NEW TYPES OF FUZZY n-ARY SUBHYPERGROUPS OF AN n-ARY HYPERGROUP

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ABSTRACT. In this paper, the new notions of "belongingness $(\in_\gamma)$" and "quasi-coincidence (q_δ)" of a fuzzy point with a fuzzy set are introduced. By means of this new idea, the concept of (α,β)-fuzzy n-ary subhypergroup of an n-ary hypergroup is given, where $\alpha,\beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$, and it is shown that, in 16 kinds of (α,β)-fuzzy n-ary subhypergroups, the significant ones are the $(\in_\gamma,\in_\gamma)$-fuzzy n-ary subhypergroups, $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups and the $(\in_\gamma \wedge q_\delta,\in_\gamma)$-fuzzy n-ary subhypergroups.

1. Introduction

Hypergroup which is based on the notion of hyperoperation has been introduced by Marty in [28] and studied extensively by many mathematicians. Hypergroup theory both extends some well-known group results and introduce new topics leading they to a wide variety of applications, as well as to a broadening of the investigation fields, see [3, 4, 5, 17, 32].

The concept of n-group was introduced by Dörnte [19] in 1928. Since then, n-ary systems have been studied in depth in different contexts. The research about n-ary hyperstructures was initiated by Davvaz and Vougiouklis who introduced these structures in [18]. Some applications of the n-ary hypergroups to lattices and to binary relations are analyzed by Leoreanu-Fotea and Davvaz in [26, 27]. In [8], Davvaz et al. provided ternary (3-ary) hyperstructures associated with chain reactions in chemistry.

After introducing the concept of fuzzy sets by Zadeh in 1965 [36], there are many papers devoted to fuzzify the classical mathematics into fuzzy mathematics. Murali [29] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [30] played a vital role to generate some different types of fuzzy subgroups. Using the notion "belongingness $(\in)$" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set, the concept of (α,β)-fuzzy subgroup where α,β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced by Bhakat and Das [1] in 1992, in which the $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup [31], also see [2]. After this, Corsini, Davvaz, Dudek, Jun, Kazancı, Yin, Yuan and Zhan applied

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this concept to different algebraic structures (for example, see [10, 15, 16, 20, 23, 33, 34, 35, 37, 38, 39, 40, 41]). Davvaz [7] applied fuzzy sets to the theory of algebraic hyperstructures and defined the concept of fuzzy subhypergroup (respectively fuzzy H_v-subgroups), also see [6]. Davvaz and Corsini in [14] introduced the notion of a fuzzy n-ary subhypergroup of an n-ary hypergroup, also see [9, 11, 12, 13, 22, 25]. Recently, Kazancı et al. [24] applied this concept to n-ary hypergroups and discussed some properties.

In this paper, we deal with a generalization of the papers [14, 24]. We first introduce new notions of "belongingness ($\in_\gamma$)" and "quasi-coincidence (q_δ)" of a fuzzy point with a fuzzy set. By means of this new idea, we then provide the concept of (α, β)-fuzzy n-ary subhypergroups of an n-ary hypergroup, where $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$, and we show that, in 16 kinds of (α, β)-fuzzy n-ary subhypergroups, the significant ones are the $(\in_\gamma, \in_\gamma)$-fuzzy n-ary subhypergroups, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups and the $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroups. It is noteworthy that the notion of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups are useful and important generalizations of fuzzy n-ary subhypergroups and $(\in, \in \vee q)$-fuzzy n-ary subhypergroups studied in [14, 24], and hence many results in [14, 24] will become easy corollaries of our results given in this paper.

2. **Preliminaries**

In this section, we summarize some basic concepts which will be used throughout the paper. For more details, we refer to the references [14, 18, 24, 36].

We will concern primarily with a basic non-empty set H. Denote by H^n the cartesian product $H \times \cdots \times H$, where H appears n times. An element of H^n will be denoted by $(x_1, \cdots, x_n)$, where $x_i \in H$, for all $1 \le i \le n$. In general, a mapping $f : H^n \to \wp^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity* of the hyperoperation f. Let f be an n-ary hyperoperation on H and $A_1, \cdots, A_n$ be subsets in H. Define

$$f(A_1, \cdots, A_n) = \cup\{f(x_1, \cdots, x_n) | x_i \in A_i, 1 \le i \le n\}.$$

In the sequel, we shall denote the sequence $x_i, x_{i+1}, \cdots, x_j$ by x_i^j. For $j < i$, $x_i^j = \emptyset$. Thus,

$$f(x_1, \cdots, x_i, y_{i+1}, \cdots, y_j, z_{j+1}, \cdots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Also $f(a_1^i, x*)$ means $f(a_1^i, \underbrace{x, \cdots, x}_{n-i})$ for $a_1^i, x \in H$ and $1 \le i \le n$.

H with an n-ary hyperoperation $f : H^n \to \wp^*(H)$ is called an *n-ary hypergroupoid* and will be denoted by (H, f). An n-ary hypergroupoid (H, f) is said to be an *n-ary semi-hypergroup* if the following associative axiom holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

for all $i, j \in \{1, 2, \cdots, n\}$ and $x_1^{2n-1} \in H$. An n-ary semi-hypergroup is said to be an *n-ary hypergroup* [18] if, for all $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \le i \le n$, there exists $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$.

Definition 2.1. *Let (H, f) be an n-ary hypergroup and K be a non-empty subset of H. We say that K is an n-*ary subhypergroup *of H if the following conditions hold:*

(i) *K is closed under the n-ary hyperoperation f;*
(ii) For all $a_1^{i-1}, a_{i+1}^n, b \in K$ and $1 \le i \le n$, there exists $x_i \in K$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$.

Next we recall some fuzzy logic concepts. Let X be a non-empty set. A fuzzy subset μ of X is defined as a mapping from X to $[0,1]$. We define $\mu_{\bar{\gamma}} = \{x \in X \mid \mu(x) > \gamma\}$. Denote the set of all fuzzy subsets of X by $\mathbb{F}(X)$. For $\mu, \nu \in \mathbb{F}(X)$, by $\mu \subseteq \nu$ we mean that $\mu(x) \le \nu(x)$, for all $x \in X$. We shall use the following abbreviated notation: the sequence $\mu(x_i), \mu(x_{i+1}), \cdots, \mu(x_j)$ will be denoted by $\mu_{x_i}^{x_j}$. For $j < i$, $\mu_{x_i}^{x_j} = 0$.

For any $r \in (0,1]$, a fuzzy subset μ of X having the form

$$\mu(y) = \begin{cases} r & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a *fuzzy point with support x and value r* and is denoted by x_r.

For a fuzzy point x_r and a fuzzy subset μ of X, we say that

(1) $x_r \in \mu$ if $\mu(x) \ge r$.
(2) $x_r \, q \, \mu$ if $\mu(x) + r > 1$.
(3) $x_r \in \vee q \, \mu$ if $x_r \in \mu$ or $x_r \, q \, \mu$.
(4) $x_r \, \overline{\alpha} \, \mu$ if $x_r \, \alpha \, \mu$ does not hold for $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

Definition 2.2. *Let (H, f) be an n-ary hypergroup and μ a fuzzy subset of H. Then, μ is said to be a* fuzzy n-ary subhypergroup *of H if the following axioms hold*

(1) $\min\{\mu_{x_1}^{x_n}\} \le \inf_{z \in f(x_1^n)} \mu(z)$, *for all* $x_1^n \in H$,

(2) for all $a_1^{i-1}, a_{i+1}^n, b \in H$ *and* $1 \le i \le n$, *there exists* $x_i \in H$ *such that* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ *and* $\min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\} \le \mu(x_i)$.

Definition 2.3. *Let (H, f) be an n-ary hypergroup and μ a fuzzy subset of H. Then, μ is said to be an* $(\in, \in\vee q)$-fuzzy n-ary subhypergroup *of H if the following axioms hold*

(1) $(x_1)_{r_1}, (x_2)_{r_2}, \cdots, (x_n)_{r_n} \in \mu$ *implies* $z_{r_1 \wedge r_2 \wedge \ldots \wedge r_n} \in \vee q \, \mu$, *for all* $z \in f(x_1^n)$,

(2) $(a_1)_{r_1}, \cdots, (a_{i-1})_{r_{i-1}}, (a_{i+1})_{r_{i+1}}, \cdots, (a_n)_{r_n}, b_t \in \mu$ *and* $1 \le i \le n$ *implies* $(x_i)_{r_1 \wedge r_2 \wedge \cdots \wedge r_{(i-1)} \wedge r_{(i+1)} \wedge \cdots \wedge r_n \wedge t} \in \vee q \, \mu$ *for some* $x_i \in H$ *with* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$.

Definition 2.4. *Given $\gamma, \delta \in (0,1]$ and $\gamma < \delta$, let (H, f) be an n-ary hypergroup and μ a fuzzy subset of H. Then, μ is said to be a* fuzzy n-ary subhypergroup with thresholds (γ, δ) *of H if the following axioms hold*

(1) $\min\{\mu_{x_1}^{x_n}\} \wedge \delta \le \inf_{z \in f(x_1^n)} \mu(z) \vee \gamma$, *for all* $x_1^n \in H$,

(2) for all $a_1^{i-1}, a_{i+1}^n, b \in H$ *and* $1 \le i \le n$, *there exists* $x_i \in H$ *such that* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ *and* $\min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\} \wedge \delta \le \mu(x_i) \vee \gamma$.

