MIXED VARIATIONAL INCLUSIONS INVOLVING INFINITE FAMILY OF FUZZY MAPPINGS

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Abstract. In this paper, we introduce and study a mixed variational inclusion problem involving infinite family of fuzzy mappings. An iterative algorithm is constructed for solving a mixed variational inclusion problem involving infinite family of fuzzy mappings and the convergence of iterative sequences generated by the proposed algorithm is proved. Some illustrative examples are also given.

1. Introduction

The concept of fuzzy set theory was introduced by Zadeh [27] in 1965. The applications of the fuzzy set theory can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science and operational research [28]. For related topics, we refer to [22, 10].

Hilbert [6] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mapping which is a fuzzy analogue of Nadler’s fixed point theorem for set-valued mappings. In 1989, Chang and Zhu [3] first introduced and studied a class of variational inequalities with fuzzy mappings and extended the results of Lassonde [14], Shih and Tan [21], Takahashi [24] and Yen [26].

Since then, several classes of variational inequalities (inclusions) with fuzzy mappings were considered and studied by Chang and Huang [2], Noor [17], Huang [9], Park and Jeong [18, 19], Ding and Park [5], Ding [4], Wu and Xu [25], Kumam and Petrol [11], Lee et al. [15] and Lan [12], etc..

In this paper, we introduce and study a mixed variational inclusion problem involving infinite family of fuzzy mappings in real uniformly smooth Banach spaces, which includes as special cases, many known classes of variational inequalities with fuzzy mappings.

In the next section, we recall some basic definitions and formulate the main problem, i.e. a mixed variational inclusion problem involving infinite family of fuzzy mappings. Some examples and some special cases are given.

In section 3, we construct an iterative algorithm for solving a mixed variational inclusion problem involving infinite family of fuzzy mappings. Section 4 is devoted...
to the proof of an existence and convergence result for a mixed variational inclusion problem involving infinite family of fuzzy mappings. In the last section, conclusion of the paper is given.

2. Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$, $E^*$ is the topological dual of $E$, $d$ is the metric induced by the norm $\| \cdot \|$, $CB(E)$ (respectively, $2^E$) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of $E$ and $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{D}(A,B) = \max \{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(A,y) \},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$ and $d(A,y) = \inf_{x \in A} d(x,y)$. We also assume that $\langle \cdot,\cdot \rangle$ is the duality pairing between $E$ and $E^*$ and $J: E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^*: \langle x,f \rangle = \|x\|\|f\|, \text{ and } \|f\| = \|x\| \},$$

for all $x \in E$.

The function $\gamma_E: [0, \infty) \to [0, \infty)$ defined by

$$\gamma_E(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\},$$

is called the modulus of smoothness of $E$. We remark that a Banach space $E$ is uniformly smooth if $\lim_{t \to 0} \frac{\gamma_E(t)}{t} = 0$.

Definition 2.1. A mapping $g: E \to E$ is said to be

(i) $\delta$-strongly accretive, if for any $x,y \in E$, $j(x-y) \in J(x-y)$, there exists a constant $\delta > 0$ such that

$$\langle g(x) - g(y), j(x-y) \rangle \geq \delta \|x-y\|^2;$$

(ii) Lipschitz continuous, if for any $x,y \in E$, there exists a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x-y\|;$$

(iii) $\alpha$-relaxed Lipschitz continuous, if for any $x,y \in E$, $j(x-y) \in J(x-y)$, there exists a constant $\alpha > 0$ such that

$$\langle g(x) - g(y), j(x-y) \rangle \leq -\alpha \|x-y\|^2.$$

Definition 2.2. A mapping $A: E \to E$ is said to be

(i) $r$-strongly $\eta$-accretive, if for any $x,y \in E$, $j(\eta(x,y)) \in J(\eta(x,y))$, there exists a constant $r > 0$ such that

$$\langle A(x) - A(y), j(\eta(x,y)) \rangle \geq r \|x-y\|^2;$$

(ii) strictly $\eta$-accretive, if $A$ is $\eta$-accretive and the equality holds if and only if $x = y$. 
Definition 2.3. A set-valued mapping \( M : E \to 2^E \) is said to be \( m \)-relaxed \( \eta \)-accretive, if for any \( x, y \in E, u \in M(x), v \in M(y) \), there exists a constant \( m > 0 \) such that
\[
\langle u - v, j(\eta(x, y)) \rangle \geq -m \|x - y\|^2.
\]

Definition 2.4. Let \( A : E \to E \), \( \eta : E \times E \to E \) be the single-valued mappings. Then the set-valued mapping \( M : E \to 2^E \) is said to be \((A, \eta)\)-accretive, if

(i) \( M \) is \( m \)-relaxed \( \eta \)-accretive;

(ii) \( (A + \rho M)(E) = E \), for every \( \rho > 0 \).

