

ON LOCAL BOUNDEDNESS OF I -TOPOLOGICAL VECTOR SPACES

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ABSTRACT. The notion of generalized locally bounded I -topological vector spaces is introduced. Some of their important properties are studied. The relationship between this kind of spaces and the locally bounded I -topological vector spaces introduced by Wu and Fang [Boundedness and locally bounded fuzzy topological vector spaces, Fuzzy Math. 5 (4) (1985) 87–94] is discussed. Moreover, we also use the family of generalized fuzzy quasi-norms to characterize the generalized locally bounded I -topological vector spaces, and give some applications of this characterization.

1. Introduction

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981 [3]. According to the terminology of standardization in [2], it has now been renamed as I -topological vector space (briefly, I -tvs), where $I = [0, 1]$.

It is well-known that, the local boundedness is an important concept in theory of classical topological vector spaces. So it is natural and necessary to find its counterpart in research of I -tvs. Out of this consideration, Wu and Fang [8] introduced the notion of locally bounded I -tvs and studied some of its properties. We know that the locally bounded I -topological vector spaces defined in [8] are a special subclass of (QL) -type I -topological vector spaces [7]. This has led to the following questions: Whether we can investigate the local boundedness of general I -tvs (Do not have to be (QL) -type)? This article will give a definite answer.

This paper is organized as follows. In section 2, we briefly recall some basic concepts and lemmas that we will use in the sequel. In section 3, we introduce the notion of generalized locally bounded I -topological vector space, which contains that of locally bounded I -topological vector space introduced by Wu and Fang [8] as a special case. In section 4, we introduce the notion of a family of generalized fuzzy quasi-norms, and use it to characterize generalized locally bounded L -tvs. In section 5, as applications of the results obtained in section 4, we characterize the Hausdorff separation property, convergence of a net of fuzzy points in generalized locally bounded I -tvs by means of the family of generalized fuzzy quasi-norms.

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2. Basic Concepts and Lemmas

Throughout this paper, I^X denotes a family of all fuzzy subsets of X . A fuzzy subset which takes the constant value r on X ($0 \leq r \leq 1$) is denoted by \underline{r} . A fuzzy subset of X is called a fuzzy point [4], denoted by x_λ , if it takes value 0 at $y \in X \setminus \{x\}$ and its value at x is λ . The set of all fuzzy points on X is denoted by $\text{Pt}(I^X)$. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy subset U , denoted by $x_\lambda \tilde{\in} U$, if $U(x) > 1 - \lambda$ (see [4]). An I -topology on X [4] is a subset τ of I^X which contains the constant fuzzy sets $\underline{0}$ and $\underline{1}$ and is closed with respect to finite infima and arbitrary suprema. The pair (X, τ) is called an I -topological space. An I -topology τ on X is called fully stratified [4] if $\underline{r} \in \tau$ for each $r \in [0, 1]$.

Definition 2.1. [4] Let (X, τ) be an I -topological space and $x_\lambda \in \text{Pt}(I^X)$. A fuzzy subset U of X is called Q -neighborhood of x_λ iff there exists $G \in \tau$ such that $x_\lambda \tilde{\in} G \subseteq U$.

The family consisting of all Q -neighborhoods of x_λ , is called the system of Q -neighborhoods of x_λ and is denoted by $\mathcal{N}_Q(x_\lambda)$.

Definition 2.2. [4] Let (X, τ) be an I -topological space and $x_\lambda \in \text{Pt}(I^X)$. A family \mathcal{U}_{x_λ} of Q -neighborhoods of x_λ is called a Q -neighborhood base of x_λ iff for each $A \in \mathcal{N}_Q(x_\lambda)$ there exists $U \in \mathcal{U}_{x_\lambda}$ such that $U \subseteq A$.

In the following, let X be a vector space over the field \mathbb{K} (\mathbb{R} or \mathbb{C}) and θ the zero element of X .

Definition 2.3. [3] Let $A, B \in I^X$ and $k \in \mathbb{K}$. Then $A + B$ and kA are defined respectively by

$$\begin{aligned} (A + B)(x) &= \bigvee \{A(s) \wedge B(t) : s + t = x\}, \\ (kA)(x) &= A(x/k), \quad \text{whenever } k \neq 0; \\ (0A)(x) &= \begin{cases} \bigvee_{z \in X} A(z), & \text{if } x = \theta, \\ 0, & \text{if } x \neq \theta. \end{cases} \end{aligned}$$

In particular, for $x_\lambda, y_\mu \in \text{Pt}(I^X)$, we have

$$x_\lambda + y_\mu = (x + y)_{\lambda \wedge \mu}, \quad kx_\lambda = (kx)_\lambda.$$

Definition 2.4. [3] A stratified I -topology τ on X is said to be a I -vector topology, if the following two mappings are both continuous:

$$\begin{aligned} f : X \times X &\rightarrow X, \quad (x, y) \mapsto x + y, \\ g : \mathbb{K} \times X &\rightarrow X, \quad (k, x) \mapsto kx, \end{aligned}$$

where \mathbb{K} is equipped with the I -topology induced by the usual topology, $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product I -topologies.

A vector space X with an I -vector topology τ , denoted by (X, τ) , is called a I -topological vector space (for short, an I -tvs).

Definition 2.5. [7] An I -tvs (X, τ) is said to be (QL) -type of, if there exists a family \mathcal{U} of fuzzy subsets on X such that for each $\lambda \in (0, 1]$,

$$\mathcal{U}_\lambda = \{U \cap \underline{r} \mid U \in \mathcal{U}, r \in (1 - \lambda, 1]\}$$

is a Q -neighborhood base of θ_λ in (X, τ) . The family \mathcal{U} is called a Q -prebase for τ .

Definition 2.6. [8] A fuzzy set B in (X, τ) is said to be λ -bounded ($\lambda \in (0, 1]$ is given), if for each Q -neighborhood U of θ_λ in X , there exists $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B \cap \underline{r} \subseteq tU$.

B is said to be *bounded*, if it is λ -bounded for each $\lambda \in (0, 1]$.

Definition 2.7. [8] An I -tvs (X, τ) is said to be locally bounded if it has a bounded neighborhood of θ .

Remark 2.8. It is not difficult to prove that $U \in I^X$ is a bounded neighborhood of θ in an I -tvs (X, τ) iff $\{tU \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. Therefore, each locally bounded I -tvs is necessarily (QL) -type I -tvs.