3. (α,β)-fuzzy n-ary Subhypergroups

In what follows let $\gamma,\delta\in[0,1]$ be such that $\gamma<\delta$. For a fuzzy point x_r and a fuzzy subset μ of X, we say that

(1) $x_r\in_\gamma\mu$ if $\mu(x)\geq r>\gamma$.
(2) $x_r\,q_\delta\,\mu$ if $\mu(x)+r>2\delta$.
(3) $x_r\in_\gamma\vee q_\delta\,\mu$ if $x_r\in_\gamma\mu$ or $x_r\,q_\delta\,\mu$.
(4) $x_r\in_\gamma\wedge q_\delta\,\mu$ if $x_r\in_\gamma\mu$ and $x_r\,q_\delta\,\mu$.
(5) $x_r\,\overline{\alpha}\,\mu$ if $x_r\,\alpha\,\mu$ does not hold for $\alpha\in\{\in_\gamma,q_\delta,\in_\gamma\vee q_\delta,\in_\gamma\wedge q_\delta\}$.

Now, we introduce the concept of (α,β)-fuzzy n-ary subhypergroups of an n-ary hypergroup (H,f), where $\alpha,\beta\in\{\in_\gamma,q_\delta,\in_\gamma\wedge q_\delta,\in_\gamma\vee q_\delta\}$, as follows.

Definition 3.1. *Let (H,f) be an n-ary hypergroup and μ be a fuzzy subset of H. Then, μ is said to be an (α,β)-*fuzzy n-ary subhypergroup *of H if, for all $r_1,r_2,\cdots,r_n,t\in(\gamma,1]$ the following axioms hold*
(F1) $(x_1)_{r_1},(x_2)_{r_2},\cdots,(x_n)_{r_n}\,\alpha\,\mu$ implies $z_{r_1\wedge r_2\wedge\cdots\wedge r_n}\beta\,\mu$, for all $z\in f(x_1^n)$,
(F2) $(a_1)_{r_1},\cdots,(a_{i-1})_{r_{i-1}},(a_{i+1})_{r_{i+1}},\cdots,(a_n)_{r_n},b_t\,\alpha\,\mu$ and $1\leq i\leq n$ implies $(x_i)_{r_1\wedge r_2\wedge\cdots\wedge r_{(i-1)}\wedge r_{(i+1)}\wedge\cdots\wedge r_n\wedge t}\beta\,\mu$ for some $x_i\in H$ with $b\in f(a_1^{i-1},x_i,a_{i+1}^n)$.

Thus, there are 16 kinds of (α,β)-fuzzy n-ary subhypergroups in all. Next, we will discuss the properties of these 16 kinds of (α,β)-fuzzy n-ary subhypergroups. In the sequel, H will denote any given n-ary hypergroup unless otherwise stated.

The following propositions follows directly from Definition 3.1.

Proposition 3.2. *Each $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H is an $(\in_\gamma\wedge q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H.*

Proposition 3.3. *Each $(\in_\gamma\wedge q_\delta,\in_\gamma)$-fuzzy n-ary subhypergroup of H is an $(\in_\gamma\wedge q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H.*

Proposition 3.4. *Every $(\in_\gamma,\in_\gamma)$-fuzzy n-ary subhypergroup of H is both an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H and an $(\in_\gamma\wedge q_\delta,\in_\gamma)$-fuzzy n-ary subhypergroup of H.*

3.1. $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy n-ary Subhypergroups.

As it is not difficult to see, each (α,β)-fuzzy n-ary subhypergroup of H is an $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H. Hence, in the theory of (α,β)-fuzzy n-ary subhypergroups the central role is played by $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroups. In this section, we will study the properties of $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroups. Before proceeding, we first provide an example of $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup as follows.

Example 3.5. *Let $(H,\cdot)$ be a group in which $|H|>2$ and S be a subgroup of H. Define $f(x_1^n)=Sx_1\cdots x_n$. Then, (H,f) is an n-ary hypergroup. Define a fuzzy subset μ of H by*

$$\mu(x)=\begin{cases}0.7 & \text{if } x\in S,\\ 0.2 & \text{otherwise,}\end{cases}$$

for all $x\in S$. Then, μ is an $(\alpha,\in_{0.2}\vee q_{0.6})$-fuzzy n-ary subhypergroup of H.

Theorem 3.6. *Let $2\delta = 1 + \gamma$ and μ be an (α, β)-fuzzy n-ary subhypergroup of H. Then, the set $\mu_{\hat{\gamma}}$ is an n-ary subhypergroup of H, where $\alpha \neq \in_\gamma \wedge q_\delta$.*

Proof. Assume that μ is an (α, β)-fuzzy n-ary subhypergroup of H. Let $x_1^n \in \mu_{\hat{\gamma}}$. Then, $\mu(x_1) > \gamma, \mu(x_2) > \gamma, \cdots, \mu(x_n) > \gamma$. We consider the following two cases.

Case 1: $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then, $(x_1)_{\mu(x_1)}, (x_2)_{\mu(x_2)}, \cdots, (x_n)_{\mu(x_n)}\, \alpha\, \mu$. By (F1), $z_{\min\{\mu_{x_1}^{x_n}\}} \beta \mu$, for all $z \in f(x_1^n)$. We have the following subcases.

Case 1a: $\beta = \in_\gamma$. Then, $z_{\min\{\mu_{x_1}^{x_n}\}} \in_\gamma \mu$. It follows that $\mu(z) \geq \min\{\mu_{x_1}^{x_n}\} > \gamma$, and so $z \in \mu_{\hat{\gamma}}$ for all $z \in f(x_1^n)$.

Case 1b: $\beta = q_\delta$. Then, $z_{\min\{\mu_{x_1}^{x_n}\}}\, q_\delta\, \mu$. It follows that $\mu(z) + \min\{\mu_{x_1}^{x_n}\} > 2\delta$, and so $\mu(z) > 2\delta - \min\{\mu_{x_1}^{x_n}\} \geq 2\delta - 1 = \gamma$. Hence, $\mu(z) > \gamma$ and so $z \in \mu_{\hat{\gamma}}$, for all $z \in f(x_1^n)$.

Case 1c: $\beta \in \{\in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$. Analogous to the proof of Cases 1a and 2a, we have $z \in \mu_{\hat{\gamma}}$ for all $z \in f(x_1^n)$.

Case 2: $\alpha = q_\delta$. Then, $(x_1)_1, (x_2)_1, \cdots, (x_n)_1\, \alpha\, \mu$ since $2\delta = 1+\gamma$. Analogous to the proof of Case 1, we may prove that $z \in \mu_{\hat{\gamma}}$, for all $z \in f(x_1^n)$.

Thus, in any case, we have $z \in \mu_{\hat{\gamma}}$, for all $z \in f(x_1^n)$. In a similar way, we may prove that, for all $a_1^{i-1}, a_{i+1}^n, b \in \mu_{\hat{\gamma}}$ and $1 \leq i \leq n$, there exists $x_i \in \mu_{\hat{\gamma}}$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$.

Therefore, $\mu_{\hat{\gamma}}$ is an n-ary subhypergroup of H. □

Note that if μ is an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H, then the set $\mu_{\hat{\gamma}}$ is not an n-ary subhypergroup of H in general. In fact, let $(H, \cdot)$ be a group with identity e such that there exists a non-empty subset S of H satisfying $S \cup \{e\}$ is not a subgroup of H. Define $f(x_1^n) = ex_1 \cdots x_n$, for all $x_1^n \in H$ and a fuzzy subset μ of H by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = e, \\ 0.4 & \text{if } x \in S, \\ 0.2 & \text{otherwise,} \end{cases}$$

for all $x \in S$. Then, (H, f) is an n-ary hypergroup and μ is an $(\in_{0.2} \wedge q_{0.6}, \beta)$-fuzzy n-ary subhypergroup of H but $\mu_{\widehat{0.2}} = S \cup \{e\}$ is not an n-ary subhypergroup of H.

Theorem 3.7. *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of H. Then, A is an n-ary subhypergroup of H if and only if the fuzzy subset μ of H such that $\mu(x) \geq \delta$, for all $x \in A$ and $\mu(x) \leq \gamma$ otherwise is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, where $\alpha \neq \in_\gamma \wedge q_\delta$.*

Proof. Assume that A is an n-ary hypergroup of H. Let $x_1^n \in H$ and $r_1, r_2, \cdots, r_n \in (\gamma, 1]$ such that $(x_1)_{r_1}, (x_2)_{r_2}, \cdots, (x_n)_{r_n}\, \alpha\, \mu$. Then, we have the following three cases.

Case 1: $\alpha = \in_\gamma$. Then, $(x_i)_{r_i} \in_\gamma \mu$ for $i \in \{1, 2, \cdots, n\}$, and so $\mu(x_i) \geq r_i > \gamma$ for $i \in \{1, 2, \cdots, n\}$. Thus, $\mu(x_i) \geq \delta$ for $i \in \{1, 2, \cdots, n\}$, i.e., $x_1^n \in A$.

Case 2: $\alpha = q_\delta$. Then, $(x_i)_{r_i}\, q_\delta\, \mu$ for $i \in \{1, 2, \cdots, n\}$, i.e., $\mu(x_i) + r_i > 2\delta$ for $i \in \{1, 2, \cdots, n\}$ and so $\mu(x_i) > 2\delta - r_i \geq 2\delta - 1 = \gamma$ for $i \in \{1, 2, \cdots, n\}$. It follows that $\mu(x_i) \geq \delta$ for $i \in \{1, 2, \cdots, n\}$, i.e., $x_1^n \in A$.