Definition 2.5. A mapping \( N : E^\infty = E \times E \times E \ldots \to E \) is said Lipschitz continuous in the \( i^{th} \) argument with constants \( \beta_i \), if
\[
\|N(\cdots, x_i, \cdots) - N(\cdots, y_i, \cdots)\| \leq \beta_i \|x_i - y_i\|, \text{ for all } x_i, y_i \in E, i = 1, 2, \ldots .
\]

A mapping \( F \) from a set \( E \) to the collection \( F(E) = \{B : E \to [0, 1], a \text{ function}\} \) of fuzzy sets over \( E \) is called a fuzzy mapping, which means that for each \( x \in E \) a fuzzy set \( F(x) \), denoted by \( F_x \), is a function from \( E \) to \([0, 1] \). For each \( y \in E \), \( F_x(y) \) denotes the membership-grade of \( y \) in \( F_x \).

A fuzzy mapping \( F : E \to F(E) \) is said to be closed if for each \( x \in E \), the function \( y \to F_x(y) \) is upper semi-continuous, that is, for any given net \( \{y_\alpha\} \subseteq E \), satisfying \( y_\alpha \to y_0 \in E \), we have
\[
\limsup_{\alpha} F_x(y_\alpha) \leq F_x(y_0).
\]

For \( B \in F(E) \) and \( \lambda \in [0, 1] \), the set \( (B)_\lambda = \{x \in E : B(x) \geq \lambda\} \) is called a \( \lambda \)-cut set of \( B \). Let \( F : E \to F(E) \) be a closed fuzzy mapping satisfying the following condition:

Condition (\(*\)) : If there exists a function \( a : E \to [0, 1] \) such that for each \( x \in E \), the set \( (F_x)_a(x) = \{y \in E : F_x(y) \geq a(x)\} \) is a nonempty bounded subset of \( E \).

Remark 2.6. Let \( E \) be a normed vector space. If \( F \) is a closed fuzzy mapping satisfying the condition (\(*\)), then for each \( x \in E \), the set \( (F_x)_a(x) \) belongs to the collection \( CB(E) \) of all nonempty closed and bounded subsets of \( E \). In fact, let \( \{y_\alpha\} \subseteq (F_x)_a(x) \) be a net and \( y_\alpha \to y_0 \in E \), then \( (F_x)(y_0) \geq a(x) \) for each \( x \). Since \( F \) is closed, we have
\[
F_x(y_0) \geq \lim_{\alpha} F_x(y_\alpha) \geq a(x),
\]
which implies that \( y_0 \in (F_x)_a(x) \) and so \( (F_x)_a(x) \in CB(E) \).

Let \( T_i : E \to F(E), \ i = 1, 2, \ldots \) be an infinite family of closed fuzzy mappings satisfying condition (\(*\)). Then there exist functions \( a_i : E \to [0, 1], \ i = 1, 2, \ldots \) such that for each \( u \in E \), we have \( (T_1u)_{a_1(u)}, (T_2u)_{a_2(u)}, \ldots \in CB(E) \). Therefore, we can define set-valued mappings \( \bar{T}_i : E \to CB(E), \ i = 1, 2, \ldots \) by \( \bar{T}_i(u) = (T_1u)_{a_1(u)}, \bar{T}_2(u) = (T_2u)_{a_2(u)}, \ldots \) for all \( u \in E \). In the sequel, \( \bar{T}_1, \bar{T}_2, \bar{T}_3, \ldots \) are called the set-valued mappings induced by the fuzzy mappings \( T_1, T_2, T_3, \ldots \) respectively.