Definition 2.9. [1] Let $\mathcal{U} \subset I^X$. A family $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ of nonempty subsets of \mathcal{U} is called the stratified structure of \mathcal{U} if the following two equalities hold: $\mathcal{U} = \bigcup_{\lambda \in (0, 1]} \mathcal{U}_\lambda$ and $\mathcal{U}_\lambda = \bigcup_{\mu \in (0, \lambda)} \mathcal{U}_\mu$.

Definition 2.10. [1] Let (X, τ) be an I -tvs. A family \mathcal{U} of fuzzy subsets on X is called a local base with the stratified structure $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ of (X, τ) if $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ is the stratified structure of \mathcal{U} and \mathcal{U}_λ is a Q -neighborhood base of θ_λ in (X, τ) for each $\lambda \in (0, 1]$.

Lemma 2.11. [6] Each I -tvs (X, τ) has an Q -neighborhood base of θ_λ consisting of balanced fuzzy sets on X for each $\lambda \in (0, 1]$.

Lemma 2.12. [1] Let (X, τ) be an I -tvs and \mathcal{U} be a local base with stratified structure $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ of (X, τ) . Then it has the following properties:

- (1) If $U \in \mathcal{U}_\lambda, V \in \mathcal{U}_\lambda$ or $V = \underline{r}$ ($1 - \lambda < r \leq 1$), then there exists $W \in \mathcal{U}_\lambda$ such that $W \subseteq U \cap V$;
- (2) If $U \in \mathcal{U}_\lambda$, then there exists $V \in \mathcal{U}_\lambda$ such that $V + V \subseteq U$;
- (3) If $U \in \mathcal{U}_\lambda$, then there exists $V \in \mathcal{U}_\lambda$ such that $kV \subseteq U$ for all $|k| \leq 1$;
- (4) If $U \in \mathcal{U}_\lambda$, then for each $x \in X$, there is an $\alpha > 0$ such that $x_\lambda \in \alpha U$.

Conversely, let X be a vector space over \mathbb{K} and \mathcal{U} be a family of fuzzy subsets on X with the stratified structure $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ which satisfies the above conditions (1)–(4). Then there exists a unique I -topology τ on X such that (X, τ) is an I -tvs and \mathcal{U} is a local base with the stratified structure $\{\mathcal{U}_\lambda\}_{\lambda \in (0, 1]}$ of (X, τ) .

Lemma 2.13. [8] Let (X, τ) be an I -tvs and $A, B \in I^X$. Then the following statements hold:

- (1) Every fuzzy point x_α on X is bounded.
- (2) If B is λ -bounded and $B_0 \subseteq B$, then B_0 is also λ -bounded.
- (3) If A and B are λ -bounded, then $A + B$ is also λ -bounded.
- (4) If B is λ -bounded, then kB is also λ -bounded for all $k \in \mathbb{K}$.
- (5) If B is λ -bounded, then the balanced hull $\text{ba}(B)$ of B is also λ -bounded.

3. Generalized Locally Bounded I -tvs

Definition 3.1. An I -tvs (X, τ) is said to be generalized locally bounded if for each $\lambda \in (0, 1]$, there exists a λ -bounded Q -neighborhood U_λ of θ_λ , and $0 < \mu < \lambda \leq 1$ implies $U_\lambda \subseteq U_\mu$.

Lemma 3.2. Let (X, τ) be an I -tvs. If U_λ is a λ -bounded Q -neighborhood of θ_λ on X , then $\mathcal{U}_\lambda = \{(1/n)U_\lambda \cap \underline{r} : n \in \mathbb{N}, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ .

Proof. Since U_λ is a Q -neighborhood of θ_λ , it is easy to see that for each $n \in \mathbb{N}$ and $r \in (1 - \lambda, 1]$, $(1/n)U_\lambda \cap \underline{r}$ is also a Q -neighborhood of θ_λ . Let W be an arbitrary Q -neighborhood of θ_λ . By Lemma 2.11, we may assume that it is balanced. Since U_λ is λ -bounded, there exist $r \in (1 - \lambda, 1]$ and $t > 0$ such that $U_\lambda \cap \underline{r} \subseteq tW$. Take $n_0 \in \mathbb{N}$ with $n_0 \geq t$. Then we have $\frac{1}{n_0}U_\lambda \cap \underline{r} \subseteq \frac{t}{n_0}W \subseteq W$. Hence \mathcal{U}_λ is a Q -neighborhood base of θ_λ . \square

Lemma 3.3. Let (X, τ) be a generalized locally bounded I -tvs. Then for each $\lambda \in (0, 1]$, there exists a balanced and λ -bounded Q -neighborhood B_λ of θ_λ , and $0 < \mu < \lambda \leq 1$ implies $B_\mu \subseteq B_\lambda$.

Proof. Since (X, τ) is a generalized locally bounded I -tvs, for each $\lambda \in (0, 1]$ there exists a λ -bounded Q -neighborhood U_λ of θ_λ , and $0 < \mu < \lambda \leq 1$ implies $U_\lambda \subseteq U_\mu$. Put $B_\lambda = \text{ba}(U_\lambda)$. By Lemma 2.13, we know that B_λ is λ -bounded. Obviously, it is also a balanced Q -neighborhood of θ_λ , and $0 < \mu < \lambda \leq 1$ implies $B_\lambda \subseteq B_\mu$. \square

Theorem 3.4. Let (X, τ) be an I -tvs. Then (X, τ) is generalized locally bounded iff there exists a family $\{B_\lambda\}_{\lambda \in (0, 1]}$ of fuzzy sets on X satisfying the following two conditions:

- (i) $0 < \mu < \lambda \leq 1$ implies $B_\lambda \subseteq B_\mu$;
- (ii) $\mathcal{B}_\lambda = \{tB_\lambda \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$.

Proof. Necessity: Suppose that (X, τ) is a generalized locally bounded I -tvs. Then for each $\lambda \in (0, 1]$, there exists a λ -bounded Q -neighborhood B_λ of θ_λ , and $0 < \mu < \lambda \leq 1$ implies $B_\lambda \subseteq B_\mu$. And then, by Lemma 3.2, we can conclude that $\mathcal{B}_\lambda = \{tB_\lambda \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ .

