REVISION OF SIGN DISTANCE METHOD FOR RANKING OF FUZZY NUMBERS

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Abstract. Recently, Abbasbandy and Asady have been proposed a modification of the distance based approach, namely "sign distance method". However, in this paper, it is shown that this method has some drawbacks, i.e., the result is not consistent with human intuition for special cases and this method cannot always logically infer ranking order of the images of the fuzzy numbers. In this paper, we present a revised method which can avoid these problems for ranking fuzzy numbers. Also, we present several properties for revised sign distance method while the original method does not have some of them.

1. Introduction

Ranking fuzzy numbers plays a very important role in linguistic decision making and some other fuzzy application systems. Instance, Kumar et al. [22] have used ranking functions to proposed a method for solving the bi-objective warehouse problems in a fuzzy environment. Also, Mahdavi et al. [27] have proposed a fuzzy number ranking method to determine a fuzzy shortest path for a biobjective shortest path problem in networks with fuzzy arc lengths. Hitherto, many researchers have been proposed various methods for ranking fuzzy numbers. For instance, in 1976 and 1977, Jain [19, 20] proposed a method using the concept of maximizing set to order the fuzzy numbers. Bass and Kwakernaak in 1977 [7] suggested a canonical way to extend the natural ordering of real numbers to fuzzy numbers. In 1978 Dubios and Prade [14] proposed a method based on the maximizing sets to order fuzzy numbers. Adamo 1980 [3] employed the concept of α-cut set in order to introduce α-preference rule. Yager in 1981 [36, 37] proposed four indices which may be used for ordering fuzzy quantities in $[0, 1]$. Bortolan and Degani 1985 [8] have been compared and reviewed some methods to rank fuzzy numbers. In 1988, E. Lee and R.J. Li [24] investigated a method for ranking fuzzy numbers based on the uniform and proportional probability distributions. Chen and Hwang [9] proposed fuzzy multiple attribute decision making in 1992. An index for ordering fuzzy numbers was proposed by Choobineh and H. Li [11] in 1993. K.M. Lee et al. [23] ranked fuzzy numbers with a satisfaction function in 1994. Fortemps and Roubens [16] presented ranking and defuzzication methods using the concept of area compensation in 1996. Cheng 1998 [10] defined the coefficient variance to

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improve E. Lee and R.J. Li's ranking approach [24]. In 2002, Chu and Tsao [12] found some problems in Cheng's method and proposed a method using the area between the centroid and original points to rank fuzzy numbers. Furthermore, in 2006, Y.M. Wang et al. [35] found that the centroid formulae for ranking fuzzy numbers provided by Cheng [10] is incorrect and leads to some misapplications in Chu and Tsao [12]. They presented the correct centroid formulae for ranking fuzzy numbers and justified them from the viewpoint of analytical geometry [35]. After that, Y.J. Wang and S.H. Lee [32] made a revision on ranking fuzzy numbers to improve Chu and Tsao's approach [12]. In 2007, Asady and Zendehnam [6] proposed a defuzzification using minimizer of the distance between the two fuzzy numbers and obtain the nearest point with respect to a fuzzy number. Then, by considering the nearest point, they presented a ranking method for the fuzzy numbers, namely "distance minimization method"[6]. In 2009, Abbasbandy and Hajjari in [2] found a problem of distance minimization method and present a new method by using the concept of magnitude to rank fuzzy numbers. Recently, Asady [5] revised the distance minimization method by using the concept of epsilon-neighborhood of the fuzzy numbers. In 2009, Z.X. Wang et al. [33] proposed an approach to overcome the limitations of the existing studies and simplify the computational procedures based on the LR deviation degree of a fuzzy number. However, there were some problems with the ranking method, as pointed out by Asady [4] in 2010. Also, Asady in his paper [4] proposed a revised method which can avoid these problems of Z.X. Wang et al.'s method [33] for ranking fuzzy numbers. In 2009, Y.M. Wang and Luo [34] proposed an area ranking of fuzzy numbers based on positive and negative ideal points. In 2011, Mohamadi Nejad and Mashinchi [28] presented a new method based on the left and the right sides of fuzzy numbers to rank fuzzy numbers. All the above methods, provide the results of comparison in the form of a real value. But in 2007, Sevastianov [29] used the Dempster-Shafer theory of evidence with its probabilistic interpretation to justify and construct a method which provides the result of comparison in the form of an interval number, which was called "belief interval (BI)".

Regarding to the above-mentioned methods, everybody knows that there is no unique and natural order in a family of all fuzzy numbers and order is generally chosen with respect to particular applications. However, a ranking method should has some reasonable properties. The lists of some reasonable properties can be found in [30, 31]. One of the reasonable properties for ranking methods is that we must be able to logically infer ranking order of the images of the fuzzy numbers (opposite with respect to origin, [21, 38]), that is if $u \preceq v$ then $-v \preceq -u$.

In 2006, Abbasbandy and Asady [1], have been proposed "sign distance method" for ranking fuzzy numbers to overcome shortcomings of some methods, such as: CV index [10], distance between fuzzy sets [10], centroid point and original point [12] and weighted mean value [37]. They considered a fuzzy origin for fuzzy numbers, then according to the distance of fuzzy numbers with respect to this origin they ranked various fuzzy numbers. They revealed that the sign distance method is both efficient to evaluate and able to overcome the shortcomings of the above-mentioned methods. Also, the calculation of the sign distance method is far simpler than the other approaches (see [1]). However, in this paper, we will show that this method has some drawbacks, i.e., the result of sign distance method is not always consistent with human intuition and this method cannot logically infers ranking order of the images of the fuzzy numbers for special cases. In this paper, we shall revise the sign distance method to avoid the mentioned problems. Also, we will prove that unlike sign distance method, we can effectively rank images of fuzzy numbers by the revised sign distance method . We prepare our discussion in 5 sections.