Let \( N : E^\infty = E \times E \times E \ldots \to E, \ g : E \to E \) and \( \eta : E \times E \to E \) be three mappings such that \( g \neq 0 \). Let \( T_i : E \to F(E) \) be the fuzzy mappings and
a_i : E → [0, 1] be the given functions, i = 1, 2, . . . . Let M : E × E → 2^E be a
set-valued mapping such that for each u ∈ E, M(·, u) is an (A, η)-accretive mapping
in the first argument with g(u) ∈ dom M(·, u). We consider the following problem:
Find u, x_1, x_2, . . . ∈ E, such that
\[(T_1u)(x_1) \geq a_1(u), \quad (T_2u)(x_2) \geq a_2(u), . . . \quad \text{and} \quad 0 \in N(x_1, x_2, . . .) + M(g(u), u). \tag{1}\]
Problem (1) is called mixed variational inclusion problem involving infinite family
of fuzzy mappings.
In support of problem (1), we give the following examples.

Example 2.7. Let E = [0, 1] and suppose
(i) the closed fuzzy mappings T_i : E → F(E), i = 1, 2, . . . are defined by
\[T_{iu}(x_i) = \begin{cases} \frac{u + x_i}{2}, & \text{if } u \in [0, \frac{1}{2}), x_i \in [0, 1]; \\ ((1 - \frac{1}{2})u + \frac{1}{2})x_i, & \text{if } u \in [\frac{1}{2}, 1], x_i \in [0, 1]. \end{cases} \]
(ii) the functions a_i : E → E, i = 1, 2, . . . are defined by
\[a_i(u) = \begin{cases} \frac{u}{2}, & \text{if } u \in [0, \frac{1}{2}); \\ 0, & \text{if } u \in [\frac{1}{2}, 1]. \end{cases} \]
(iii) for all x_i ∈ [0, 1], the mapping N : E^∞ : E × E × . . . → E, is defined by
\[N(x_1, x_2, . . .) = \begin{cases} 1, & \text{if } x_1 > x_2 > x_3 > . . .; \\ \frac{1}{2}, & \text{if } x_1 < x_2 < x_3 < . . .; \\ 0, & \text{if } x_1 = x_2 = x_3 = . . .; \\ \frac{1}{3}, & \text{otherwise.} \end{cases} \]
(iv) the mappings g, A : E → E are defined by
\[\begin{aligned} (a) \quad & g(u) = u^2, \text{ for all } u \in [0, 1], \quad g \not\equiv 0; \\ \text{and} \quad & A(u) = 0, \text{ for all } u \in [0, 1]. \end{aligned} \]
(v) the mapping η : E × E → E is defined by
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\[ \eta(x, y) = \begin{cases} x - y, & \text{if } x > y, \\ y - x, & \text{if } y > x, \\ 0, & \text{if } x = y, \end{cases} \]

for all \( x, y \in [0, 1] \).

(vi) the set-valued mapping \( M : E \times E \to 2^E \) is defined by

\[ M(g(u), u) = \{ g(u) \} \cup \{ u \}, \]

such that \( g(u) \in \text{dom} M(\cdot, u) \). Then it is easy to check that \( M \) is an \((A, \eta)\)-accretive in the first argument for \( \rho = 1 \).

In view of assumptions (i)-(vi), it is clear that all the conditions of problem (1) are satisfied.

The following problem plays an important role in fuzzy games, which can be obtained from problem (1).

**Example 2.8.** Consider a 'group of players' identified with an interval \( \Omega \) of the straight line. The 'fuzzy coalitions' of players are identified by the measurable functions from \( \Omega \) to \([0, 1] \).

We associate each player with its action \( f(\omega, \cdot) \) where \( f : \Omega \times L \to \mathbb{R}^n \), \( L \) is a nonempty subset of \( \mathbb{R}^n \), and each fuzzy coalition \( c(\omega) \) with its action \( \int_{\Omega} f(\omega, x)c(\omega)d\omega \).