Sufficiency: Suppose that $\{B_\lambda\}_{\lambda \in (0, 1]}$ is a family of fuzzy sets on X satisfying conditions (i) and (ii). Then it is not difficult to see that B_λ is a Q -neighborhood of θ_λ , and for each $W \in \mathcal{N}_Q(\theta_\lambda)$, there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $tB_\lambda \cap \underline{r} \subseteq W$. Hence B_λ is a λ -bounded Q -neighborhood of θ_λ . This shows that (X, τ) is generalized locally bounded. \square

Theorem 3.5. Every locally bounded I -tvs is necessarily generalized locally bounded.

Proof. Let (X, τ) be a locally bounded I -tvs. Then it has a bounded neighborhood U of θ . By Remark 2.8, we know that $\mathcal{U}_\lambda = \{tU \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. Put $B_\lambda = U$ for each $\lambda \in (0, 1]$, then

$\mathcal{B}_\lambda = \{tB_\lambda \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ . Hence, by Theorem 3.4, we conclude that (X, τ) is generalized locally bounded. \square

The following example indicates that the converse of Theorem 3.5 is invalid in general.

Example 3.6. Let (X, T) be a classical nontrivial locally bounded topological vector space. Assume that U is a bounded neighborhood of θ and $U \neq X$ (since T is nontrivial). Let $c \in (0, 1)$ be given. For each $\lambda \in (0, 1]$, we define \mathcal{U}_λ as follows:

$$\mathcal{U}_\lambda = \begin{cases} \{\underline{r} \mid r \in (1 - \lambda, 1]\}, & \lambda \in (0, c], \\ \{tU \wedge \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}, & \lambda \in (c, 1]. \end{cases}$$

Then similar to Example 3.1 in [10], we can prove that there exists a unique I -topology $N_c(T)$ on X such that $(X, N_c(T))$ is an I -tvs and \mathcal{U}_λ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. We shall prove that $(I^X, N_c(T))$ is generalized locally bounded.

In fact, if $\lambda \in (0, c]$, then $\underline{1}$ is a λ -bounded Q -neighborhood of θ_λ ; If $\lambda \in (c, 1]$, then U is a λ -bounded Q -neighborhood of θ_λ .

However, we can show that $(X, N_c(T))$ is not of (QL) -type, and so it is not locally bounded.

In fact, if $(X, N_c(T))$ is a (QL) -type I -tvs, then there exists a family of fuzzy sets $\mathcal{B} = \{B\}$ such that $\mathcal{B}_\lambda = \{B \cap \underline{r} \mid B \in \mathcal{B}, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. Obviously, for each $\lambda \in (0, 1]$, $B \in \mathcal{B}$ is also a Q -neighborhood of θ_λ . Notice that $\mathcal{U}_\lambda = \{\underline{r} : r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, c]$. Hence, for each $B \in \mathcal{B}$, there exists $r_0 \in (1 - \lambda, 1]$ such that $r_0 \subseteq B$, which implies that $B(x) \geq r_0 > 1 - \lambda$ for all $\lambda \in (0, c]$ and $x \in X$, and so $B = X$. Hence $\mathcal{B}_\lambda = \{\underline{r} : r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$. Since U is a neighborhood of θ , U is a Q -neighborhood of θ_λ for each $\lambda \in (0, 1]$. So, there exists $r \in (1 - \lambda, 1]$ such that $\underline{r} \subseteq U$, which implies that $U = X$. This contradicts $U \neq X$. Therefore, $(X, N_c(T))$ is not locally bounded.

4. Characterization of Generalized Locally Bounded I -tvs

In this section, we first introduce the notion of a family of generalized fuzzy quasi-norms and investigate some of its properties, then characterize generalized locally bounded I -tvs in terms of a family of generalized fuzzy quasi-norms.

Definition 4.1. A family of mappings $\|\cdot\|_{(\lambda)} : \text{Pt}(I^X) \rightarrow [0, \infty]$ ($\lambda \in (0, 1]$) is called a family of generalized fuzzy quasi-norms on X , if it satisfies the following conditions:

- (GQN-1) for each $\lambda \in (0, 1]$, $\|\theta_\lambda\|_{(\lambda)} = 0$ and $\|x_\lambda\|_{(\lambda)} < +\infty$ for all $x \in X$;
- (GQN-2) $\|kx_\alpha\|_{(\lambda)} = |k| \|x_\alpha\|_{(\lambda)}$ for each $x_\alpha \in \text{Pt}(I^X)$ and $k \in \mathbb{K}$ with $k \neq 0$;
- (GQN-3) for each $\lambda \in (0, 1]$, there exist $s \in (1 - \lambda, 1]$ and $k_0 \geq 1$ such that
$$\|x_\alpha + y_\alpha\|_{(\lambda)} \leq k_0 [\|x_\alpha\|_{(\lambda)} + \|y_\alpha\|_{(\lambda)}] \quad \text{for all } \alpha \in (1 - s, 1], \quad x, y \in X;$$
- (GQN-4) $\|x_\alpha\|_{(\lambda)} = \inf_{0 < \beta < \alpha} \|x_\beta\|_{(\lambda)}$, and

$$0 < \mu < \lambda \leq 1 \text{ implies that } \|x_\alpha\|_{(\mu)} \leq \|x_\alpha\|_{(\lambda)} \quad \text{for all } x_\alpha \in \text{Pt}(I^X);$$

(GQN-5) for each $\lambda \in (0, 1]$ there exist $\lambda_0 \in (0, \lambda)$, $s \in (1 - \lambda_0, 1]$ and $n_0 \geq 1$ such that

$$\|x_\alpha\|_{(\lambda)} \leq n_0 \|x_\alpha\|_{(\lambda_0)} \quad \text{for all } \alpha \in (1 - s, 1] \text{ and } x \in X.$$

Lemma 4.2. *Let $\{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0, 1]}$ be a family of generalized fuzzy quasi-norms on X . For each $\lambda \in (0, 1]$ and $t > 0$, define a fuzzy set $U_{\lambda, t}$ on X by*

$$U_{\lambda, t}(x) = \sup \{1 - \alpha \mid \|x_\alpha\|_{(\lambda)} < t\}. \quad (1)$$

Then $U_{\lambda, t}$ has the following properties:

(P-1) $x_\alpha \tilde{\in} U_{\lambda, t}$ if and only if $\|x_\alpha\|_{(\lambda)} < t$;

(P-2) $U_{\lambda, t} = \bigcup_{0 < s < t} U_{\lambda, s}$;

(P-3) $sU_{\lambda, t} = U_{\lambda, st}$ for each $s > 0$;

(P-4) $0 < \mu < \lambda \leq 1 \implies U_{\lambda, t} \subseteq U_{\mu, t}$;

(P-5) $U_{\lambda, t}$ is semi-balanced, i.e., $kU_{\lambda, t} \subseteq U_{\lambda, t}$ for all $k \in \mathbb{K}$ with $0 < |k| \leq 1$.