Case 3: $\alpha = \in_\gamma \vee q_\delta$. Then,

$$(x_{i_1})_{r_{i_1}}, \cdots, (x_{i_{k-1}})_{r_{i_{k-1}}} \in_\gamma \mu \ \ and \ \ (x_{i_k})_{r_{i_k}}, \cdots, (x_{i_n})_{r_{i_n}}\, q_\delta\, \mu$$

for any $k \in \{2, \cdots, n\}$, where $\{i_1, i_2, \cdots, i_n\}$ is a permutation of $\{1, 2, \cdots, n\}$. Hence, $\mu(x_{i_1}) \geq r_{i_1} > \gamma, \cdots, \mu(x_{i_{k-1}}) \geq r_{i_{k-1}} > \gamma$ and $\mu(x_{i_k}) + i_k > 2\delta, \cdots, \mu(x_{i_n}) + i_n > 2\delta$. Analogous to the proof of Cases 1 and 2, $\mu(x_{i_1}) \geq \delta, \cdots, \mu(x_{i_n}) \geq \delta$, i.e., $x_1^n \in A$.

Thus, in any case, $x_1^n \in A$. Hence, $f(x_1^n) \subseteq A$, which implies that $\mu(z) \geq \delta$, for all $z \in f(x_1^n)$. If $r_1 \wedge r_2 \wedge \cdots \wedge r_n \leq \delta$, then $\mu(z) \geq \delta \geq r_1 \wedge r_2 \wedge \cdots \wedge r_n > \gamma$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \mu$. If $r_1 \wedge r_2 \wedge \cdots \wedge r_n > \delta$, then $\mu(z) + r_1 \wedge r_2 \wedge \cdots \wedge r_n > \delta + \delta = 2\delta$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n}\, q_\delta\, \mu$. Therefore, $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \vee q_\delta\, \mu$, for all $z \in f(x_1^n)$.

In a similar way, we may show that

$$(a_1)_{r_1}, \cdots, (a_{i-1})_{r_{i-1}}, (a_{i+1})_{r_{i+1}}, \cdots, (a_n)_{r_n}, b_t\, \alpha\, \mu$$

and $1 \leq i \leq n$ implies $(x_i)_{r_1 \wedge r_2 \wedge \cdots \wedge r_{(i-1)} \wedge r_{(i+1)} \wedge \cdots \wedge r_n \wedge t} \in_\gamma \vee q_\delta\, \mu$ for some $x_i \in H$ with $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$.

Therefore, μ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H.

Conversely, assume that μ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H. It is easy to see that $A = \mu_{\bar{\gamma}}$. Hence, it follows from Theorem 3.6, A is an n-ary hypergroup of H. □

Theorem 3.8. *Let μ be a fuzzy subset of H. If μ is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, then the following conditions hold:*

(F3) $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$, *for all* $x_1^n \in H$,

(F4) *for all* $a_1^{i-1}, a_{i+1}^n, b \in H$ *and* $1 \leq i \leq n$, *there exists* $x_i \in H$ *such that* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ *and* $\max\{\mu(x_i), \gamma\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\}$.

Proof. Let μ be an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H and $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \gamma\} < \min\{\mu_{x_1}^{x_n}, \delta\}$, then, for all r such that

$$2\delta - \max\{\mu(z), \gamma\} > r > 2\delta - \min\{\mu_{x_1}^{x_n}, \delta\},$$

we have

$$2\delta - \mu(z) \geq 2\delta - \max\{\mu(z), \gamma\} > r > \max\{2\delta - \mu(x_1), 2\delta - \mu(x_2), \cdots, 2\delta - \mu(x_n), \delta\}$$

and so $\mu(x_1) + r > 2\delta$, $\mu(x_2) + r > 2\delta$, $\cdots$, $\mu(x_n) + r > 2\delta$, $\mu(z) + r < 2\delta$ and $\mu(z) < \delta < r$. Hence, $(x_1)_r\, q_\delta\, \mu, (x_2)_r\, q_\delta\, \mu, \cdots, (x_n)_r\, q_\delta\, \mu$ but $z_r\, \overline{\in_\gamma \vee q_\delta}\, \mu$, a contradiction. Thus, $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$, for all $x_1^n \in H$. Therefore, (F3) is satisfied.

Now, let $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$. Let $x_i \in H$ be such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ and $\max\{\mu(x_i), \gamma\} < \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\}$. Then, for all r such that

$$2\delta - \max\{\mu(x_i), \gamma\} > r > 2\delta - \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\},$$

we have

$$\begin{aligned} &2\delta - \mu(x_i) > 2\delta - \max\{\mu(x_i), \gamma\} > r \\ &> \max\{2\delta - \mu(a_1), \cdots, 2\delta - \mu(a_{i-1}), 2\delta - \mu(a_{i+1}), \cdots, 2\delta - \mu(a_n), \mu(b), \delta\}, \end{aligned}$$

and so $\mu(a_1)+r>2\delta,\cdots,\mu(a_{i-1})+r>2\delta$, $\mu(a_{i+1})+r>2\delta,\cdots,\mu(a_n)+r>2\delta$, $\mu(b)+r>2\delta, \mu(x_i)+r<2\delta$ and $\mu(x_i)<\delta<r$. Hence,

$$(a_1)_r\, q_\delta\, \mu,\cdots,(a_{i-1})_r\, q_\delta\, \mu,(a_{i+1})_r\, q_\delta\, \mu,\cdots,(a_n)_r\, q_\delta\, \mu, b_r\, q_\delta\, \mu$$

but $(x_i)_r\, \overline{\in_\gamma \vee q_\delta}\, \mu$, a contradiction. Therefore,

$$\max\{\mu(x_i),\gamma\}\geq \min\{\mu_{a_1}^{a_{i-1}},\mu_{a_{i+1}}^{a_n},\mu(b),\delta\}$$

and so (F4) is satisfied. □

Theorem 3.9. *Let $2\delta=1+\gamma$ and A be an n-ary subhypergroup of H. Let μ be a fuzzy subset of H such that $\mu(x)\leq\gamma$, for all $x\in H-A$ and $\mu(x)>\gamma$ otherwise. Then, μ is an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if μ is constant on A or $\mu(x)\geq\delta$, for all $x\in A$.*

Proof. Let μ be an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H. If μ is not constant on A, we assert that there exists $x\in A$ such that $\mu(x)\geq\delta$. In fact, if $\mu(x)<\delta$ for all $x\in A$, since μ is not constant on A, there exist $x,y\in A$ such that $\mu(x)\neq\mu(y)$. We consider the following cases.

Case 1: $\delta>\mu(x)>\mu(y)$. Since μ is an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H, there exists $a\in H$ with $y\in f(a,x*)$ such that $\max\{\mu(a),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$, by Theorem 3.8. It follows that $\mu(a)\geq\mu(y)>\gamma$ since $y\in A$. Now, for all r such that

$$1=2\delta-\gamma\geq 2\delta-\mu(y)>r>2\delta-\mu(x)>\delta>\mu(y),$$

we have $x_r\, q_\delta\, \mu$, $a_1\, q_\delta\, \mu$, but $y_r\, \overline{\in_\gamma\vee q_\delta}\, \mu$, a contradiction.

Case 2: $\mu(x)<\mu(y)<\delta$. Since μ is an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H, there exists $b\in H$ with $x\in f(b,y*)$ such that $\max\{\mu(b),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$, by Theorem 3.8. It follows that $\mu(a)\geq\mu(x)>\gamma$ since $\delta>\mu(y)>\mu(x)>\gamma$. Now, for all r such that $1=2\delta-\gamma\geq 2\delta-\mu(x)>r>2\delta-\mu(y)>\delta>\mu(x)$, we have $y_r\, q_\delta\, \mu$, $b_1\, q_\delta\, \mu$, but $x_r\, \overline{\in_\gamma\vee q_\delta}\, \mu$, a contradiction.

Therefore, there exists $x\in A$ such that $\mu(x)\geq\delta$. Now, suppose that $y\in A$ such that $\mu(y)<\delta$. Since μ is an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H, there exists $a\in H$ with $y\in f(a,x*)$ such that $\max\{\mu(a),\gamma\}\geq\min\{\mu(x),\mu(y),\delta\}$. It follows that $\mu(a)\geq\mu(y)>\gamma$ since $\mu(x)\geq\delta>\mu(y)>\gamma$. Now, for all r such that $1=2\delta-\gamma\geq 2\delta-\mu(y)>r>\delta>\mu(y)$, we have $x_r\, q_\delta\, \mu$, $a_1\, q_\delta\, \mu$, but $y_r\, \overline{\in_\gamma\vee q_\delta}\, \mu$, a contradiction. Therefore, $\mu(x)\geq\delta$, for all $x\in A$.

Conversely, assume that μ is constant on A or $\mu(x)\geq\delta$, for all $x\in A$. Then, from Theorem 3.7, in both cases, μ is clearly an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H. □

Corollary 3.10. *Let $2\delta=1+\gamma$ and μ be an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy n-ary subhypergroup of H. Then, μ is constant on $\mu_{\hat{\gamma}}$ or $\mu(x)\geq\delta$, for all $x\in\mu_{\hat{\gamma}}$.*

Proof. The proof is straightforward, by Theorems 3.6 and 3.9. □

Proposition 3.11. *Each $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H. If $2\delta = 1+\gamma$, then each $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup μ of H is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H.*

Proof. If μ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, then it is clearly that it is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H since $x_r\, q_\delta\, \mu$ implies $x_r \in_\gamma \vee q_\delta \mu$, for all $x \in H$ and $r \in (\gamma, 1]$.