The problem of fuzzy game is to find \( x \in E \) and \( c \in L^\infty(\Omega, [0, 1]) \) such that

\[ \int_{\Omega} f(\omega, x)c(\omega)d\omega = 0. \]

The above problem can be obtained from problem (1). If we take \( E = \mathbb{R}^n \) and define \( N(\cdot, \cdot) = N(\cdot, \cdot) : \Omega \times L \to E \) by

\[ N(\omega, x) = \int_{\Omega} f(\omega, x)c(\omega)d\omega, \]

and all the other functions involved in the formulation of problem (1) are equal to zero. For more details we refer to the book ‘Optima and Equilibria’ by Aubin [1].

**Some specific cases of problem (1)**

(i) If \( N(\cdot, \cdot) = N(\cdot, \cdot); T_1 = T, T_2 = F; T_3 = T_4 = \ldots = 0; a_1, a_2 : E \to [0, 1]; a_3 = a_4 = \ldots = 0; \eta(u, v) = u - v, \) for all \( u, v \in E \) and \( M(\cdot, \cdot) = M(\cdot) \) be an \( m \)-accretive mapping. Then the problem (2.1) reduces to the problem of finding \( u, w, q \in E \) such that

\[ T_u(w) \geq a_1(u), \quad F_u(q) \geq a_2(u) \text{ and } 0 \in N(u, q) + M(g(u)). \]

(ii) If \( T_i : E \to CB(E), i = 1, 2, \ldots \) are classical set-valued mappings, we can define fuzzy mappings \( T_i : E \to F(E) \) by

\[ u \to \chi_{T_i(u)}, \quad i = 1, 2, \ldots, \]

where \( \chi_{T_i(u)} \) are characteristic functions of \( T_i(u), i = 1, 2, \ldots \).
Proposition 3.3.\hspace*{1em}

Taking \(a_i(u) = 1, i = 1, 2, \ldots\) for all \(u \in E\), \(\eta(u, v) = u - v\) and \(M : E \times E \to 2^E\) to be an \(m\)-accretive mapping. The problem (1) is equivalent to the following problem considered by W.Y-Heng [7].

Find \(u \in E, x_i \in T_i(u), i = 1, 2, \ldots\) such that

\[
0 \in N(x_1, x_2, \ldots) + M(g(u), u). \tag{3}
\]

For suitable choice of mappings involved in the formulation of the problem (1), a number of known classes of variational inequalities (inclusions) studied in [5, 8], can be obtained from the problem (1).

3. Iterative Algorithm

In this section, we construct an iterative algorithm for solving problem (1).

Definition 3.1. Let \(A : E \to E\) be the strictly \(\eta\)-accretive mapping and \(M : E \times E \to 2^E\) be an \((A, \eta)\)-accretive mapping in the first argument. The resolvent operator \(J_{\eta, M}^{\rho, A} : E \to E\) is defined by

\[
J_{\eta, M}^{\rho, A}(v) = (A + \rho M(\cdot, u))^{-1}(v), \text{ for all } u, v \in E.
\]

Proposition 3.2. [13] Let \(E\) be a real Banach space and \(\eta : E \times E \to E\) be \(\tau\)-Lipschitz continuous mapping. Let \(A : E \to E\) be \(r\)-strongly \(\eta\)-accretive mapping and \(M : E \times E \to 2^E\) be an \((A, \eta)\)-accretive mapping in the first argument. Then the resolvent operator \(J_{\eta, M}^{\rho, A} : E \to E\) is \(\frac{\tau}{r\rho m}\)-Lipschitz continuous, i.e.,

\[
\|J_{\eta, M}^{\rho, A}(x) - J_{\eta, M}^{\rho, A}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \text{ for all } x, y \in E,
\]

where \(\rho \in (0, \frac{r}{m})\) is a constant.

Proposition 3.3. [20] Let \(E\) be a uniformly smooth Banach space and \(J : E \to 2^{E^*}\) be a normalized duality mapping. Then, for any \(x, y \in E\), \(j(x + y) \in J(x + y)\),

(i) \(\|x + y\|^2 \leq \|x\|^2 + 2\langle j(x) + j(y), x + y \rangle\),

(ii) \(\langle x - y, j(x) - j(y) \rangle \leq 2D^2\|x - y\|^2 / 4\), where \(D = \sqrt{\|x\|^2 + \|y\|^2 / 2}\).