(P-6) for each $\lambda \in (0, 1]$, there exist $\lambda_0 \in (0, \lambda)$, $s \in (1 - \lambda_0, 1]$ and $n_0 \geq 1$ such that $U_{\lambda_0, \frac{1}{n_0}} \cap \underline{s} \subseteq U_{\lambda, 1}$.

Proof. (P-1) If $x_\alpha \tilde{\in} U_{\lambda, t}$, i.e., $U_{\lambda, t}(x) > 1 - \alpha$, then by (1) there exists $\alpha_0 \in (0, \alpha)$ such that $\|x_{\alpha_0}\|_{(\lambda)} < t$. By (GQN-4), we have $\|x_\alpha\|_{(\lambda)} \leq \|x_{\alpha_0}\|_{(\lambda)} < t$. Conversely, if $\|x_\alpha\|_{(\lambda)} < t$, by (GQN-4), there exists $0 < \beta < \alpha$ such that $\|x_\beta\|_{(\lambda)} < t$, and so $U_{\lambda, t} \geq 1 - \beta > 1 - \alpha$, i.e., $x_\alpha \tilde{\in} U_{\lambda, t}$.

(P-2) By the definition of $U_{\lambda, t}$, it is easy to see that $U_{\lambda, s} \subseteq U_{\lambda, t}$ for each $s \in (0, t)$. Hence $\bigcup_{0 < s < t} U_{\lambda, s} \subseteq U_{\lambda, t}$. On the other hand, if $x_\alpha \tilde{\in} U_{\lambda, t}$, by (P-1), it is easy to prove that there exists $0 < s < t$ such that $x_\alpha \tilde{\in} U_{\lambda, s}$, and so $x_\alpha \tilde{\in} \bigcup_{0 < s < t} U_{\lambda, s}$. Hence $U_{\lambda, t} \subseteq \bigcup_{0 < s < t} U_{\lambda, s}$. Therefore (P-2) holds.

(P-3) For each $s, t > 0$ and $x \in X$, we have

$$\begin{aligned} U_{\lambda, st}(x) &= \sup \{1 - \alpha \mid \|x_\alpha\|_{(\lambda)} < st\} = \sup \{1 - \alpha \mid \|((1/s)x)_\alpha\|_{(\lambda)} < t\} \\ &= U_{\lambda, t}(x/s) = (sU_{\lambda, t})(x). \end{aligned}$$

Hence (P-3) holds.

(P-4) If $0 < \mu < \lambda \leq 1$ and $x_\alpha \tilde{\in} U_{\lambda, t}$, then by (GQN-4) and (P-1) we have $\|x_\alpha\|_{(\mu)} \leq \|x_\alpha\|_{(\lambda)} < t$, and so $x_\alpha \tilde{\in} U_{\mu, t}$. Hence (P-4) holds.

(P-5) If $0 < |k| \leq 1$, then by (P-1) it is easy to know that the following implications hold:

$$x_\alpha \tilde{\in} kU_{\lambda, t} \implies \frac{1}{k}x_\alpha \tilde{\in} U_{\lambda, t} \implies \|x_\alpha\|_{(\lambda)} < |k|t \leq t \implies x_\alpha \tilde{\in} U_{\lambda, t}.$$

Hence $kU_{\lambda, t} \subseteq U_{\lambda, t}$. This shows that $U_{\lambda, t}$ is semi-balanced.

(P-6) By (GQN-5), for each $\lambda \in (0, 1]$, there exist $\lambda_0 \in (0, \lambda)$, $s \in (1 - \lambda_0, 1]$ and $n_0 \geq 1$ such that $\|x_\alpha\|_{(\lambda)} \leq n_0 \|x_\alpha\|_{(\lambda_0)}$ for all $\alpha \in (1 - s, 1]$ and $x \in X$. Thus, we can prove that $U_{\lambda_0, \frac{1}{n_0}} \cap \underline{s} \subseteq U_{\lambda, 1}$. In fact, by (P-1), if $x_\alpha \tilde{\in} U_{\lambda_0, \frac{1}{n_0}} \cap \underline{s}$, then

$\|x_\alpha\|_{(\lambda_0)} < \frac{1}{n_0}$ and $\alpha \in (1-s, 1]$, and so $\|x_\alpha\|_{(\lambda)} < 1$, i.e., $x_\alpha \in U_{\lambda,1}$. This shows that $U_{\lambda_0, \frac{1}{n_0}} \cap \underline{s} \subseteq U_{\lambda,1}$. \square

Theorem 4.3. *Let $\mathcal{P} = \{\|\cdot\|_{(\lambda)} : \text{Pt}(I^X) \rightarrow [0, \infty]\}_{\lambda \in (0,1]}$ be a family of generalized fuzzy quasi-norms on X . Then there exists a unique I -topology τ on X such that (X, τ) is a generalized locally bounded I -tvs, and*

$$\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid t > 0, r \in (1-\lambda, 1]\}$$

is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$, where $U_{\lambda,t}$ is defined by (1). The topology τ is said to be generated by the family of generalized fuzzy quasi-norms \mathcal{P} .