Conversely, let $2\delta = 1+\gamma$. If μ is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, then, by Corollary 3.10, we have the following two cases:

Case 1: μ is constant on $\mu_{\hat{\gamma}}$. Let $\mu(x) = r$, for all $x \in \mu_{\hat{\gamma}}$ and let $r_i \in (\gamma, 1]$ and $x_i \in H$ be such that $(x_i)_{r_i} \in_\gamma \vee q_\delta\, \mu$, for all $i \in \{1, 2, \cdots, n\}$. Then, it is clear that $x_i \in \mu_{\hat{\gamma}}$, and $r \geq r_i$ or $r + r_i > 2\delta$, for all $i \in \{1, 2, \cdots, n\}$. Thus, for any $z \in f(x_1^n)$, we have $z \in \mu_{\hat{\gamma}}$, i.e., $\mu(z) = r$, since $\mu_{\hat{\gamma}}$ is an n-ary subhypergroup of H, by Theorem 3.6. Now, if $r + r_i > 2\delta$, for all $i \in \{1, 2, \cdots, n\}$, then $\mu(z) + (r_1 \wedge r_2 \wedge \cdots \wedge r_n) = r + (r_1 \wedge r_2 \wedge \cdots \wedge r_n) > 2\delta$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} q_\delta \mu$; otherwise $\mu(z) = r \geq r_1 \wedge r_2 \wedge \cdots \wedge r_n > \gamma$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \mu$. It follows that $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \vee q_\delta \mu$. Similarly, we have $(a_1)_{r_1}, \cdots, (a_{i-1})_{r_{i-1}}, (a_{i+1})_{r_{i+1}}, \cdots, (a_n)_{r_n}, b_t \in_\gamma \vee q_\delta\, \mu$ and $1 \leq i \leq n$ implies $(x_i)_{r_1 \wedge r_2 \wedge \cdots \wedge r_{(i-1)} \wedge r_{(i+1)} \wedge \cdots \wedge r_n \wedge t} \in_\gamma \vee q_\delta \mu$ for some $x_i \in H$ with $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, μ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H.

Case 2: $\mu(x) \geq \delta$, for all $x \in \mu_{\hat{\gamma}}$. Then, by Theorems 3.6 and 3.7, μ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, as required.

□

Theorem 3.12. *A fuzzy subset μ of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if it satisfies (F3) and (F4).*

Proof. (F1)⇒ (F3). Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \gamma\} < r = \min\{\mu_{x_1}^{x_n}, \delta\}$, then $\mu(x_1) \geq r > \gamma, \mu(x_2) \geq r > \gamma, \cdots, \mu(x_n) \geq r > \gamma, \mu(z) < r$ and $\mu(z) + r < 2r \leq 2\delta$, i.e., $(x_1)_r \in_\gamma \mu, (x_2)_r \in_\gamma \mu, \cdots, (x_n)_r \in_\gamma \mu$ but $z_r\, \overline{\in_\gamma \vee q_\delta}\, \mu$, a contradiction. Thus, $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$, for all $x_1^n \in H$. Hence, (F3) is satisfied.

(F3)⇒ (F1) Let $(x_1)_{r_1}, (x_2)_{r_2}, \cdots, (x_n)_{r_n} \in_\gamma \mu$. If $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \overline{\in_\gamma \vee q_\delta}\, \mu$ for some $z \in f(x_1^n)$, then $\mu(x_1) \geq r_1 > \gamma, \mu(x_2) \geq r_2 > \gamma, \cdots, \mu(x_n) \geq r_n > \gamma, \mu(z) < r_1 \wedge r_2 \wedge \cdots \wedge r_n$ and $\mu(z) + r_1 \wedge r_2 \wedge \cdots \wedge r_n \leq 2\delta$. It follows that $\mu(z) < \min\{\mu_{x_1}^{x_n}\}$ and $\mu(z) < \delta$. Hence, $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \leq \max\{\mu(z), \gamma\} < \min\{\mu_{x_1}^{x_n}, \delta\}$, a contradiction. Hence, (F1) is satisfied.

(F2)⇒ (F4). Let $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$. If for all $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$, we have $\max\{\mu(x_i), \gamma\} < r = \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\}$, then

$$\mu(a_1) \geq r > \gamma, \cdots, \mu(a_{i-1}) \geq r > \gamma,\ \mu(a_{i+1}) \geq r > \gamma, \cdots,$$
$$\mu(a_n) \geq r > \gamma,\ \mu(b) \geq r > \gamma,$$

$\mu(x_i) < r$ and $\mu(x_i) + r < 2r \leq 2\delta$, i.e., $(a_1)_r \in_\gamma \mu, \cdots, (a_{i-1})_r \in_\gamma \mu, (a_{i+1})_r \in_\gamma \mu, \cdots, (a_n)_r \in_\gamma \mu, b_r \in_\gamma \mu$ but $(x_i)_r\, \overline{\in_\gamma \vee q_\delta}\, \mu$, a contradiction. Hence, (F4) is valid.

(F4)$\Rightarrow$ (F2) Let $(a_1)_{r_1}, \cdots, (a_{i-1})_{r_{i-1}}, (a_{i+1})_{r_{i+1}}, \cdots, (a_n)_{r_n}, b_t \in_\gamma \mu$ and $1 \leq i \leq n$. If for all $x_i \in H$ with

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n),\ (x_i)_{r_1\wedge r_2 \wedge \cdots \wedge r_{(i-1)} \wedge r_{(i+1)} \wedge \cdots \wedge r_n \wedge b_t} \overline{\in_\gamma \vee q_\delta}\ \mu,$$

then $\mu(a_1) \geq r_1 > \gamma, \cdots, \mu(a_{i-1}) \geq r_{i-1} > \gamma, \mu(a_{i+1}) \geq r_{i+1} > \gamma, \cdots, \mu(a_n) \geq r_n > \gamma, \mu(b) \geq t > \gamma, \mu(x_i) < r_1 \wedge \cdots \wedge r_{i-1} \wedge r_{i+1} \wedge \cdots \wedge r_n \wedge t$ and $\mu(x_i) + \mu(x_i) < r_1 \wedge \cdots \wedge r_{i-1} \wedge r_{i+1} \wedge \cdots \wedge r_n \wedge t \leq 2\delta$. It follows that $\mu(x_i) < \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\}$ and $\mu(x_i) < \delta$, i.e., $\max\{\mu(x_i), \gamma\} < \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\}$, a contradiction. Hence, (F2) is satisfied.

Therefore, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if conditions (F3) and (F4) hold. □

As a direct consequence of Theorems 3.8 and 3.12, we have the following result.

Proposition 3.13. *Each $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup μ of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H.*

The following example shows that the converse of Proposition 3.13, may not be true.

Example 3.14. *Let $H = \{x, y, z\}$ be a set with a 3-ary hyperoperation f as follows:*

$f(x,x,x) = x$	$f(y,y,x) = \{x,z\}$	$f(z,x,x) = z$
$f(x,x,y) = y$	$f(y,y,y) = \{y,z\}$	$f(z,x,y) = \{y,z\}$
$f(x,x,z) = z$	$f(y,y,z) = H$	$f(z,x,z) = \{x,y\}$
$f(x,y,x) = y$	$f(y,x,x) = y$	$f(z,y,x) = \{y,z\}$
$f(x,y,y) = \{x,z\}$	$f(y,x,y) = \{x,z\}$	$f(z,y,y) = H$
$f(x,y,z) = \{y,z\}$	$f(y,x,z) = \{y,z\}$	$f(z,y,z) = H$
$f(x,z,x) = z$	$f(y,z,x) = \{y,z\}$	$f(z,z,x) = \{x,y\}$
$f(x,z,y) = \{y,z\}$	$f(y,z,y) = H$	$f(z,z,y) = H$
$f(x,z,z) = \{x,y\}$	$f(y,z,z) = H$	$f(z,z,z) = \{y,z\}$.

Then, (H, f) is a 3-ary hypergroup (see [24]*). Define a fuzzy subset μ of H by*

$$\mu(x) = 0.6,\ \mu(y) = 0.3 \text{ and } \mu(z) = 0.4.$$

Then, μ is an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy 3-ary subhypergroup of H, but it is not an $(q_{0.6}, \in_{0.4} \vee q_{0.6})$-fuzzy 3-ary subhypergroup of H since $x_{0.7}\, q_{0.6}\, \mu$, $z_{0.9}\, q_{0.6}\, \mu$ and $y \in f(x,z,z)$ but $y_{0.7}\, \overline{\in_{0.4} \vee q_{0.6}}\, \mu$.

Remark 3.15. *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup μ of H, we can conclude that*

(1) μ is a fuzzy n-ary subhypergroup of H when $\gamma = 0$ and $\delta = 1$ (see [14]*);*

(2) μ is an $(\in, \in \vee q)$-fuzzy n-ary subhypergroup of H when $\gamma = 0$ and $\delta = 0.5$ (see [24]*);*

(3) μ is a fuzzy n-ary subhypergroup of H with thresholds (γ, δ) (see [24]*).*

For any $\mu \in \mathbb{F}(H)$, we define $\mu_r = \{x \in H | x_r \in_\gamma \mu\}$, $\mu_r^\delta = \{x \in H | x_r\, q_\delta\, \mu\}$ and $[\mu]_r^\delta = \{x \in H | x_r \in_\gamma \vee q_\delta\ \mu\}$ for all $r \in (\gamma, 1]$. It is clear that $[\mu]_r^\delta = \mu_r \cup \mu_r^\delta$.

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroups of H and crisp n-ary hypergroups of H.