Lemma 3.4. \((u, x_1, x_2, \ldots)\) is a solution of problem (1) if and only if it is a solution of the following equation:

\[
g(u) = J_{\eta, M}^{\rho, A}[A(g(u)) - \rho N(x_1, x_2, \ldots)]. \tag{4}
\]

Proof. The conclusion can be drawn directly from the definition of the resolvent operator \(J_{\eta, M}^{\rho, A}\). \(\square\)

Using Lemma 3.2 and Nadler’s theorem [16], we construct an iterative algorithm for solving problem (1).

Iterative Algorithm 3.5. Let \(T_i : E \to \mathcal{F}(E), i = 1, 2, \ldots\) be an infinite family of closed fuzzy mappings satisfying the condition (\(\ast\)) and let \(\tilde{T}_i : E \to CB(E), i = 1, 2, \ldots\) be an infinite family of set-valued mappings induced by the fuzzy mappings \(T_i, i = 1, 2, \ldots\), respectively. Let \(N = E^\infty = E \times E \times E \to E\), \(\eta : E \times E \to E\),
g, A : E → E be the mappings such that g ≠ 0 and M : E × E → 2^E be a set-valued mapping such that for each u ∈ E, M(·, u) is an (A, η)-accretive mapping in the first argument such that g(u) ∈ domM(·, u).

Let Q : E → 2^E be a set-valued mapping such that for each u ∈ E, Q(u) ⊆ g(E), where Q is defined by

\[ Q(u) = \bigcup_{x_1 \in \tilde{T}_1(u)} \bigcup_{x_2 \in \tilde{T}_2(u)} \bigcup_{x_3 \in \tilde{T}_3(u)} \cdots \bigcup \{ A(g(u)) - \rho N(x_1, x_2, \cdots) \}. \tag{5} \]

For given u_0 ∈ E, x_0^0 ∈ \tilde{T}_1(u_0), x_2^0 ∈ \tilde{T}_2(u_0), \ldots and let

\[ t_0 = J_{\eta, M}^{p, A}(A(g(u_0)) - \rho N(x_1^0, x_2^0, \cdots)) \subseteq Q(u_0) \subseteq g(E). \]

Hence, there exists u_1 ∈ E such that t_0 = g(u_1). Further, since \tilde{T}_i(u_i) ∈ CB(E), i = 1, 2, \ldots by Nadler’s [16], there exist x_1^1 ∈ \tilde{T}_1(u_1), x_2^1 ∈ \tilde{T}_2(u_1), \ldots such that

\[ \| x_1^1 - x_1^0 \| \leq D(\tilde{T}_1(u_0), \tilde{T}_1(u_1)); \]
\[ \| x_2^1 - x_2^0 \| \leq D(\tilde{T}_2(u_0), \tilde{T}_2(u_1)); \]

and so on.

Let \[ t_1 = J_{\eta, M}^{p, A}(A(g(u_1)) - \rho N(x_1^1, x_2^1, \cdots)) \subseteq Q(u_1) \subseteq g(E). \]

Hence there exists u_2 ∈ E such that t_1 = g(u_2). Continuing the above process inductively, we can define iterative sequences \{ u_n \}, \{ x_n^i \}, i = 1, 2, \ldots satisfying

\[ g(u_{n+1}) = J_{\eta, M}^{p, A}(A(g(u_n)) - \rho N(x_1^n, x_2^n, \cdots)); \tag{6} \]

and

\[ x_n^i ∈ \tilde{T}_i(u_n), \| x_n^i - x_{n-1}^i \| ≤ D(\tilde{T}(u_n), \tilde{T}(u_{n-1})), \]

for all i = 1, 2, \ldots and n = 0, 1, 2, \ldots.