Proof. Define a family \mathcal{U} of fuzzy sets on X as follows:

$$\mathcal{U} = \{U_{\alpha,t} \cap \underline{r} \mid \alpha \in (0, 1), r \in (0, 1], t > 0\}.$$

We first prove that $\{\mathcal{U}_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of \mathcal{U} , where \mathcal{U}_λ is defined by

$$\mathcal{U}_\lambda = \{U_{\alpha,t} \cap \underline{r} \mid \alpha \in (0, \lambda), r \in (1-\lambda, 1], t > 0\}. \quad (2)$$

In fact, it is evident that $\bigcup_{\lambda \in (0,1]} \mathcal{U}_\lambda \subseteq \mathcal{U}$. If $U_{\alpha,t} \cap \underline{r} \in \mathcal{U}$, then $0 < \alpha < 1$ and $0 < r \leq 1$. Take $\mu \in (\max\{\alpha, 1-r\}, 1]$, then $\alpha < \mu$ and $r \in (1-\mu, 1]$, and so $U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\mu$. This shows that $\mathcal{U} \subseteq \bigcup_{\lambda \in (0,1]} \mathcal{U}_\lambda$. Hence $\mathcal{U} = \bigcup_{\lambda \in (0,1]} \mathcal{U}_\lambda$. In addition, by (2), it is evident that $\bigcup_{\mu \in (0,\lambda)} \mathcal{U}_\mu \subseteq \mathcal{U}_\lambda$. If $U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$, then $0 < \alpha < \lambda$, $1-\lambda < r \leq 1$. Take $\mu \in (\max\{\alpha, 1-r\}, \lambda)$, then $U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\mu$. This shows that $\mathcal{U}_\lambda \subseteq \bigcup_{0 < \mu < \lambda} \mathcal{U}_\mu$. Hence $\mathcal{U}_\lambda = \bigcup_{0 < \mu < \lambda} \mathcal{U}_\mu$.

Next, we prove that $\{\mathcal{U}_\lambda\}_{\lambda \in (0,1]}$ satisfies conditions (1)–(4) in Lemma 2.12

(1) Let $U_i = U_{\alpha_i, t_i} \cap \underline{r}_i \in \mathcal{U}_\lambda$, where $0 < \alpha_i < \lambda$, $t_i > 0$ and $r_i \in (1-\lambda, 1]$ ($i = 1, 2$). Put $\alpha = \max\{\alpha_1, \alpha_2\}$, $r = \min\{r_1, r_2\}$ and $t = \min\{t_1, t_2\}$. Then $V = U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$, and by (P-2) and (P-4), it is evident that $V \subseteq U_1 \cap U_2$.

Let $U = U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$ where $0 < \alpha < \lambda$, $t > 0$ and $r \in (1-\lambda, 1]$, and let $V = \underline{s}$ with $s \in (1-\lambda, 1]$. Put $q = \min\{r, s\}$. Then it is obvious that $W = U_{\alpha,t} \cap \underline{q} \in \mathcal{U}_\lambda$ and $W \subseteq U \cap V$.

(2) Let $U = U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$, where $0 < \alpha < \lambda$, $t > 0$ and $r \in (1-\lambda, 1]$. For the above α , by (GQN-3), there exists $s \in (1-\alpha, 1]$ and $k_0 \geq 1$ such that

$$\|x_\beta + y_\beta\|_{(\alpha)} \leq k_0 [\|x_\beta\|_{(\alpha)} + \|y_\beta\|_{(\alpha)}] \quad (3)$$

for all $\beta \in (1-s, 1]$ and $x, y \in X$. Put $V = U_{\alpha, \frac{t}{2k_0}} \cap \underline{q}$, where $q = \min\{r, s\}$. Obviously, $V \in \mathcal{U}_\lambda$. We shall prove that $V + V \subseteq U$.

In fact, if $z_\beta \in V + V$, there exist $x, y \in X$ with $x + y = z$ such that $x_\beta, y_\beta \in U_{\alpha, \frac{t}{2k_0}}$ and $q > 1 - \beta$. By (P-1), it follows that $\|x_\beta\|_{(\alpha)} < t/2k_0$ and $\|y_\beta\|_{(\alpha)} < t/2k_0$. Note that $s \geq q > 1 - \beta$, i.e., $\beta \in (1-s, 1]$. So, by (3), we get

$$\|z_\beta\|_{(\alpha)} = \|x_\beta + y_\beta\|_{(\alpha)} \leq k_0 [\|x_\beta\|_{(\alpha)} + \|y_\beta\|_{(\alpha)}] < t,$$

i.e., $z_\beta \tilde{\in} U_{\alpha,t}$. Note that $r \geq q > 1 - \beta$. Hence $z_\beta \tilde{\in} U$. This shows that $V + V \subseteq U$.

(3) Let $U = U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$, where $0 < \alpha < \lambda$, $t > 0$ and $r \in (1 - \lambda, 1]$. Put $\mu = \min\{U_{\alpha,t}(\theta), r\}$ and $V = U_{\alpha,t} \cap \mu$. It is evident that $V \in \mathcal{U}_\lambda$. By (P-5), we know that $U_{\alpha,t}$ is semi-balanced, and so V is also semi-balanced. Note that $(0 \cdot V)(\theta) = \sup_{z \in X} U_{\alpha,t}(z) \wedge \mu \geq U_{\alpha,t}(\theta) \wedge \mu = \mu$. Hence $0 \cdot V = \theta_\mu \subseteq V$. This shows that V is balanced, and so $kV \subseteq V \subseteq U$ for all $k \in \mathbb{K}$ with $|k| \leq 1$.

(4) Let $U = U_{\alpha,t} \cap \underline{r} \in \mathcal{U}_\lambda$, where $0 < \alpha < \lambda$, $t > 0$ and $r \in (1 - \lambda, 1]$. For $x \in X$, by (GQN-1) and (GQN-4), $\|x_\lambda\|_{(\alpha)} \leq \|x_\alpha\|_{(\alpha)} < +\infty$. Put $s = (\|x_\lambda\|_{(\alpha)} + 1) / t$. By (GQN-2), we have

$$\left\| \frac{1}{s} x_\lambda \right\|_{(\alpha)} = \frac{1}{s} \|x_\lambda\|_{(\alpha)} = \frac{\|x_\lambda\|_{(\alpha)}}{\|x_\lambda\|_{(\alpha)} + 1} \cdot t < t.$$

Hence, $(1/s)x_\lambda \tilde{\in} U_{\alpha,t}$. Note that $r > 1 - \lambda$. So $(1/s)x_\lambda \tilde{\in} U_{\alpha,t} \cap \underline{r}$, i.e., $x_\lambda \tilde{\in} sU$.

Thus, by Lemma 2.12, there exists a unique I -topology τ on X such that (X, τ) is an I -tvs, and \mathcal{U} is a local base with the stratified structure $\{\mathcal{U}_\lambda\}_{\lambda \in (0,1]}$ of (X, τ) , and so \mathcal{U}_λ is a Q -neighborhood base of θ_λ for each $\lambda \in (0, 1]$.