Theorem 3.16. *Let $\mu \in \mathbb{F}(H)$.*

(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if $\mu_r(\neq \emptyset)$ is an n-ary hypergroup of H, for all $r \in (\gamma, \delta]$.*

(2) *If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if $\mu_r^\delta(\neq \emptyset)$ is an n-ary hypergroup of H, for all $r \in (\delta, 1]$.*

(3) *If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if $[\mu]_r^\delta(\neq \emptyset)$ is an n-ary hypergroup of H, for all $r \in (\gamma, 1]$.*

Proof. We only prove (3). The other properties can be similarly proved. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H and $x_1^n \in [\mu]_r^\delta$ for some $r \in (\gamma, 1]$. Then, $(x_i)_r \in_\gamma \vee q_\delta \mu$, for all $i \in \{1, 2, \cdots n\}$, i.e., $\mu(x_i) \geq r > \gamma$ or $\mu(x_i) \geq 2\delta - r > \gamma$, for all $i \in \{1, 2, \cdots n\}$. Hence, $\gamma < \min\{\mu_{x_1}^{x_n}, \delta\}$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, we have $\mu(z) \geq \min\{\mu_{x_1}^{x_n}, \delta\}$, for all $z \in f(x_1^n)$. We consider the following cases:

Case 1: $r \in (\gamma, \delta]$. Then, $2\delta - r \geq \delta \geq r$. We have the following subcases.

Case 1a: There exists $i \in \{1, 2, \cdots, n\}$ such that $\mu(x_i) \geq r$, then $\mu(z) \geq \min\{\mu_{x_1}^{x_n}, \delta\} \geq r$. Hence, $z_r \in_\gamma \mu$.

Case 1b: $\mu(x_i) > 2\delta - r$, for all $i \in \{1, 2, \cdots, n\}$. Then, $\mu(z) \geq \min\{\mu_{x_1}^{x_n}, \delta\} = \delta \geq r$. Hence, $z_r \in_\gamma \mu$.

Case 2: $r \in (\delta, 1]$. Then, $2\delta - r < \delta < r$. We have the following subcases.

Case 2a: $\mu(x_i) \geq r$, for all $i \in \{1, 2, \cdots, n\}$. Then, $\mu(z) \geq \min\{\mu_{x_1}^{x_n}, \delta\} = \delta > 2\delta - r$. Hence, $z_r \, q_\delta \, \mu$.

Case 2b: There exists $i \in \{1, 2, \cdots, n\}$ such that $\mu(x_i) > 2\delta - r$, then $\mu(z) \geq \min\{\mu_{x_1}^{x_n}, \delta\} > 2\delta - r$. Hence, $z_r \, q_\delta \, \mu$.

Thus, in any case, $z_r \in_\gamma \vee q_\delta \mu$, i.e., $z \in [\mu]_r^\delta$, for all $z \in f(x_1^n)$. Similarly, we can show that for all $a_1^{i-1}, a_{i+1}^n, b \in [\mu]_r^\delta$ and $1 \leq i \leq n$, there exists $x_i \in [\mu]_r^\delta$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, $[\mu]_r^\delta$ is an n-ary subhypergroup of H.

Conversely, assume that $[\mu]_r^\delta \neq \emptyset$ is an n-ary subhypergroup. Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\mu(z) = \max\{\mu(z), \gamma\} \geq r = \min\{\mu_{x_1}^{x_n}, \delta\}$, then $\mu(x_1) \geq r > \gamma, \mu(x_2) \geq r > \gamma, \cdots, \mu(x_n) \geq r > \gamma, \mu(z) < r$ and $\mu(z) + r < 2r \leq 2\delta$, i.e., $(x_1)_r \in_\gamma \mu, (x_2)_r \in_\gamma \mu, \cdots, (x_n)_r \in_\gamma \mu$ but $z_r \overline{\in_\gamma \vee q_\delta} \mu$, i.e., $x_1^n \in [\mu]_r^\delta$ but $z \notin [\mu]_r^\delta$, a contradiction. Hence, $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$. Similarly, we can show that condition (F4) holds. Therefore, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, by Theorem 3.12. □

As a direct consequence of Theorem 3.16, we have the following result.

Corollary 3.17. *Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta$, $\gamma' < \delta'$, $\gamma < \gamma'$ and $\delta' < \delta$. Then, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-fuzzy n-ary subhypergroup of H.*

The following example shows that the converse of Corollary 3.17 is not true in general.

Example 3.18. *Consider the 3-ary hypergroup* (H, f) *as defined in Example 3.14. Define a fuzzy subset* μ *of* H *by*

$$\mu(x) = 0.6,\ \mu(y) = 0.6 \text{ and } \mu(z) = 0.7.$$

Then, by Theorem 3.16, μ *is an* $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$*-fuzzy* n*-ary subhypergroup of* H*, but not an* $(\in_{0.4}, \in_{0.4} \vee q_{0.7})$*-fuzzy* n*-ary subhypergroup of* H*, since* $\mu_{0.7} = \{z\}$ *is not an* n*-ary subhypergroup of* H*.*

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.16, then we have the following result.

Corollary 3.19. Let $\mu \in \mathbb{F}(H)$.

(1) μ *is an* $(\in, \in \vee q)$*-fuzzy* n*-ary subhypergroup of* H *if and only if* $\mu_r (\neq \emptyset)$ *is an* n*-ary hypergroup of* H*, for all* $r \in (0, 0.5]$ *(see* [24]*).*

(2) μ *is an* $(\in, \in \vee q)$*-fuzzy* n*-ary subhypergroup of* H*if and only if* $\mu_r^{0.5} (\neq \emptyset)$ *is an* n*-ary hypergroup of* H*, for all* $r \in (0.5, 1]$*.*

(3) μ *is an* $(\in, \in \vee q)$*-fuzzy* n*-ary subhypergroup of* H *if and only if* $[\mu]_r^{0.5} (\neq \emptyset)$ *is an* n*-ary hypergroup of* H*, for all* $r \in (0, 1]$*.*

Theorem 3.20. *A fuzzy subset* μ *of* H *is an* $(\in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy* n*-ary subhypergroup of* H *if and only if the following conditions hold:*

(F5) $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$ *or* $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$, *for all* $x_1^n \in H$,

(F6) *for all* $a_1^{i-1}, a_{i+1}^n, b \in H$ *and* $1 \leq i \leq n$*, there exists* $x_i \in H$ *such that* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ *and* $\max\{\mu(x_i), \gamma\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b), \delta\}$ *or* $\max\{\mu(x_i), \delta\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\}$.

Proof. (F1)$\Rightarrow$ (F5) Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that

$$\max\{\mu(z), \delta\} < r = \min\{\mu_{x_1}^{x_n}\},$$

then $\mu(x_i) \geq r > \delta$ for $i \in \{1, 2, \cdots, n\}$ and $\mu(z) < r$. Hence, $(x_i)_\delta \in_\gamma \wedge q_\delta \mu$ for $i \in \{1, 2, \cdots, n\}$. It follows from (F1) that $z_\delta \in_\gamma \vee q_\delta \mu$, for all $z \in f(x_1^n)$, i.e., $\mu(z) \geq \delta$ or $\mu(z) + \delta > 2\delta$, for all $z \in f(x_1^n)$. Thus, $\max\{\mu(z), \gamma\} = \mu(z) \geq \delta = \min\{\mu_{x_1}^{x_n}, \delta\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}, \delta\}$. Hence, (F5) is satisfied.

(F5)$\Rightarrow$ (F1) Let $(x_i)_{r_i} \in_\gamma \wedge q_\delta \mu$, for all $i \in \{1, 2, \cdots, n\}$. If $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \overline{\in_\gamma \vee q_\delta} \mu$ for some $z \in f(x_1^n)$, then $\mu(x_i) \geq r_i > \gamma$ and $\mu(x_i) + r_i > 2\delta$ for all $i \in \{1, 2, \cdots, n\}$ and $\mu(z) + r_1 \wedge r_2 \wedge \cdots \wedge r_n \leq 2\delta$. It follows that $\mu(x_1) > \delta, \mu(x_2) > \delta, \cdots, \mu(x_n) > \delta$ and $\mu(z) < \delta$. Hence, $\max\{\inf_{z \in f(x_1^n)} \mu(z), \gamma\} \leq \max\{\mu(z), \gamma\} < \min\{\mu_{x_1}^{x_n}, \delta\}$ and $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \leq \max\{\mu(z), \delta\} < \min\{\mu_{x_1}^{x_n}\}$, a contradiction. Hence, (F1) is satisfied.

In a similar way, we may show that (F2)$\Leftrightarrow$ (F6)

Therefore, μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H if and only if conditions (F5) and (F6) hold. □

The following example shows that the converse of Proposition 3.2 is not true in general.

Example 3.21. *Consider the 3-ary hypergroup (H, f) as defined in Example 3.14. Define a fuzzy subset μ of H by*

$$\mu(x) = 0.9,\ \mu(y) = 0.5 \text{ and } \mu(z) = 0.4.$$

Then, μ is an $(\in_{0.3} \wedge q_{0.6}, \in_{0.3} \vee q_{0.6})$-fuzzy n-ary subhypergroup of H, but it is not an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-fuzzy n-ary subhypergroup of H since $z \in f(y, y, y)$ and $y_{0.5} \in_{0.3} \mu$ but $z_{0.5} \overline{\in_{0.3} \vee q_{0.6}} \mu$.

3.2. $(\in_\gamma \wedge q_\delta, \beta)$**-fuzzy n-ary Subhypergroups.** In this section, we study the properties of $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroups, where $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$. Before proceeding, we first provide an example of $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup as follows.

Example 3.22. *Consider the n-ary hypergroup (H, f) as defined in Example 3.5. Define a fuzzy subset μ of H by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0.6 & \text{otherwise,} \end{cases}$$

for all $x \in S$. Then, μ is an $(\in_{0.2} \wedge q_{0.6}, \beta)$-fuzzy n-ary subhypergroup of H.