In Section 2, we give some definitions and preliminaries. In Section 3, we describe the sign distance method. In Section 4, we first give some examples to state the problems of the sign distance method and then present a revision of sign distance method to overcome these problems. Finally, conclusions are given in Section 5.

2. **Preliminaries**

The basic definitions of a fuzzy number are given in [15, 38] as follows.

Definition 2.1. A fuzzy number is a fuzzy set $u : \mathbb{R} \to I = [0, 1]$ which satisfies:

1: u is upper semi-continuous,
2: $u(x) = 0$ outside some interval $[a, d]$,
3: There are real numbers b and c such that $a \leqslant b \leqslant c \leqslant d$ and
  i: u is monotonic increasing on $[a, b]$,
  ii: u is monotonic decreasing on $[c, d]$,
  iii: $u(x) = 1$, $b \leqslant x \leqslant c$.

The membership function u can be expressed as

$$u(x) = \begin{cases} u_L(x), & a \leqslant x \leqslant b, \\ 1, & b \leqslant x \leqslant c, \\ u_R(x), & c \leqslant x \leqslant d, \\ 0, & \text{otherwise}, \end{cases}$$

where $u_L : [a, b] \to [0, 1]$ and $u_R : [c, d] \to [0, 1]$ are left and right membership functions of fuzzy number u. An equivalent parametric form is also given in [26] as follows.

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leqslant r \leqslant 1$ which satisfy the following requirements:

1: $\underline{u}$ is a bounded left-continuous non-decreasing function over $[0, 1]$,
2: $\overline{u}$ is a bounded left-continuous non-increasing function over $[0, 1]$,
3: $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

The collection of all fuzzy numbers is denoted by E. An important class of fuzzy numbers are the trapezoidal fuzzy numbers. The trapezoidal fuzzy number u is the fuzzy set in E which is characterized by $u = (u_l, u_c^1, u_c^2, u_r)$ with $u_l \leqslant u_c^1 \leqslant u_c^2 \leqslant u_r$ where the membership function is as

$$u(x) = \begin{cases} \frac{x-u_l}{u_c^1-u_l}, & u_l \leqslant x \leqslant u_c^1, \\ 1, & u_c^1 \leqslant x \leqslant u_c^2, \\ \frac{u_r-x}{u_r-u_c^2}, & u_c^2 \leqslant x \leqslant u_r, \\ 0, & \text{otherwise}, \end{cases}$$

and its parametric form is

$$\underline{u}(r) = u_l + r(u_c^1 - u_l), \qquad \overline{u}(r) = u_r - r(u_r - u_c^2).$$

It needs to point out that when $u_c^1 = u_c^2$, the fuzzy number u is a triangular fuzzy number. Also, membership function of crisp number α is

$$u(x) = \begin{cases} 1, & x = \alpha, \\ 0, & x \neq \alpha, \end{cases}$$

and its parametric form is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leqslant r \leqslant 1$.

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and an arbitrary crisp number k, we define fuzzy addition and scalar multiplication as

1: $(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \qquad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$
2: $(\underline{ku})(r) = k\underline{u}(r), \qquad (\overline{ku})(r) = k\overline{u}(r), \quad k \geqslant 0,$
3: $(\underline{ku})(r) = k\overline{u}(r), \qquad (\overline{ku})(r) = k\underline{u}(r), \quad k < 0.$

Definition 2.3. Image (opposite) of the $u = (u_l, u_c^1, u_c^2, u_r)$ is defined by $-u = (-u_r, -u_c^2, -u_c^1, -u_l)$, see [21, 38].

Definition 2.4. [25] A fuzzy number $u = (\underline{u}(r), \overline{u}(r))$, $r \in [0, 1]$ is said to be symmetric fuzzy number if it satisfies $\underline{u}(r) = -\overline{u}(r)$.

Remark 2.5. If $u = (\underline{u}(r), \overline{u}(r))$, $r \in [0, 1]$ is a symmetric fuzzy number, then the image of u is u itself, i.e., $-u = u$.

Definition 2.6. For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$, the function

$$D_p(u, v) = \left[\int_0^1 |\underline{u}(r) - \underline{v}(r)|^p \, dr + \int_0^1 |\overline{u}(r) - \overline{v}(r)|^p \, dr \right]^{\frac{1}{p}}, \qquad p \geqslant 1,$$

is the distance between u and v. Indeed, the function D_p is a metric in E [13, 17, 18].

3. **Sign Distance Method**

In this section, we briefly describe the sign distance method proposed by Abbasbandy and Asady [1]. We consider u_0 as a fuzzy origin as follows:

$$u_0(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

and consequently $\underline{u_0}(r) = \overline{u_0}(r) = 0$. Therefore, for each $u \in E$

$$D_p(u, u_0) = \left[\int_0^1 |\underline{u}(r)|^p \, dr + \int_0^1 |\overline{u}(r)|^p \, dr \right]^{\frac{1}{p}}, \qquad p \geqslant 1.$$

Remark 3.1. For any $u \in E$, it is clear that $D_p(u, u_0) = D_p(-u, u_0)$.