Finally, we prove that (X, τ) is generalized locally bounded. Note that $0 < \mu < \lambda \leq 1$ implies $U_{\lambda,1} \subseteq U_{\mu,1}$ and $tU_{\lambda,1} = U_{\lambda,t}$ (see (P-3) and (P-4)). By Theorem 3.1, we now need only to prove that for each $\lambda \in (0, 1]$,

$$\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$$

is a Q -neighborhood base of θ_λ in (X, τ) .

Note that $\mathcal{U}_\lambda = \{U_{\alpha,t} \cap \underline{r} \mid 0 < \alpha < \lambda, t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ in (X, τ) . Hence, by (P-6), it is easy to see that every member in \mathcal{B}_λ is a Q -neighborhood of θ_λ in (X, τ) . On the other hand, by (P-3) and (P-4), $tU_{\lambda,1} = U_{\lambda,t} \subseteq U_{\alpha,t}$ for each $\alpha \in (0, \lambda)$. Hence, \mathcal{B}_λ is also a Q -neighborhood base of θ_λ in (X, τ) . Thus, by Theorem 3.1, we conclude that (X, τ) is a generalized locally bounded L -tvs. \square

Theorem 4.4. *Let (X, τ) be a generalized locally bounded I -tvs. Then the topology τ can be generated by a family of generalized fuzzy quasi-norms $\{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]}$ on X , i.e.,*

$$\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$$

is a Q -neighborhood base of θ_λ in (X, τ) for each $\lambda \in (0, 1]$, where $U_{\lambda,t}$ is defined by (1).

Proof. Since (X, τ) is generalized locally bounded, by Lemma 3.3, there exists a balanced and λ -bounded Q -neighborhood B_λ of θ_λ for each $\lambda \in (0, 1]$, and $0 < \mu < \lambda \leq 1$ implies $B_\lambda \subseteq B_\mu$. For each $\lambda \in (0, 1]$, define a mapping $\|\cdot\|_{(\lambda)} : \text{Pt}(I^X) \rightarrow [0, \infty]$ by

$$\|x_\alpha\|_{(\lambda)} = \inf \{t > 0 \mid x_\alpha \tilde{\in} tB_\lambda\}. \quad (4)$$

We shall verify that $\{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]}$ is a family of generalized fuzzy quasi-norms on X .

(GQN-1) Since U_λ is a Q -neighborhood of θ_λ , $\theta_\lambda \tilde{\in} B_\lambda$, which implies $\theta_\lambda = t\theta_\lambda \tilde{\in} tB_\lambda$ for all $t > 0$, and so $\|\theta_\lambda\|_{(\lambda)} = 0$. By (4) of Lemma 2.12 we easily

know that for each $x \in X$ there exists $s > 0$, such that $x_\lambda \tilde{\in} sB_\lambda$. So, by (4) we have $\|x_\lambda\|_{(\lambda)} \leq s < +\infty$.

(GQN-2) Let $\lambda \in (0, 1]$ and $k \in \mathbb{K}$ with $k \neq 0$. Since B_λ is balanced, $x_\alpha \tilde{\in} (t/k)B_\lambda \Leftrightarrow x_\alpha \tilde{\in} (t/|k|)B_\lambda$, and so

$$\begin{aligned} \|kx_\alpha\|_{(\lambda)} &= \inf\{t > 0 \mid kx_\alpha \tilde{\in} tB_\lambda\} = \inf\{t > 0 \mid x_\alpha \tilde{\in} (t/k)B_\lambda\} \\ &= \inf\{t > 0 \mid x_\alpha \tilde{\in} (t/|k|)B_\lambda\} = |k| \inf\{s > 0 \mid x_\alpha \tilde{\in} sB_\lambda\} \\ &= |k| \|x_\alpha\|_{(\lambda)}. \end{aligned}$$

(GQN-3) Note that B_λ is a λ -bounded Q -neighborhood of θ_λ for each $\lambda \in (0, 1]$. By Lemma 3.2, we know that that $\mathcal{U}_\lambda = \{(1/n)B_\lambda \cap \underline{s} \mid n \in \mathbb{N}, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ . So, by Lemma 2.12, there exist $s \in (1 - \lambda, 1]$ and $k_0 \geq 1$ such that

$$(1/k_0)B_\lambda \cap \underline{s} + (1/k_0)B_\lambda \cap \underline{s} \subseteq B_\lambda. \quad (5)$$

From (5), we can prove that

$$\|x_\alpha + y_\alpha\|_{(\lambda)} \leq k_0(\|x_\alpha\|_{(\lambda)} + \|y_\alpha\|_{(\lambda)}), \quad \text{for all } \alpha \in (1 - s, 1], x, y \in X. \quad (6)$$

In fact, $\forall \alpha \in (1 - s, 1]$ and $x, y \in X$, without loss of generality, we can assume that $\|x_\alpha\|_{(\lambda)} = a < \infty$, $\|y_\alpha\|_{(\lambda)} = b < \infty$. By (4), for each $\varepsilon > 0$, there exist $t_1, t_2 > 0$ such that $t_1 < a + \varepsilon/2$, $t_2 < b + \varepsilon/2$ and $x_\alpha \tilde{\in} t_1 B_\lambda$, $y_\alpha \tilde{\in} t_2 B_\lambda$. Note that $s > 1 - \alpha$, it follows from (5) that

$$\begin{aligned} \frac{1}{k_0}x_\alpha + \frac{1}{k_0}y_\alpha &\tilde{\in} \left(\frac{t_1}{k_0}B_\lambda\right) \cap \underline{s} + \left(\frac{t_2}{k_0}B_\lambda\right) \cap \underline{s} \\ &= (t_1 + t_2) \left[\left(\frac{t_1}{k_0(t_1 + t_2)}B_\lambda\right) \cap \underline{s} + \left(\frac{t_2}{k_0(t_1 + t_2)}B_\lambda\right) \cap \underline{s} \right] \\ &\subseteq (t_1 + t_2) \left[\left(\frac{1}{k_0}B_\lambda\right) \cap \underline{s} + \left(\frac{1}{k_0}B_\lambda\right) \cap \underline{s} \right] \subseteq (t_1 + t_2)B_\lambda, \end{aligned}$$

which shows that $x_\alpha + y_\alpha \tilde{\in} k_0(t_1 + t_2)B_\lambda$. By (4), we get

$$\|x_\alpha + y_\alpha\|_{(\lambda)} \leq k_0(t_1 + t_2) < k_0(a + b + \varepsilon).$$

By the arbitrariness of ε , (6) is proved.