Theorem 3.23. *Let μ be an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H. Then, the set $\mu_{\hat{\delta}}$ is an n-ary subhypergroup of H.*

Proof. Assume that μ is an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H. Let $x_1^n \in \mu_{\hat{\delta}}$. Then, $\mu(x_1) > \delta, \mu(x_2) > \delta, \cdots, \mu(x_n) > \delta$. We have the following cases.

Case 1: $\beta \in \{\in_\gamma, \in_\gamma \wedge q_\delta\}$. Then, for all r such that $\delta < r < \min\{\mu_{x_1}^{x_n}\}$, we have $(x_1)_r \in_\gamma \wedge q_\delta \mu, (x_2)_r \in_\gamma \wedge q_\delta \mu, \cdots, (x_n)_r \in_\gamma \wedge q_\delta \mu$, and so $z_r \beta \mu$, for all $z \in f(x_1^n)$. Hence, $\mu(z) \geq r > \delta$.

Case 2: $\beta = q_\delta$. Then, $(x_1)_\delta \in_\gamma \wedge q_\delta \mu, (x_2)_\delta \in_\gamma \wedge q_\delta \mu, \ldots, (x_n)_\delta \in_\gamma \wedge q_\delta \mu$, and so $z_\delta \beta \mu$, for all $z \in f(x_1^n)$. Hence, $\mu(z) + \delta > 2\delta$ and so $\mu(z) > \delta$.

Case 3: $\beta = \in_\gamma \vee q_\delta$. Analogous to the proof of Cases 1 and 2, we have $\mu(z) > \delta$, for all $z \in f(x_1^n)$.

Thus, in any case, $\mu(z) > \delta$ and so $z \in \mu_{\hat{\delta}}$ for all $z \in f(x_1^n)$. In a similar way, we may show that for all $a_1^{i-1}, a_{i+1}^n, b \in \mu_{\hat{\delta}}$ and $1 \leq i \leq n$, there exists $x_i \in \mu_{\hat{\delta}}$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, $\mu_{\hat{\delta}}$ is an n-ary subhypergroup of H. □

Theorem 3.24. *Let A be a non-empty subset of H and μ a fuzzy subset of H be constant on A such that $\mu(x) = r > \delta$, for $x \in A$, and $\mu(x) \leq \delta$ otherwise, where $r \in (0, 1]$. Then, A is an n-ary subhypergroup of H if and only if μ is an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H.*

Proof. Suppose that A is an n-ary subhypergroup of H and μ is a fuzzy subset of H such that $\mu(x) = r > \delta$, for all $x \in \mu_{\hat{\delta}}$ and $\mu(x) \leq \delta$ otherwise. Let $x_1^n \in H$ be such that $(x_1)_{r_1} \in_\gamma \wedge q_\delta \mu$, $(x_2)_{r_2} \in_\gamma \wedge q_\delta \mu$, $\cdots$, $(x_n)_{r_n} \in_\gamma \wedge q_\delta \mu$. Then, $\mu(x_1) \geq r_1$ and $\mu(x_1) + r_1 > 2\delta$, $\mu(x_2) \geq r_2$ and $\mu(x_2) + r_2 > 2\delta, \cdots, \mu(x_n) \geq r_n$ and $\mu(x_n) + r_n > 2\delta$, which give that $\mu(x_1) = r > \delta$, $\mu(x_2) = r > \delta, \cdots, \mu(x_n) = r > \delta$. Indeed, from $\mu(x_i) \geq r_i$ and $\mu(x_i) + r_i > 2\delta$, we have $\mu(x_i) + \mu(x_i) \geq \mu(x_i) + r_i > 2\delta$, and so $\mu(x_i) > \delta$, for $i = 1, \ldots, n$. It follows that $x_i \in A$ and so $\mu(x_i) = r$.

Hence, $x_1^n \in A$ and so $z \in A$, for all $z \in f(x_1^n)$. It follows that $\mu(z) = r$ and so $\mu(z) \geq r_1 \wedge r_2 \wedge \cdots \wedge r_n$ and $\mu(z) + r_1 \wedge r_2 \wedge \cdots \wedge r_n > 2\delta$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \mu$ and $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} q_\delta \, \mu$. Hence, $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \beta \, \mu$. In a similar way, we may show that condition (F2) holds. Therefore, μ is an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H.

Conversely, assume that μ is an $(\in_\gamma \wedge q_\delta, \beta)$-fuzzy n-ary subhypergroup of H. It is easy to see that $A = \mu_{\hat{\delta}}$. Hence, it follows from Theorem 3.23, that A is an n-ary subhypergroup of H. □

Theorem 3.25. *Let μ be a fuzzy subset of H. If μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H, then the following conditions hold:*

(F7) $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$, *for all $x_1^n \in H$,*

(F8) *for all $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, there exists $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ and* $\max\{\mu(x_i), \delta\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\}$.

Proof. Let μ be an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H. We only show (F7). (F8) can be similarly proved. Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \delta\} < \min\{\mu_{x_1}^{x_n}\}$, then for all r such that $2\delta - \max\{\mu(z), \delta\} > r > 2\delta - \min\{\mu_{x_1}^{x_n}\}$, we have

$$\min\{2\delta - \mu(z), \delta\} > r > \max\{2\delta - \mu(x_1), 2\delta - \mu(x_2), \cdots, 2\delta - \mu(x_n)\}$$

and so $\mu(z) + r < 2\delta, \mu(x_1) + r > 2\delta > 2r, \mu(x_2) + r > 2\delta > 2r, \cdots, \mu(x_n) + r > 2\delta > 2r$. Hence, $(x_1)_r \in_\gamma \wedge q_\delta \, \mu, (x_2)_r \in_\gamma \wedge q_\delta \, \mu, \cdots, (x_n)_r \in_\gamma \wedge q_\delta \, \mu$, but $z_r \, q_\delta \, \mu$, a contradiction. Therefore, $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$. Hence, (F7) is satisfied. □

Theorem 3.26. *Let A be an n-ary subhypergroup of H and μ a fuzzy subset of H such that $\mu(x) \leq \delta$, for all $x \in H - A$ and $\mu(x) > \delta$ otherwise. Then, μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H if and only if μ is constant on A.*

Proof. Suppose that μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H. If μ is not constant on A, then there exist $x, y \in \mu_{\hat{\delta}}$ such that $\mu(x) \neq \mu(y)$. We have the following cases.

- **Case 1:** $\mu(x) > \mu(y) > \delta$. Since μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H, there exists $a \in H$ with $y \in f(a, x*)$ such that $\max\{\mu(a), \delta\} \geq \min\{\mu(x), \mu(y)\} = \mu(y) > \delta$ by Theorem 3.23. If follows that $\mu(a) > \delta$. Now, for all r such that $\delta > 2\delta - \mu(y) > r > 2\delta - \mu(x)$, we have $x_r \in_\gamma \wedge q_\delta \, \mu$, $a_{\mu(a)} \in_\gamma \wedge q_\delta \, \mu$, but $y_{r \wedge \mu(a)} = y_r \, \overline{q_\delta} \, \mu$, a contradiction.
- **Case 2:** $\mu(y) > \mu(x) > \delta$. Since μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H, there exists $b \in H$ with $x \in f(b, y*)$ such that $\max\{\mu(a), \delta\} \geq \min\{\mu(x), \mu(y)\} = \mu(x) > \delta$ by Theorem 3.23. If follows that $\mu(b) > \delta$. Now, for all r such that $\delta > 2\delta - \mu(x) > r > 2\delta - \mu(y)$, we have $y_r \in_\gamma \wedge q_\delta \, \mu$, $b_{\mu(b)} \in_\gamma \wedge q_\delta \, \mu$, but $x_{r \wedge \mu(b)} = x_r \, \overline{q_\delta} \, \mu$, a contradiction.

Therefore, μ is constant on A.

Conversely, assume that μ is constant on A. Then, μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H, by Theorem 3.24. □

Corollary 3.27. *Let μ be an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H. Then, μ is constant on $\mu_{\hat{\delta}}$.*

Proof. It is straightforward, by Theorems 3.23 and 3.26. □

From Theorems 3.23, 3.24 and Corollary 3.27, we have the following result.

Proposition 3.28. *A fuzzy subset μ of H is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H if and only if it is an $(\in_\gamma \wedge q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H.*

Proof. By Definition 3.1, it is clear that every $(\in_\gamma \wedge q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H. Now, assume that μ is an $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup μ of H. By Theorems 3.23 and 3.27, $\mu_{\hat{\delta}}$ is an n-ary subhypergroup of H and μ is constant on $\mu_{\hat{\delta}}$, and then Theorem 3.24 implies that μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H. □

Theorem 3.29. *A fuzzy subset μ of H is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H if and only if the following conditions hold:*

(F7) $\max\{\inf_{z\in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$, *for all* $x_1^n \in H$,

(F8) *for all* $a_1^{i-1}, a_{i+1}^n, b \in H$ *and* $1 \leq i \leq n$, *there exists* $x_i \in H$ *such that* $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ *and* $\max\{\mu(x_i), \delta\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\}$.

Proof. (F1)⇒ (F7) Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \delta\} < r = \min\{\mu_{x_1}^{x_n}\}$, then $\mu(z) < r$, $\mu(x_1) \geq r > \delta, \mu(x_1) + r \geq 2r > 2\delta, \mu(x_2) \geq r > \delta, \mu(x_2) + r \geq 2r > 2\delta, \cdots, \mu(x_n) \geq r > \delta, \mu(x_n) + r \geq 2r > 2\delta$, i.e., $(x_1)_r \in_\gamma \wedge q_\delta \mu, (x_2)_r \in_\gamma \wedge q_\delta \mu, \cdots, (x_n)_r \in_\gamma \wedge q_\delta \mu$, but $z_r \overline{\in_\gamma} \mu$, a contradiction. Therefore, $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z\in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$. Hence, (F7) is satisfied.