Definition 3.2. [1] Let $\gamma : E \to \{-1, 1\}$ be a function that is defined as follows:

$$\gamma(u) = \begin{cases} 1, & \text{if } \int_0^1 \{\underline{u}(r) + \overline{u}(r)\} \, dr \geqslant 0, \\ -1, & \text{if } \int_0^1 \{\underline{u}(r) + \overline{u}(r)\} \, dr < 0. \end{cases} \tag{1}$$

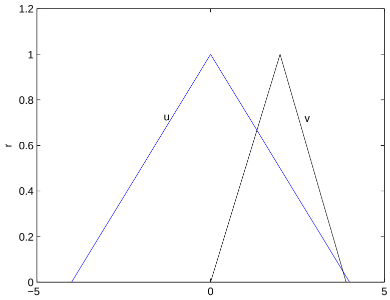

FIGURE 1. Graphical Representation of Fuzzy Numbers u and v Presented in Example 4.1

Definition 3.3. [1] For $u \in E$, index $d_p(u, u_0) = \gamma(u) \cdot D_p(u, u_0)$ is called sign distance.

Remark 3.4. If u is a symmetric fuzzy number, then $\gamma(u) = \gamma(-u) = 1$ and $d_p(u, u_0) = d_p(-u, u_0) \geqslant 0$ for each $p \geqslant 1$.

Definition 3.5. [1] For u and $v \in E$, we define the ranking of u and v on E by d_p, i.e.

$$d_p(u, u_0) > d_p(v, u_0) \text{ if and only if } u \succ v,$$
$$d_p(u, u_0) < d_p(v, u_0) \text{ if and only if } u \prec v,$$
$$d_p(u, u_0) = d_p(v, u_0) \text{ if and only if } u \sim v.$$

Then, we formulate the order " $\succeq$ " and " $\preceq$ " as $u \succeq v$ if and only if $u \succ v$ or $u \sim v$ and $u \preceq v$ if and only if $u \prec v$ or $u \sim v$.

4. Problems and Revision of Sign Distance Method

In this section, we first give two simple examples to present the problems of sign distance method [1].

Example 4.1. Consider the triangular fuzzy numbers $u = (-4, 0, 0, 4)$ and $v = (0, 2, 2, 3.9)$ indicated in Figure 1. Intuitively, the ranking order is $u \prec v$. The results obtained by sign distance method with various values of p are shown in Table 1. Obviously, the results obtained by sign distance method are unreasonable and are not consistent with human intuition. Therefore, sign distance method has inconsistency in ranking fuzzy numbers for special cases.

Example 4.2. Consider the trapezoidal fuzzy number $u = (-2, -1, 1, 2)$ and the triangle fuzzy number $v = (-2, -1, -1, 0)$ indicated in Figure 2. Intuitively, the ranking orders for u, v and their images are $v \prec u$ and $-u \prec -v$. The ranking orders obtained by sign distance method with various values of p for u, v and their images are presented in Table 2. Obviously, by sign distance method, we cannot logically infer ranking order of the images of these fuzzy numbers.

p	u	v	Result
1	4.0000	3.9500	$v \prec u$
2	3.2660	3.2151	$v \prec u$
3	3.1748	3.1187	$v \prec u$
4	3.1811	3.1204	$v \prec u$
5	3.2120	3.1464	$v \prec u$

TABLE 1. The Results Obtained by Sign Distance Method with Various Values of p for Example 4.1

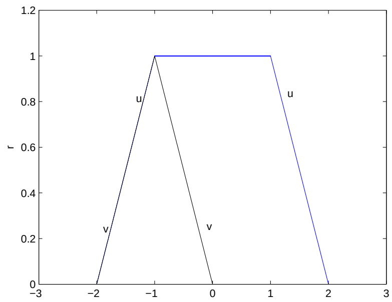

FIGURE 2. Graphical Representation of Fuzzy Numbers u and v Presented in Example 4.2

In the real world, we can illustrate many examples as the above situations again. Consequently, by sign distance method we may obtain the results which are inconsistent with human intuition and also we may not effectively rank images of fuzzy numbers, that are the main problems of this method.

Definition 4.3. We say that a ranking method M satisfies the image property if $u \preceq v$ by M implies $-v \preceq -u$ by M and conversely.

In the following, we indicate the problem of sign distance method as a theorem.

Theorem 4.4. *Let u and v be symmetric and non-symmetric fuzzy numbers, respectively. Also, suppose that $d_p(u, u_0) = \alpha$ and $d_p(v, u_0) = \beta$, where $p \geqslant 1$ is fixed.*

p	u	v	$-u$	$-v$	Results
1	3.0000	-2.0000	3.0000	2.0000	$v \preceq u$ and $-v \preceq -u$
2	2.1602	-1.6330	2.1602	1.6330	$v \preceq u$ and $-v \preceq -u$
3	1.9574	-1.5874	1.9574	1.5874	$v \preceq u$ and $-v \preceq -u$
4	1.8765	-1.5905	1.8765	1.5905	$v \preceq u$ and $-v \preceq -u$
5	1.8384	-1.6055	1.8384	1.6055	$v \preceq u$ and $-v \preceq -u$

TABLE 2. The Results Obtained by Sign Distance Method with Various Values of p for Example 4.2

If $0 < \beta < \alpha$ *or* $\beta < 0 < \alpha$ *such that* $|\beta| < |\alpha|$, *then sign distance method does not satisfy the image property.*

