } y \notin H(\alpha, \beta)(x) &\Rightarrow \mu_{H(\alpha, \beta)(x)}(y) = 0 \ \& \ \nu_{H(\alpha, \beta)(x)}(y) = 1 \\ &\Rightarrow \forall \langle \lambda, \mu \rangle \geq \langle \alpha, \beta \rangle, \mu_{H(\lambda, \mu)(x)}(y) = 0 \ \& \ \nu_{H(\lambda, \mu)(x)}(y) = 1 \\ \Rightarrow \tilde{f}(x) &= \bigvee_{\langle \lambda, \mu \rangle \in I^2} \langle \lambda, \mu \rangle \cdot H(\lambda, \mu)(x) = \bigvee_{\substack{\lambda < \alpha \\ \mu > \beta}} \langle \lambda, \mu \rangle \cdot H(\lambda, \mu)(x) \\ &\leq \{ \langle y, \alpha, \beta \rangle \mid y \in Y \} \Rightarrow y \notin \tilde{f}_{\langle \alpha, \beta \rangle}(x). \end{aligned}$$

Hence, $\tilde{f}_{\langle \alpha, \beta \rangle} \subseteq H(\alpha, \beta)$. It is obvious that the conclusion (i) holds.

According to the conclusion (i) and Theorem 4.6, the conclusions (ii) and (iii) can be easily proved. \square

5. Conclusions

The intuitionistic fuzzy mappings are viewed as a generalization of fuzzy mappings, which provides an important basis to deal with the fuzzy optimization problems in a better way. In this paper, we defined intuitionistic fuzzy mappings and presented its' relationship relationship with intuitionistic fuzzy relations. The result shows that an intuitionistic fuzzy mapping can uniquely determine an intuitionistic fuzzy relation. On the contrary, given an intuitionistic fuzzy relation, one can uniquely establish an intuitionistic fuzzy mapping. Meantime, we also found that all of intuitionistic fuzzy mappings constitute a soft algebra with respect to the basic operations. Furthermore, the Atanassov's operator was applied to intuitionistic fuzzy mappings and some relevant properties were studied. Finally, the decomposition theorem and representation theorem of intuitionistic fuzzy mappings were given.

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