(F7)⇒ (F1) Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $(x_i)_{r_i} \in_\gamma \wedge q_\delta \mu$, for all $i \in \{1, 2, \cdots, n\}$ but $z_{r_1\wedge r_2\wedge\cdots\wedge r_n} \overline{\in_\gamma} \mu$, then $\mu(x_1) \geq r_1$ and $\mu(x_1) + r_1 > 2\delta$, $\mu(x_2) \geq r_1$ and $\mu(x_2) + r_1 > 2\delta, \cdots$, $\mu(x_n) \geq r_n$ and $\mu(x_n) + r_n > 2\delta$ but $\mu(z) < r_1 \wedge r_2 \wedge \cdots \wedge r_n$. It follows that $\min\{\mu_{x_1}^{x_n}\} \geq r_1 \wedge r_2 \wedge \cdots \wedge r_n > \mu(z)$ and $\mu(x_1) > \delta, \mu(x_2) > \delta, \cdots, \mu(x_n) > \delta$, and so $\min\{\mu_{x_1}^{x_n}\} > \max\{\mu(z), \delta\}$, a contradiction. Hence, (F1) is satisfied.

In a similar way we may show that (F2)⇔ (F8). □

Proposition 3.30. *Each $(\in_\gamma \wedge q_\delta, q_\delta)$-fuzzy n-ary subhypergroup of H is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H.*

The following examples show that the converse of Propositions 3.3 and 3.30 are not true in general.

Example 3.31. *Consider the fuzzy n-ary subhypergroup defined in Example 3.18. It is not difficult to check that μ is an $(\in_{0.4} \wedge q_{0.6}, \in_{0.4} \vee q_{0.6})$-fuzzy n-ary subhypergroup of H, but it is not an $(\in_{0.4} \wedge q_{0.6}, \in_{0.4})$-fuzzy n-ary subhypergroup of H, since $z_{0.7} \in_{0.4} \wedge q_{0.6}\ \mu$ and $y \in f(z, z, z)$ but $y_{0.7} \overline{\in_{0.4}} \mu$.*

Example 3.32. *Consider the fuzzy n-ary subhypergroup defined in Example 3.14. Define a fuzzy subset μ of H by*

$$\mu(x) = 0.9,\ \mu(y) = 0.7 \text{ and } \mu(z) = 0.7.$$

Then, μ is an $(\in_{0.3} \wedge q_{0.6}, \in_{0.3})$-fuzzy n-ary subhypergroup of H, but it is not an $(\in_{0.3} \wedge q_{0.6}, q_{0.6})$-fuzzy n-ary subhypergroup of H since $y \in f(y, x, x)$, $y_{0.6} \in_{0.3} \wedge q_{0.6}\, \mu$ and $x_{0.5} \in_{0.3} \wedge q_{0.6}\, \mu$, but $y_{0.6\wedge 0.5}\, \overline{q_{0.6}}\, \mu$.

The next theorem provides the relationship between $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroups of H and crisp n-ary hypergroups of H.

Theorem 3.33. *Let $\mu \in \mathbb{F}(H)$.*

(1) μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H if and only if $\mu_r (\neq \emptyset)$ is an n-ary subhypergroup of H, for all $r \in (\delta, 1]$.

(2) μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H if and only if $\mu_r^\delta (\neq \emptyset)$ is an n-ary subhypergroup of H, for all $r \in (\gamma, \delta]$.

Proof. (1) Let μ be an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H and $x_1^n \in \mu_r$ for some $r \in (\delta, 1]$. Then, $(x_1)_r \in_\gamma \mu, (x_2)_r \in_\gamma \mu, \cdots, (x_n)_r \in_\gamma \mu$, i.e., $\mu(x_1) \geq r, \mu(x_2) \geq r, \cdots, \mu(x_n) \geq r$. Since μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H, we have $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\} \geq r$, for all $z \in f(x_1^n)$. It follows from $r \in (\delta, 1]$ that $\mu(z) \geq r > \gamma$. Hence, $z \in \mu_r$, for all $z \in f(x_1^n)$ proving (F7). Similarly, we may show that for all $a_1^{i-1}, a_{i+1}^n, b \in \mu_r$ and $1 \leq i \leq n$, there exists $x_i \in \mu_r$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, μ_r is an n-ary hypergroup of H.

Conversely, assume that $\mu_r \neq \emptyset$ is an n-ary subhypergroup of H. Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \delta\} < r = \min\{\mu_{x_1}^{x_n}\}$, then $r > \delta, (x_1)_r \in_\gamma \mu, (x_2)_r \in_\gamma \mu, \cdots, (x_n)_r \in_\gamma \mu$ but $z_r \overline{\in_\gamma} \mu$, i.e., $x_1^n \in \mu_r$ but $z \notin \mu_r$, a contradiction. Hence, $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$. Similarly, we can show that condition (F8) holds. Therefore, μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H.

(2) Let μ be an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H and $x_1^n \in \mu_r^\delta$ for some $r \in (\gamma, \delta]$. Then, $(x_1)_r\, q_\delta\, \mu, (x_2)_r\, q_\delta\, \mu, \cdots, (x_n)_r\, q_\delta\, \mu$, i.e., $\mu(x_1) + r > 2\delta, \mu(x_2) + r > 2\delta, \cdots, \mu(x_n) + r > 2\delta$. Since μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H, we have $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\} > 2\delta - r$, for all $z \in f(x_1^n)$. It follows from $r \in (\gamma, \delta]$ that $\delta \leq 2\delta - r$, and so $\mu(z) > 2\delta - r$, i.e., $z \in \mu_r^\delta$, for all $z \in f(x_1^n)$. Similarly, we may show that for all $a_1^{i-1}, a_{i+1}^n, b \in \mu_r^\delta$ and $1 \leq i \leq n$, there exists $x_i \in \mu_r^\delta$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, μ_r^δ is an n-ary hypergroup of H.

Conversely, $\mu_r^\delta \neq \emptyset$ is an n-ary subhypergroup of H. Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \delta\} < \min\{\mu_{x_1}^{x_n}\}$, then for all r such that $2\delta - \max\{\mu(z), \delta\} > r > 2\delta - \min\{\mu_{x_1}^{x_n}\}$, we have $\min\{2\delta - \mu(z), \delta\} > r > \max\{2\delta - \mu(x_1), 2\delta - \mu(x_2), \cdots, 2\delta - \mu(x_n)\}$, and so $r < \delta, \mu(z) + r < 2\delta, \mu(x_1) + r > 2\delta, \mu(x_2) + r > 2\delta, \cdots, \mu(x_n) + r > 2\delta$,. Hence, $(x_1)_r\, q_\delta\, \mu, (x_2)_r\, q_\delta\, \mu, \cdots, (x_n)_r\, q_\delta\, \mu$ but $z_r\, \overline{q_\delta}\, \mu$, i.e., $x_1^n \in \mu_r^\delta$ but $z \notin \mu_r^\delta$, a contradiction. Hence, $\max\{\mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z \in f(x_1^n)} \mu(z), \delta\} \geq \min\{\mu_{x_1}^{x_n}\}$. Similarly, we can show that condition (F8) holds. Therefore, μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. □

As a direct consequence of Theorem 3.33, we have the following result.

Corollary 3.34. *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such hat $\gamma < \delta$, $\gamma' < \delta'$ and $\delta' < \delta$. Then, every $(\in_{\gamma'} \wedge q_{\delta'}, \in_{\gamma'})$-fuzzy n-ary subhypergroup of H is an $(\in_{\gamma} \wedge q_{\delta}, \in_{\gamma})$-fuzzy n-ary subhypergroup of H.*

The following example shows that the converse of Corollary 3.34 is not true in general.

Example 3.35. *Consider the 3-ary hypergroup (H, f) and μ as defined in Example 3.21. Then, by Theorem 3.33, μ is an $(\in_{0.3} \wedge q_{0.5}, \in_{0.3})$-fuzzy n-ary subhypergroup of H, but not an $(\in_{0.3} \wedge q_{0.4}, \in_{0.3})$-fuzzy n-ary subhypergroup of H, since $\mu_{0.5} = \{x, y\}$ is not an n-ary subhypergroup of H.*

3.3. **Other Types of (α, β)-fuzzy n-ary Subhypergroups.**

Theorem 3.36. *A fuzzy subset μ of H is an $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroup of H if and only if the following conditions hold:*

(F9) $\max\{\inf_{z\in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\}$, *for all $x_1^n \in H$,*

(F10) *for all $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, there exists $x_i \in H$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ and* $\max\{\mu(x_i), \gamma\} \geq \min\{\mu_{a_1}^{a_{i-1}}, \mu_{a_{i+1}}^{a_n}, \mu(b)\}$.

Proof. (F1)$\Rightarrow$ (F9) Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \gamma\} < r = \min\{\mu_{x_1}^{x_n}\}$, then $\mu(x_1) \geq r > \gamma, \mu(x_2) \geq r > \gamma, \cdots, \mu(x_n) \geq r > \gamma$ and $\mu(z) < r$, i.e., $(x_1)_r \in_{\gamma} \mu, (x_2)_r \in_{\gamma} \mu, \cdots, (x_n)_r \in_{\gamma} \mu$ but $z_r \overline{\in_{\gamma}} \mu$, a contradiction. Thus, $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z\in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\}$. Hence, (F9) is satisfied.