Proof. Since u is symmetric, we have by Remark 3.4 that $d_p(u, u_0) = d_p(-u, u_0) = \alpha$. On the other hand, based on the Definitions 3.2 and 3.3, $d_p(v, u_0) = \beta$ implies that $d_p(-v, u_0) = -\beta$. At first, we assume that $0 < \beta < \alpha$. Obviously $v \prec u$ and $-\alpha < -\beta < 0 < \alpha$ implies $-v \prec -u$. Thus, the image property does not hold. Now, suppose that $\beta < 0 < \alpha$ such that $|\beta| < |\alpha|$. Clearly $v \prec u$ and $-\alpha < 0 < -\beta < \alpha$ that implies $-v \prec -u$. Then, the sign distance method does not satisfy the image property. □

Now, we shall present "revised sign distance method" for ranking fuzzy numbers to derive a method that always satisfies the image property. The fundamental difference between revised method and original sign distance method is the definition of function γ in equation (1). We correct the definition of γ as follows:

Definition 4.5. We define the function $\gamma^* : E \to \{-1, 0, 1\}$ as follows:

$$\gamma^*(u) = \begin{cases} 1, & if \quad \int_0^1 \{\underline{u}(r) + \overline{u}(r)\}\, dr > 0, \\ 0, & if \quad \int_0^1 \{\underline{u}(r) + \overline{u}(r)\}\, dr = 0, \\ -1, & if \quad \int_0^1 \{\underline{u}(r) + \overline{u}(r)\}\, dr < 0. \end{cases}$$

Remark 4.6. For the function γ^*, the following properties hold:

a: If $inf\,\{x : x \in supp(u)\} > 0$ or $inf\,\{\underline{u}(r) : r \in [0, 1]\} > 0$ then $\gamma^*(u) = 1$.
b: If $sup\,\{x : x \in supp(u)\} < 0$ or $sup\,\{\overline{u}(r) : r \in [0, 1]\} < 0$ then $\gamma^*(u) = -1$.
c: If u is a symmetric fuzzy number then $\gamma^*(u) = 0$.
d: If u and λ are arbitrary fuzzy and real numbers, respectively, then

$$\gamma^*(\lambda u) = \begin{cases} \gamma^*(u), & if \quad \lambda > 0, \\ 0, & if \quad \lambda = 0, \\ -\gamma^*(u), & if \quad \lambda < 0. \end{cases}$$

Remark 4.7. According to the above property (d), for arbitrary fuzzy number u, we have $\gamma^*(-u) = -\gamma^*(u)$. Note that for the function γ this property does not hold (see Remark 3.4). Due to this fact, we can conclude that in some cases, the sign distance method does not satisfy the image property.

Definition 4.8. For $u \in E$, index $d_p^*(u, u_0) = \gamma^*(u) \cdot D_p(u, u_0)$ is called revised sign distance.

Following [1], we have:

Definition 4.9. For u and $v \in E$, we define by d_p^* the ranking of u and v on E, i.e.

$d_p^*(u, u_0) > d_p^*(v, u_0)$ if and only if $u \succ v$,
$d_p^*(u, u_0) < d_p^*(v, u_0)$ if and only if $u \prec v$,
$d_p^*(u, u_0) = d_p^*(v, u_0)$ if and only if $u \sim v$.

Also $u \succeq v$ if and only if $u \succ v$ or $u \sim v$, and $u \preceq v$ if and only if $u \prec v$ or $u \sim v$.

In the following theorem, we show that by revised sign distance, we can always effectively rank images of fuzzy numbers.

Theorem 4.10. *For any $u, v \in E$, $u \preceq v$ by d_p^* if and only if $-v \preceq -u$ by d_p^*. That means the revised sign distance method satisfies the image property.*

Proof. By Remarks 4.7 and 3.1, we have

$$\begin{aligned} u \preceq v \quad &\Leftrightarrow \quad d_p^*(u, u_0) \leqslant d_p^*(v, u_0) \\ &\Leftrightarrow \quad \gamma^*(u) \cdot D_p(u, u_0) \leqslant \gamma^*(v) \cdot D_p(v, u_0) \\ &\Leftrightarrow \quad -\gamma^*(-u) \cdot D_p(-u, u_0) \leqslant -\gamma^*(-v) \cdot D_p(-v, u_0) \\ &\Leftrightarrow \quad \gamma^*(-u) \cdot D_p(-u, u_0) \geqslant \gamma^*(-v) \cdot D_p(-v, u_0) \\ &\Leftrightarrow \quad d_p^*(-u, u_0) \geqslant d_p^*(-v, u_0) \\ &\Leftrightarrow \quad -u \succeq -v. \end{aligned}$$

□

Remark 4.11. For any $u \in E$, we have $d_p^*(u, u_0) = -d_p^*(-u, u_0)$. Note that for the function d_p this property does not hold (see Remark 3.4).

Now, we consider two previous examples to show that the revised method can avoid problems of sign distance method.

Example 4.12. Consider the fuzzy numbers of Example 4.1. The results obtained by revised sign distance method and several new methods [2, 4, 5, 6, 33] are presented in Table 3. Clearly, unlike the sign distance method, the ranking order of revised method similar to the ranking orders of other methods are consistent with human intuition.