(GQN-4) Let $0 < \beta < \alpha \leq 1$, then $x_\beta \tilde{\in} tB_\lambda$ implies that $x_\alpha \tilde{\in} tB_\lambda$. So, by (4), we have $\|x_\alpha\|_{(\lambda)} \leq \|x_\beta\|_{(\lambda)}$.

Now, we prove that $\|x_\alpha\|_{(\lambda)} = \inf_{0 < \beta < \alpha} \|x_\beta\|_{(\lambda)}$. If $\|x_\alpha\|_{(\lambda)} = \infty$, then the above inequality implies that $\|x_\beta\|_{(\lambda)} = \infty$ for all $0 < \beta < \alpha$. Hence $\|x_\alpha\|_{(\lambda)} = \inf_{0 < \beta < \alpha} \|x_\beta\|_{(\lambda)}$; If $\|x_\alpha\|_{(\lambda)} < \infty$, by (4) we know that for each $\varepsilon > 0$, there exists $t < \|x_\alpha\|_{(\lambda)} + \varepsilon$ such that $x_\alpha \tilde{\in} tB_\lambda$, and so there exists $\mu \in (0, \alpha)$, such that $x_\mu \tilde{\in} tB_\lambda$. By (4), we get $\|x_\mu\|_{(\lambda)} \leq t < \|x_\alpha\|_{(\lambda)} + \varepsilon$. Therefore $\|x_\alpha\|_{(\lambda)} = \inf_{0 < \beta < \alpha} \|x_\beta\|_{(\lambda)}$.

Since $0 < \mu < \lambda \leq 1$ implies $B_\lambda \subseteq B_\mu$, it follows from (4) that $\|x_\alpha\|_{(\mu)} \leq \|x_\alpha\|_{(\lambda)}$. (GQN-4) is proved.

(GQN-5) We know that B_λ is a λ -bounded Q -neighborhood of θ_λ for each $\lambda \in (0, 1]$. It is evident that for each $\lambda \in (0, 1]$, there exists $\lambda_0 \in (0, \lambda)$ such that

B_λ is a Q -neighborhood of θ_{λ_0} . Since B_{λ_0} is λ_0 -bounded, by Lemma 3.2 $\mathcal{U}_{\lambda_0} = \{(1/n)B_{\lambda_0} \cap \underline{r} \mid n \in \mathbb{N}, r \in (1 - \lambda_0, 1]\}$ is a Q -neighborhood base of θ_{λ_0} , and so there exist $s \in (1 - \lambda_0, 1]$ and $n_0 \geq 1$ such that $(1/n_0)B_{\lambda_0} \cap \underline{s} \subseteq B_\lambda$. From this we can prove that

$$\|x_\alpha\|_{(\lambda)} \leq n_0 \|x_\alpha\|_{(\lambda_0)} \quad \text{for all } \alpha \in (1 - s, 1] \text{ and } x \in X. \quad (7)$$

In fact, for $\alpha \in (1 - s, 1]$ and $x \in X$, without loss of generality, let us assume that $\|x_\alpha\|_{(\lambda_0)} = a < \infty$, then by (1) for each $\varepsilon > 0$ there exists $0 < t < c + \varepsilon$ such that $x_\alpha \tilde{\in} tB_{\lambda_0}$, which implies that $(1/n_0)x_\alpha \tilde{\in} (1/n_0)tB_{\lambda_0} \cap \underline{s} \subseteq tB_\lambda$, i.e., $x_\alpha \tilde{\in} n_0 tB_\lambda$. By (4), we have $\|x_\alpha\|_{(\lambda)} \leq n_0 t < n_0(a + \varepsilon)$. By the arbitrariness of ε , we get $\|x_\alpha\|_{(\lambda)} \leq n_0 \|x_\alpha\|_{(\lambda_0)}$, (7) is proved.

This shows that $\{\|\cdot\|_\lambda\}_{\lambda \in (0,1]}$ is a family of generalized fuzzy quasi-norms on X . By Theorem 4.3, there exists a unique L -topology $\tilde{\tau}$ such that $(X, \tilde{\tau})$ is a generalized locally bounded I -tvs and $\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid r > 1 - \lambda, t > 0\}$ is a Q -neighborhood of θ_λ in $(X, \tilde{\tau})$ for each $\lambda \in (0, 1]$, where $U_{\lambda,t}$ is defined by (1). We shall prove that $\tilde{\tau} = \tau$. To this end, we need only to prove that

$$\frac{t}{2}B_\lambda \subseteq U_{\lambda,t} \subseteq tB_\lambda, \quad \text{for all } \lambda \in (0, 1] \text{ and } t > 0. \quad (8)$$

In fact, by the definition of $\|\cdot\|_{(\lambda)}$ and (P-1), it is easy to see that the following implications hold:

$$x_\alpha \tilde{\in} \frac{t}{2}B_\lambda \Rightarrow \|x_\alpha\|_{(\lambda)} \leq \frac{t}{2} < t \Rightarrow x_\alpha \tilde{\in} U_{\lambda,t}.$$

Hence $\frac{t}{2}B_\lambda \subseteq U_{\lambda,t}$. Moreover, if $x_\alpha \tilde{\in} U_{\lambda,t}$, then $\|x_\alpha\|_{(\lambda)} < t$, and so there exists $s \in (0, t)$ such that $x_\alpha \tilde{\in} sB_\lambda \subseteq tB_\lambda$, which shows that $U_{\lambda,t} \subseteq tB_\lambda$. Therefore (8) holds. \square

Remark 4.5. From Theorem 4.4, we know that a generalized locally bounded I -tvs (X, τ) can be completely characterized in terms of a family of generalized fuzzy quasi-norms $\{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]}$. So, for convenience, we also use $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$ to denote a generalized locally bounded I -tvs.