(F9)$\Rightarrow$ (F1) Let $(x_1)_{r_1}, (x_2)_{r_2}, \cdots, (x_n)_{r_n} \in_{\gamma} \mu$. If $z_{r_1\wedge r_2\wedge\cdots\wedge r_n} \overline{\in_{\gamma}} \mu$ for some $z \in f(x_1^n)$, then $\mu(x_1) \geq r_1 > \gamma, \mu(x_2) \geq r_2 > \gamma, \cdots, \mu(x_n) \geq r_n > \gamma$ and $\mu(z) < r_1 \wedge r_2 \wedge \cdots \wedge r_n$. It follows that $\mu(z) < \min\{\mu_{x_1}^{x_n}\}$. Hence, $\max\{\inf_{z\in f(x_1^n)} \mu(z), \gamma\} \leq \max\{\mu(z), \gamma\} < \min\{\mu_{x_1}^{x_n}\}$, a contradiction. Hence, (F1) is satisfied.

In a similar way we may show that (F2)$\Leftrightarrow$ (F9). □

Remark 3.37. *For any $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroup μ of H, we can conclude that μ is the fuzzy n-ary subhypergroup of H when $\gamma = 0$ (see [14]).*

The next theorem provides the relationship between $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroups of H and crisp n-ary hypergroups of H.

Theorem 3.38. *Let $\mu \in \mathbb{F}(H)$. Then, μ is an $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroup of H if and only if $\mu_r (\neq \emptyset)$ is an n-ary subhypergroup of H, for all $r \in (\gamma, 1]$.*

Proof. Let μ be an $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroup of H and $x_1^n \in \mu_r$ for $r \in (\gamma, 1]$. Then, $(x_1)_r \in_{\gamma} \mu, (x_2)_r \in_{\gamma} \mu, \cdots, (x_n)_r \in_{\gamma} \mu$, i.e., $\mu(x_1) \geq r, \mu(x_2) \geq r, \cdots, \mu(x_n) \geq r$. Since μ is an $(\in_{\gamma}, \in_{\gamma})$-fuzzy n-ary subhypergroup of H, we have $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\} \geq r$, for all $z \in f(x_1^n)$. It follows from $r \in (\gamma, 1]$ that $\mu(z) \geq r > \gamma$. Hence, $z \in \mu_r$, for all $z \in f(x_1^n)$. Similarly, we may show that for all $a_1^{i-1}, a_{i+1}^n, b \in \mu_r$ and $1 \leq i \leq n$, there exists $x_i \in \mu_r$ such that $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, μ_r is an n-ary hypergroup of H.

Conversely, assume that $\mu_r \neq \emptyset$ is an n-ary subhypergroup of H. Let $x_1^n \in H$. If there exists $z \in f(x_1^n)$ such that $\max\{\mu(z), \gamma\} < r = \min\{\mu_{x_1}^{x_n}\}$, then $r > \gamma, (x_1)_r \in_\gamma \mu, (x_2)_r \in_\gamma \mu, \cdots, (x_n)_r \in_\gamma \mu$ but $z_r \overline{\in_\gamma} \mu$, i.e., $x_1^n \in \mu_r$ but $z \notin \mu_r$, a contradiction. Hence, $\max\{\mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\}$ and so $\max\{\inf_{z\in f(x_1^n)} \mu(z), \gamma\} \geq \min\{\mu_{x_1}^{x_n}\}$ proving (F9). Similarly, we can show that condition (F10) holds. Therefore, μ is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. □

Note that if μ is an $(\in_\gamma, \in_\gamma)$-fuzzy n-ary subhypergroup of H, then $\mu_r (\neq \emptyset)$ is an n-ary subhypergroup of H for all $r \in (\gamma, 1]$, by Proposition 3.38, and so $\mu_r (\neq \emptyset)$ is an n-ary subhypergroup of H, for all $r \in (\gamma, \delta]$, implying that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy n-ary subhypergroup of H, by Proposition 3.16.

Theorem 3.39. *Let $2\delta = 1+\gamma$, A be an n-ary subhypergroup of H and μ a fuzzy subset of H such that $\mu(x) \leq \gamma$, for all $x \in H - A$ and $\mu(x) > \gamma$ otherwise. Then, μ is an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H if and only if μ is constant on $\mu_{\hat{\gamma}}$.*

Proof. Let μ be an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H. If μ is not constant on $\mu_{\hat{\gamma}}$, then there exist $x, y \in \mu_{\hat{\gamma}}$ such that $\mu(x) \neq \mu(y)$. We have the following cases.

- **Case 1:** $\mu(x) > \mu(y)$. Since μ is an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H, $\mu_{\hat{\gamma}}$ is an n-ary subhypergroup of H, by Theorem 3.6. Hence, there exists $a \in H$ with $y \in f(a, x*)$ such that $a \in \mu_{\hat{\gamma}}$, i.e., $\mu(a) > \gamma$. Now, for all r such that $1 = 2\delta - \gamma \geq 2\delta - \mu(y) > r > 2\delta - \mu(x)$, we have $x_r\, q_\delta\, \mu$, $a_1\, q_\delta\, \mu$, but $y_r\, \overline{q_\delta}\, \mu$, a contradiction.
- **Case 2:** $\mu(x) < \mu(y)$. Analogous to the proof of Case 1, we may show that there exists r and $b \in \mu_{\hat{\gamma}}$ with $x \in f(b, y*)$ such that $y_r\, q_\delta\, \mu$, $b_1\, q_\delta\, \mu$, but $x_r\, \overline{q_\delta}\, \mu$, a contradiction.

Therefore, μ is constant on $\mu_{\hat{\gamma}}$.

Conversely, assume that μ is constant on $\mu_{\hat{\gamma}}$. It is easy to see that μ is an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H. □

Corollary 3.40. *Let $2\delta = 1+\gamma$ and μ be an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H. Then, μ is constant on $\mu_{\hat{\gamma}}$.*

Proof. It is straightforward, by Theorems 3.6 and 3.39. □

Theorem 3.41. *Let $2\delta = 1+\gamma$, A be an n-ary subhypergroup of H and μ a fuzzy subset of H such that $\mu(x) \leq \gamma$, for all $x \in H - A$ and $\mu(x) > \gamma$ otherwise. Then, μ is an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H if and only if $\mu(x) = 1$, for all $x \in A$.*

Proof. Let μ be an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. Then, $\mu_{\hat{\gamma}}$ is an n-ary subhypergroup of H. If there exists $x \in \mu_{\hat{\gamma}}$ such that $\mu(x) < 1$, then for any $y \in \mu_{\hat{\gamma}}$ there exists $a \in H$ with $x \in f(a, y*)$ such that $a \in \mu_{\hat{\gamma}}$, i.e., $\mu(a) > \gamma$. Hence, it follows from $2\delta = 1+\gamma$ that $y_1\, q_\delta\, \mu$ and $a_1\, q_\delta\, \mu$, but $x_1\, \overline{\in_\gamma}\, \mu$, a contradiction. Therefore, $\mu(x) = 1$, for all $x \in \mu_{\hat{\gamma}}$.

Conversely, assume that $\mu(x) = 1$, for all $x \in A$. It is easy to see that μ is an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. □

Corollary 3.42. *Let $2\delta = 1+\gamma$ and μ be an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. Then, $\mu(x) = 1$, for all $x \in \mu_{\hat{\gamma}}$.*

Proof. It is straightforward, by Theorems 3.3 and 3.41. □

From Theorems 3.6, 3.26, 3.39 and Corollary 3.42, we have the following results.

Proposition 3.43. *Let $2\delta = 1 + \gamma$. Then, a fuzzy subset μ of H is an $(q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H if and only if μ is an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H.*

Proof. By Definition 3.1, it is clear that every $(q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H is an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. Now, assume that μ is an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H and let $r_i \in (\gamma, 1]$ and $x_i \in H$ be such that $(x_i)_{r_i} q_\delta \, \mu$, for all $i \in \{1, 2, \cdots, n\}$, then $\mu(x_i) > 2\delta - r_i \geq 2\delta - 1 = \gamma$, and so $x_i \in \mu_{\hat{\gamma}}$, for all $i \in \{1, 2, \cdots, n\}$. Hence, for any $z \in f(x_1^n)$, $z \in \mu_{\hat{\gamma}}$ and $\mu(z) = 1$, by Theorem 3.6 and Theorem 3.41. It follows that $\mu(z) \geq r_1 \wedge r_2 \wedge \cdots \wedge r_n$ and $\mu(z) + (r_1 \wedge r_2 \wedge \cdots \wedge r_n) > \mu(z) + \gamma = 1 + \gamma = 2\delta$, i.e., $z_{r_1 \wedge r_2 \wedge \cdots \wedge r_n} \in_\gamma \wedge q_\delta \mu$. Similarly, we have $(a_1)_{r_1}, \cdots, (a_{i-1})_{r_{i-1}}, (a_{i+1})_{r_{i+1}}, \cdots, (a_n)_{r_n}, b_t \, q_\delta \, \mu$ and $1 \leq i \leq n$ implies $(x_i)_{r_1 \wedge r_2 \wedge \cdots \wedge r_{(i-1)} \wedge r_{(i+1)} \wedge \cdots \wedge r_n \wedge t} \in_\gamma \wedge q_\delta \mu$ for some $x_i \in H$ with $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$. Therefore, μ is an $(q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H, as required. □

Proposition 3.44. *Let $2\delta = 1+\gamma$. Then, every $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H is both an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H and an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H.*

Proof. By Definition 3.1, it is clear that every $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H is an $(\in_\gamma \wedge q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. Now, assume that μ is an $(q_\delta, \in_\gamma)$-fuzzy n-ary subhypergroup of H. By Proposition 3.43, μ is an $(q_\delta, \in_\gamma \wedge q_\delta)$-fuzzy n-ary subhypergroup of H, implying that μ is an (q_δ, q_δ)-fuzzy n-ary subhypergroup of H, as required. □

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