Example 4.13. Now consider the fuzzy numbers of Example 4.2. The ranking order obtained by revised sign distance method with various values of p and several new methods [2, 4, 5, 6, 33] are shown in Table 4. Obviously, unlike the sign distance

Methods	u	v	Result
Revised sign distance method (p=1)	0.00	3.95	$u \prec v$
Revised sign distance method (p=2)	0.00	3.22	$u \prec v$
Revised sign distance method (p=3)	0.00	3.12	$u \prec v$
Revised sign distance method (p=4)	0.00	3.12	$u \prec v$
Revised sign distance method (p=5)	0.00	3.15	$u \prec v$
Deviation degree method [33]	0.00	2.64	$u \prec v$
Revised deviation degree method [4]	0.89	2.37	$u \prec v$
Distance minimization method [6]	0.00	1.98	$u \prec v$
Revised distance minimization method [5]	0.00	1.96	$u \prec v$
Magnitude method ($f(r) = r$) [2]	0.00	3.98	$u \prec v$

Table 3. The Results Obtained by Revised Sign Distance Method and Other Methods for Example 4.12

Methods	u, $-u$	v	$-v$	Results
Revised sign distance (p=1)	0	-2.0	2.0	$v \prec u$, $-u \prec -v$
Revised sign distance (p=2)	0	-1.6	1.6	$v \prec u$, $-u \prec -v$
Revised sign distance (p=3)	0	-1.6	1.6	$v \prec u$, $-u \prec -v$
Revised sign distance (p=4)	0	-1.6	1.6	$v \prec u$, $-u \prec -v$
Revised sign distance (p=5)	0	-1.6	1.6	$v \prec u$, $-u \prec -v$
Deviation degree [33]	0.4	0.0	1.6	$v \prec u$, $-u \prec -v$
Revised deviation degree [4]	0.8	0.3	2.0	$v \prec u$, $-u \prec -v$
Distance minimization [6]	0	-1.0	1.0	$v \prec u$, $-u \prec -v$
Revised distance minimization [5]	0	-1.00	1.0	$v \prec u$, $-u \prec -v$
Magnitude ($f(r) = r$) [2]	0	-1.0	1.0	$v \prec u$, $-u \prec -v$

Table 4. The Results Obtained by Revised Sign Distance Method and Other Methods for Example 4.13

method, the revised method similar to the other methods can correctly ranks the images of fuzzy numbers.

Remark 4.14. It should be noted that if u is an arbitrary fuzzy number such that $\gamma^*(u) \neq 0$, then for $p \geqslant 1$, we have $d_p^*(u, u_0) = d_p(u, u_0)$. Thus, for those fuzzy numbers that their value of γ^* are non-zero, the ranking result by d_p is the same as the one by d_p^*.

Now, we consider the following reasonable properties for the ordering approaches, see [30, 31].

A_1: For an arbitrary finite subset Γ of E and $u \in \Gamma$, $u \succeq u$.

A_2: For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$, $u \succeq v$ and $v \succeq u$, we should have $u \sim v$.

A_3: For an arbitrary finite subset Γ of E and $(u, v, z) \in \Gamma^3$, $u \succeq v$ and $v \succeq z$, we should have $u \succeq z$.

A_4: For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$ and

$$\inf \{x : x \in supp(u)\} > \sup \{x : x \in supp(v)\},$$

we should have $u \succeq v$.

A'_4: For an arbitrary finite subset Γ of E and $(u, v) \in \Gamma^2$ and

$$\inf \{x : x \in supp(u)\} > \sup \{x : x \in supp(v)\},$$

we should have $u \succ v$.

A_5: Let Γ and Γ' be two arbitrary finite subsets of E in which u and v are in $\Gamma \cap \Gamma'$. We obtain the ranking order $u \succ v$ by d_p^* on Γ' if and only if $u \succ v$ by d_p^* on Γ.

A_6: Let u, v, $u + z$ and $v + z$ be elements of E. If $u \succeq v$, then $u + z \succeq v + z$.

A'_6: Let u, v, $u + z$ and $v + z$ be elements of E. If $u \succ v$, then $u + z \succ v + z$.

Theorem 4.15. *The revised sign distance d_p^*, has the properties $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_5$.*

Proof. Here, we present the proof of properties $\mathbf{A}_4$ and $\mathbf{A}'_4$. The proofs of other properties are clear and very easy. To this end, we consider three possible cases:

1) $0 \notin supp(u)$ and $0 \notin supp(v)$:
In this case, the fuzzy numbers u and v are either positive or negative and thus the proof is immediate.

2) $0 \in supp(u)$:
In this case, we conclude that the fuzzy number v is negative. This means that

$$\sup\{x : x \in supp(v)\} < 0, \qquad and \qquad D_p(v, u_0) \neq 0, \quad \forall p \geqslant 1.$$

Therefore, from Remark 4.6 (b) we deduce that $\gamma^*(v) = -1$. Now, if $\gamma^*(u) \neq 0$ then based on Remark 4.14 and also Remark 2.3 of [1], the proof is completed. Otherwise, if $\gamma^*(u) = 0$ then the proof is obvious, because

$$d_p^*(v, u_0) = \gamma^*(v)\, D_p(v, u_0) < 0 = \gamma^*(u)\, D_p(u, u_0) = d_p^*(u, u_0).$$

3) $0 \in supp(v)$:
In this case, we conclude that the fuzzy number u is positive. In other words

$$\inf\{x : x \in supp(u)\} > 0, \qquad and \qquad D_p(u, u_0) \neq 0, \quad \forall p \geqslant 1.$$

Therefore, Remark 4.6 (a) implies that $\gamma^*(u) = 1$. Now, if $\gamma^*(v) \neq 0$ then based on Remark 4.14 and also Remark 2.3 of [1], the proof is completed. Otherwise, if $\gamma^*(v) = 0$ then the proof is easily obtained, because

$$d_p^*(v, u_0) = \gamma^*(v)\, D_p(v, u_0) = 0 < \gamma^*(u)\, D_p(u, u_0) = d_p^*(u, u_0).$$