5. Applications

In this section, as applications of Theorem 4.1 and Theorem 4.2, we shall use the family of generalized fuzzy quasi-norms to characterize Hausdorff separation property and convergence of a net of fuzzy points in generalized locally bounded I -tvs. We first recall some necessary concepts and results.

Definition 5.1. [4] An I -topological space (X, τ) is said to be Hausdorff, if for each $x_\lambda, y_\mu \in \text{Pt}(I^X)$ with $x \neq y$, there exist $P \in \mathcal{N}_Q(x_\lambda)$ and $Q \in \mathcal{N}_Q(y_\mu)$ such that $P \cap Q = \underline{0}$.

Lemma 5.2. [6] Let (X, τ) be an I -tvs. Then the following statements are mutually equivalent:

- (1) (L^X, δ) is Hausdorff;
- (2) For each $\lambda \in (0, 1]$, θ_λ is closed;

(3) For each $\lambda \in (0, 1]$ and $x \in X$ with $x \neq \theta$, there exists $U \in \mathcal{N}_Q(\theta_\lambda)$ such that $U(x) = 0$.

Definition 5.3. [4] Let (X, τ) be an I -topological space and $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda}$ a net of fuzzy points in X . $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda}$ is said to be convergent to $x_\lambda \in \text{Pt}(I^X)$, if for each $U \in \mathcal{N}_Q(x_\lambda)$, there exists $n_0 \in \Lambda$ such that $x_{\lambda_n}^{(n)} \tilde{\in} U$ whenever $n \in \Lambda$ with $n \geq n_0$.

Theorem 5.4. A generalized locally bounded I -tvs $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$ is Hausdorff iff $\|x_1\|_{(\lambda)} > 0$ for each $\lambda \in (0, 1]$ and each $x \in X$ with $x \neq \theta$.

Proof. Necessity: Assume that there exist some $\lambda \in (0, 1]$ and $x \in X$ with $x \neq \theta$ such that $\|x_1\|_{(\lambda)} = 0$, which implies that $\|x_1\|_{(\lambda)} < t$ for each $t > 0$. By (P-1), $x_1 \tilde{\in} U_{\lambda,t}$ for each $t > 0$. We know that $\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ , where $U_{\lambda,t}$ is defined by (1) (see Theorem 4.4). Hence, for each $U \in \mathcal{N}_Q(\theta_\lambda)$, there exist $t > 0$ and $r \in (1 - \lambda, 1]$ such that $U_{\lambda,t} \cap \underline{r} \subseteq U$. Thus, it follows from $x_1 \tilde{\in} U_{\lambda,t}$ that $U(x) > 0$. Therefore, by Lemma 5.2, $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$ is not Hausdorff.

Sufficiency: Suppose that for each $\lambda \in (0, 1]$ and each $x \in X$ with $x \neq \theta$, $\|x_1\|_{(\lambda)} > 0$. Put $t = \|x_1\|_{(\lambda)}$. By (P-1), x_1 is not quasi-coincident with $U_{\lambda,t}$, i.e., $U_{\lambda,t}(x) = 0$. Note that $U_{\lambda,t} \in \mathcal{N}_Q(\theta_\lambda)$. Hence, by Lemma 5.2, $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$ is Hausdorff. \square

Theorem 5.5. Let $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$ be a generalized locally bounded I -tvs and $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda}$ a net of fuzzy points in $(X, \{\|\cdot\|_{(\lambda)}\}_{\lambda \in (0,1]})$. Then $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda}$ is convergent to $x_\lambda \in \text{Pt}(I^X)$ iff $\lim_{n \rightarrow \infty} \|(x^{(n)} - x)_{\lambda_n}\|_{(\lambda)} = 0$ and $\liminf_{n \rightarrow \infty} \lambda_n \geq \lambda$.

Proof. Necessity: We know that for each $t > 0$ and $r \in (1 - \lambda, 1]$, $U_{\lambda,t} \cap \underline{r}$ is a Q -neighborhood of θ_λ (where $U_{\lambda,t}$ is defined by (1)), and so $x + U_{\lambda,t} \cap \underline{r}$ is a Q -neighborhood of x_λ . Since $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda} \rightarrow x_\lambda$, there exists $n_0 \in \Lambda$ such that $x_{\lambda_n}^{(n)} \tilde{\in} x + U_{\lambda,t} \cap \underline{r}$ whenever $n \in \Lambda$ with $n \geq n_0$, i.e., $(x^{(n)} - x)_{\lambda_n} \tilde{\in} U_{\lambda,t}$ and $r > 1 - \lambda_n$ whenever $n \geq n_0$, which implies that $\|(x^{(n)} - x)_{\lambda_n}\|_{(\lambda)} < t$ and $\lambda_n > 1 - r$ whenever $n \geq n_0$. This shows that $\lim_{n \rightarrow \infty} \|(x^{(n)} - x)_{\lambda_n}\|_{(\lambda)} = 0$ and $\liminf_{n \rightarrow \infty} \lambda_n \geq \lambda$.

Sufficiency: Note that $\mathcal{B}_\lambda = \{U_{\lambda,t} \cap \underline{r} \mid t > 0, r \in (1 - \lambda, 1]\}$ is a Q -neighborhood base of θ_λ . Hence, to prove that $\{x_{\lambda_n}^{(n)}\}_{n \in \Lambda} \rightarrow x_\lambda$, it suffices to prove that for each $W \in \mathcal{B}_\lambda$ there exists $n_0 \in \Lambda$ such that $x_{\lambda_n}^{(n)} \tilde{\in} x + W$ whenever $n \in \Lambda$ with $n \geq n_0$.

Let $W = U_{\lambda,t} \cap \underline{r} \in \mathcal{B}_\lambda$. Since $\lim_{n \rightarrow \infty} \|(x^{(n)} - x)_{\lambda_n}\|_{(\lambda)} = 0$ and $\liminf_{n \rightarrow \infty} \lambda_n \geq \lambda$, for $t > 0$ and $r > 1 - \lambda$ there exists $n_0 \in \Lambda$ such that $\|(x^{(n)} - x)_{\lambda_n}\|_{(\lambda)} < t$ and $\lambda_n > 1 - r$ whenever $n \in \Lambda$ with $n \geq n_0$, which implies that $x_{\lambda_n}^{(n)} \tilde{\in} x + U_{\lambda,t} \cap \underline{r} = x + W$ whenever $n \geq n_0$. This completes the proof. \square

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