Based on the above discussion, we conclude that the property A_4' holds. Consequently, the property A_4 holds, too. □

Theorem 4.16. *The revised sign distance d_p^*, for $p = 1$ has the properties $\boldsymbol{A}_6$ and $\boldsymbol{A}_6'$ if*
$\inf\{x : x \in supp(u) \cup supp(v) \cup supp(u+z) \cup supp(v+z)\} > 0,$
or
$\sup\{x : x \in supp(u) \cup supp(v) \cup supp(u+z) \cup supp(v+z)\} < 0.$

Proof. It is clear that if

$$\inf\{x : x \in supp(u) \cup supp(v) \cup supp(u+z) \cup supp(v+z)\} > 0,$$

or

$$\sup\{x : x \in supp(u) \cup supp(v) \cup supp(u+z) \cup supp(v+z)\} < 0,$$

then $\gamma^*(u)$, $\gamma^*(v)$, $\gamma^*(u+z)$ and $\gamma^*(v+z)$ are non-zero. Therefore, according to the Remark 4.14, the ranking result by d_p^* is the same as the one by d_p. Thus, regarding to the Remark 2.4 of [1], the proof is completed. □

In addition to the above properties, there are other properties for revised sign distance method as follows:

Theorem 4.17. *Let u and v be elements of E. Then we have*

- **$\mathbf{B}_1$:** *(Image property) $u \preceq v$ if and only if $-v \preceq -u$.*
- **$\mathbf{B}_2$:** *If $u \prec v$ then $v \prec u$ cannot hold.*
- **$\mathbf{B}_3$:** *If $u \preceq v$ and λ be a positive real number, then $\lambda u \preceq \lambda v$.*
- **$\mathbf{B}_4$:** *If $u \preceq v$ and λ be a negative real number, then $\lambda v \preceq \lambda u$.*
- **$\mathbf{B}_5$:** *If $u = v$ then $u \sim v$.*

$\mathbf{B}_6$: *If $u \sim v$, it is not necessary that $u = v$.*
$\mathbf{B}_7$: *If u and v are symmetric fuzzy numbers, then $u \sim v$.*
$\mathbf{B}_8$: *(Robustness) If $D_p(u,v) < \varepsilon$ and $\gamma^*(u) = \gamma^*(v) = 1$ or $\gamma^*(u) = \gamma^*(v) = -1$ then $|d_p^*(u,u_0) - d_p^*(v,u_0)| < \varepsilon$.*

Proof. From Theorem 4.10, property $\mathbf{B}_1$ holds. The property $\mathbf{B}_2$ is obvious. For the proof of $\mathbf{B}_3$, using the Remark 4.6 (d), $u \preceq v$ and $\lambda > 0$ we have

$$\begin{aligned} d_p^*(\lambda u, u_0) = \gamma^*(\lambda u)\, D_p(\lambda u, u_0) &= \gamma^*(u)\,|\lambda|\, D_p(u,u_0) \\ &= \lambda\, \gamma^*(u) D_p(u,u_0) \\ &\leq \lambda\, \gamma^*(v) D_p(v,u_0) \\ &= \gamma^*(v)\,|\lambda|\, D_p(v,u_0) \\ &= \gamma^*(\lambda v)\, D_p(\lambda v, u_0) = d_p^*(\lambda v, u_0), \end{aligned}$$

thus $\lambda u \preceq \lambda v$. Similarly $\mathbf{B}_4$ holds. The proof of properties $\mathbf{B}_5$ and $\mathbf{B}_6$ is easy. The property $\mathbf{B}_7$ follows from Remark 4.6 (c). For the proof of $\mathbf{B}_8$, since the function D_p is a metric in E, then

$$D_p(u,u_0) \leq D_p(u,v) + D_p(v,u_0) \quad \Longrightarrow \quad D_p(u,u_0) - D_p(v,u_0) \leq D_p(u,v) \tag{2}$$

$$D_p(v,u_0) \leq D_p(v,u) + D_p(u,u_0) \quad \Longrightarrow \quad D_p(v,u_0) - D_p(u,u_0) \leq D_p(v,u). \tag{3}$$

Thus, from equations (2) and (3) we have

$$|D_p(u,u_0) - D_p(v,u_0)| \leq D_p(u,v). \tag{4}$$

On the other hand, $\gamma^*(u) = \gamma^*(v) = 1$ or $\gamma^*(u) = \gamma^*(v) = -1$ imply

$$|d_p^*(u,u_0) - d_p^*(v,u_0)| = |D_p(u,u_0) - D_p(v,u_0)|. \tag{5}$$

By equations (4) and (5) and regarding to the assumption of theorem, the proof is completed. □

Corollary 4.18. *For any u, $v \in E$, if $D_p(u,v) < \varepsilon$ and*

$$\inf\{x : x \in supp(u) \cup supp(v)\} > 0,$$

or

$$\sup\{x : x \in supp(u) \cup supp(v)\} < 0,$$

then $|d_p^*(u,u_0) - d_p^*(v,u_0)| < \varepsilon$.

Proof. According to the Remark 4.6, if $\inf\{x : x \in supp(u) \cup supp(v)\} > 0$ or $\sup\{x : x \in supp(u) \cup supp(v)\} < 0$ then $\gamma^*(u) = \gamma^*(v) = 1$ or $\gamma^*(u) = \gamma^*(v) = -1$, respectively. Thus, regarding to the property $\mathbf{B}_8$ of Theorem 4.17, the proof is clear. □

To present the rationality and necessity of this revision of sign distance method, five other examples are employed to compare the revised method with the initial one and new methods. These examples are described as follows.

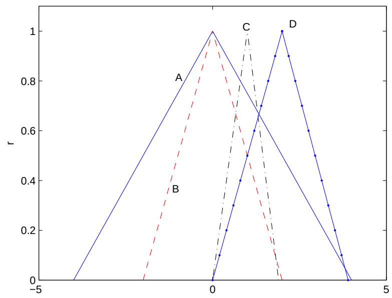

Figure 3. Graphical Representation of Fuzzy Numbers A, B, C and D Presented in Example 4.19

Methods	A	B	C	D	Results
Sign distance (p=1)	4.0	2.0	1.9	3.9	$C \prec B \prec D \prec A$
Sign distance (p=2)	3.3	1.6	1.6	3.2	$C \prec B \prec D \prec A$
Sign distance (p=3)	3.2	1.6	1.5	3.1	$C \prec B \prec D \prec A$
Revised sign distance (p=1)	0.0	0.0	1.9	3.9	$A \sim B \prec C \prec D$
Revised sign distance (p=2)	0.0	0.0	1.6	3.2	$A \sim B \prec C \prec D$
Revised sign distance (p=3)	0.0	0.0	1.5	3.1	$A \sim B \prec C \prec D$
Deviation degree [33]	0.0	0.0	1.4	3.2	$A \sim B \prec C \prec D$
Revised deviation degree [4]	0.9	0.9	1.1	1.4	$A \sim B \prec C \prec D$
Distance minimization [6]	0.0	0.0	1.0	2.0	$A \sim B \prec C \prec D$
Revised distance minimization [5]	0.0	0.0	1.0	2.0	$A \sim B \prec C \prec D$
Magnitude ($f(r) = r$) [2]	0.0	0.0	1.0	2.0	$A \sim B \prec C \prec D$

Table 5. Comparative Results of Example 4.19

Example 4.19. Consider the four fuzzy numbers $A = (-4, 0, 0, 4)$, $B = (-2, 0, 0, 2)$, $C = (0, 1, 1, 1.9)$ and $D = (0, 2, 2, 2.9)$ shown in Figure 3. By the original sign distance method with $p = 1, 2, 3$, the ranking order of these fuzzy numbers is $C \prec B \prec D \prec A$, which is unreasonable and is not consistent with human intuition. While, by the revised sign distance method and the other methods such as: Deviation degree method [33], Revised deviation degree method [4], Distance minimization method [6], Revised distance minimization method [5] and Magnitude method [2] we obtain the ranking order $A \sim B \prec C \prec D$, which is reasonable, see Table 5.

Example 4.20. Consider the trapezoidal fuzzy number $A = (-3, -1, 1, 3)$ and the triangle fuzzy numbers $B = (0, 1.4, 1.4, 3)$, $C = (0, 2.4, 2.4, 3)$, as indicated in Figure 4. By the original sign distance method with $p = 1$, the ranking order is $B \prec C \prec A$ and with $p = 2, 3$ is $B \prec A \prec C$ which both are unreasonable and are not consistent with human intuition. While, by the revised sign distance method and the other methods we get the ranking order $A \prec B \prec C$, which is reasonable, as shown in Table 6.

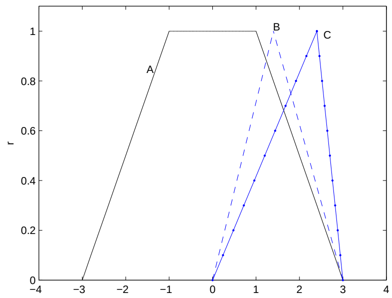

FIGURE 4. Graphical Representation of Fuzzy Numbers A, B and C Presented in Example 4.20

Methods	A	B	C	Results
Sign distance (p=1)	4.00	2.90	3.90	$B \prec C \prec A$
Sign distance (p=2)	2.94	2.39	3.04	$B \prec A \prec C$
Sign distance (p=3)	2.71	2.34	2.86	$B \prec A \prec C$
Revised sign distance (p=1)	0.00	2.90	3.90	$A \prec B \prec C$
Revised sign distance (p=2)	0.00	2.39	3.04	$A \prec B \prec C$
Revised sign distance (p=3)	0.00	2.34	2.86	$A \prec B \prec C$
Deviation degree [33]	0.00	1.74	2.47	$A \prec B \prec C$
Revised deviation degree [4]	0.86	2.17	4.71	$A \prec B \prec C$
Distance minimization [6]	0.00	1.45	1.95	$A \prec B \prec C$
Revised distance minimization [5]	0.00	1.45	1.95	$A \prec B \prec C$
Magnitude ($f(r) = r$) [2]	0.00	1.42	2.25	$A \prec B \prec C$

TABLE 6. Comparative Results of Example 4.20

In continuation, we present several examples to compare the revised sign distance method with various new methods, such as: magnitude method [2], deviation degree method [33], belief interval method [29].

Example 4.21. Figure 5 presents three fuzzy numbers $A = (-12, 1, 1, 2)$, $B = (-\frac{23}{12}, \frac{1}{12}, \frac{1}{12}, \frac{13}{12})$ and $C = (-6, 0, 1, 1)$, taken from papers [4, 5]. By magnitude method [2], we obtain $A \sim B \sim C$ which is unreasonable and is not consistent with human intuition. But, by revised sign distance method we have $A \prec B \prec C$ which is similar to the results of other methods (see Table 7). Thus, the revised sign distance method overcomes the shortcoming of the magnitude method in this example.

Example 4.22. Consider the data used in [4], i.e., the two fuzzy numbers, $A = (2, 3, 3.5)$ and $B = (2.2, 2.75, 2.75, 3.5)$, as shown in Figure 6. By the deviation degree method [33], the ranking order is $A \prec B$ which is obviously unreasonable. But, by the revised sign distance method, the ranking order is $B \prec A$. Also, by other

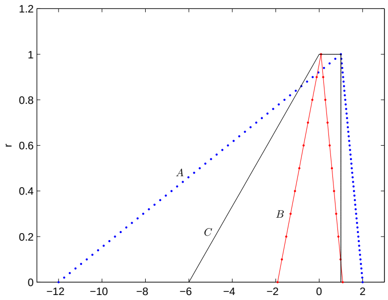

Figure 5. Graphical Representation of Fuzzy Numbers A, B and C Presented in Example 4.21

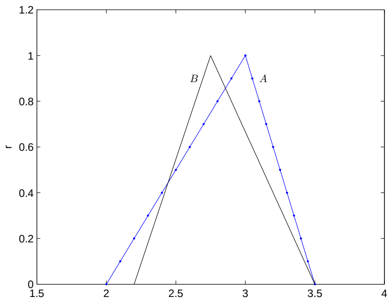

Figure 6. Graphical Representation of Fuzzy Numbers A and B Presented in Example 4.22

methods the ranking result is the same as our method (see Table 8). Therefore, the revised sign distance method overcomes the shortcoming of the inconsistency of the deviation degree method [33] in this example.

Methods	A	B	C	Results
Revised sign distance (p=1)	-7.08	-1.50	-4.00	$A \prec B \prec C$
Revised sign distance (p=2)	-6.83	-1.26	-3.61	$A \prec B \prec C$
Revised sign distance (p=3)	-7.38	-1.27	-3.80	$A \prec B \prec C$
Deviation degree [33]	0.00	6.52	2.29	$A \prec B \prec C$
Revised deviation degree [4]	2.22	4.44	3.14	$A \prec B \prec C$
Distance minimization [6]	-2.00	-0.17	-1.00	$A \prec B \prec C$
Revised distance minimization [5]	-2.00	-0.17	-1.00	$A \prec B \prec C$
Magnitude ($f(r) = r$) [2]	0.00	0.00	0.00	$A \sim B \sim C$

Table 7. Comparative Results of Example 4.21

Methods	A	B	Results
Revised sign distance (p=1)	5.750	5.600	$B \prec A$
Revised sign distance (p=2)	4.113	3.995	$B \prec A$
Revised sign distance (p=3)	3.703	3.591	$B \prec A$
Deviation degree [33]	0.719	0.732	$A \prec B$
Revised deviation degree [4]	1.069	1.027	$B \prec A$
Distance minimization [6]	2.875	2.800	$B \prec A$
Revised distance minimization [5]	2.875	2.800	$B \prec A$
Magnitude ($f(r) = r$) [2]	2.958	2.767	$B \prec A$

Table 8. Comparative Results of Example 4.22

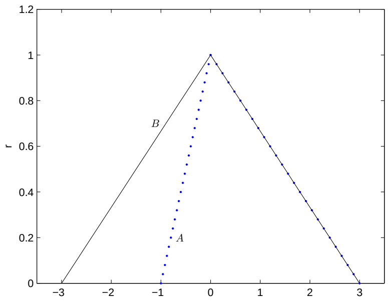

Figure 7. Graphical Representation of Fuzzy Numbers A and B Presented in Example 4.23

In the following example, we show that the results obtained by the revised sign distance method, unlike the results of original method, are in good agreement with those obtained by the belief interval method proposed by Sevastianov [29]

Example 4.23. Consider two triangle fuzzy numbers $A = (-1, 0, 0, 3)$ and $B = (-3, 0, 0, 3)$, as indicated in Figure 7. By the belief interval method [29], we obtain $BI(A \prec B) = [0, 0.667]$, $BI(A \sim B) = [0, 0.667]$ and $BI(B \prec A) = [0.333, 1]$. Also, the ranking order obtained by revised sign distance method and original sign distance method are presented in Table 9. Obviously, the results obtained by the revised sign distance method, unlike the results of original method, are in good agreement with those obtained by the belief interval method.

Methods	A	B	Results
Sign distance method (p=1)	2.00	3.00	$A \prec B$
Sign distance method (p=2)	1.83	2.45	$A \prec B$
Sign distance method (p=3)	1.91	2.38	$A \prec B$
Revised sign distance method (p=1)	2.00	0.00	$B \prec A$
Revised sign distance method (p=2)	1.83	0.00	$B \prec A$
Revised sign distance method (p=3)	1.91	0.00	$B \prec A$

Table 9. Comparative Results of Example 4.23

5. Conclusion

In this paper, we pointed out the shortcomings of "sign distance method" and in order to solve the problems we have presented a revised method for ranking fuzzy numbers. The revised method has two advantages in comparing with the original method. The first advantage is that the revised method can effectively ranks fuzzy numbers and their images. The second advantage is that the ranking order of the revised method is more consistent with our intuitions than the original one. Thus, employing the revised sign distance method is more logical than using sign distance method for ranking fuzzy numbers.

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