4. Existence of Solutions and Convergence of Iterative Sequences

**Theorem 4.1.** Let E be a uniformly smooth Banach space with modulus of smoothness \( \gamma_E(t) \leq C t^2 \), for C > 0. Let \( T_i : E → F(E) \), i = 1, 2, \ldots be an infinite family of closed fuzzy mappings satisfying condition (*) and let \( \tilde{T}_i : E → CB(E) \) be the set-valued mappings induced by the fuzzy mappings \( T_i \), i = 1, 2, \ldots such that \( \tilde{T}_i \) are \( \tilde{D} \)-Lipschitz continuous mappings in all the arguments with constants \( \lambda_{\tilde{T}_i} \), i = 1, 2, \ldots. Let N : E^∞ = E × E × E × \ldots → E be a Lipschitz continuous mapping in all the arguments with constants \( \beta_i, i = 1, 2, \ldots \), and \( \eta : E → E \) be \( \tau \)-Lipschitz continuous mapping. Let g : E → E is α-relaxed Lipschitz and \( \lambda_A \)-Lipschitz continuous with g ≠ 0 and let (g – I) is δ-strongly accretive, where I is the identity mapping. Let A : E → E be r-strongly η-accretive and \( \lambda_A \)-Lipschitz continuous mapping. Let M : E × E → 2^E be set-valued mapping such that for each u ∈ E, M(·, u) is an (A, η)-accretive mapping in the first argument with g(u) ∈ domM(·, u) and let for each u ∈ E, Q(u) ⊆ g(E), where Q is defined by (5). Suppose that there exist \( \rho > 0 \) and \( \mu > 0 \) such that for each \( x, y, z ∈ E \),

\[ \| J_{\eta, M}^{p, A}(z) - J_{\eta, M}^{p, A}(y) \| \leq \mu \| x - y \|, \tag{7} \]
and
\[ \lambda_A(1 + \sqrt{1 - 2\alpha + 64C^2\alpha}) + \rho \sum_{i=1}^{\infty} \beta_i \lambda_i^2 < \frac{(\sqrt{1 + 2\alpha} - \mu)(r - \rho m)}{r}. \]

\[ \sqrt{1 + 2\alpha} > \mu, \quad \frac{r}{r - \rho m} > m. \]  

(8)

Then the iterative sequences \( \{u_n\}, \{x^n_1\}, \{x^n_2\}, \ldots \) generated by Algorithm 3.5 converge strongly to \( u, x_1, x_2, \ldots \) respectively and \( (u, x_1, x_2, \ldots) \) is a solution of problem (1).

**Proof.** Since \( (g - I) \) is \( \delta \)-strongly accretive mapping, we have
\[ \|u_{n+1} - u_n\|^2 = \|g(u_{n+1}) - g(u_n) + u_{n+1} - u_n - (g(u_{n+1}) - g(u_n))\|^2 \]
\[ \leq \|g(u_{n+1}) - g(u_n)\|^2 - 2\delta((g - I)(u_{n+1}) - (g - I)(u_n)) \]
\[ \leq \|g(u_{n+1}) - g(u_n)\|^2 - 2\delta\|u_{n+1} - u_n\|^2, \]
which implies that
\[ \|u_{n+1} + u_n\| \leq \frac{1}{\sqrt{1 + 2\alpha}}\|g(u_{n+1}) - g(u_n)\|. \]

(9)

By Proposition 3.1, Algorithm 3.1 and (4.1), we have
\[ \|g(u_{n+1}) - g(u_n)\| = \|J^{p, A}_{n, M}(u_{n+1})[A(g(u_n)) - \rho N(x^n_1, x^n_2, \ldots)] \]
\[ - J^{p, A}_{n, M}(u_{n-1})[A(g(u_{n-1})) - \rho N(x^{n-1}_1, x^{n-1}_2, \ldots)]\]
\[ = \|J^{p, A}_{n, M}(u_{n+1})[A(g(u_n)) - \rho N(x^n_1, x^n_2, \ldots)] \]
\[ - J^{p, A}_{n, M}(u_{n-1})[A(g(u_{n-1})) - \rho N(x^{n-1}_1, x^{n-1}_2, \ldots)] \]
\[ + \|J^{p, A}_{n, M}(u_{n-1})[A(g(u_{n-1})) - \rho N(x^{n-1}_1, x^{n-1}_2, \ldots)] \]
\[ \leq \frac{\tau}{r - \rho m}(\|A(g(u_n)) - A(g(u_{n-1})) - \rho N(x^n_1, x^n_2, \ldots) \]
\[ - N(x^{n-1}_1, x^{n-1}_2, \ldots))(\|u_{n-1} - u_n\| + \mu\|u_{n-1} - u_n\| \]
\[ \leq \frac{\tau A}{r - \rho m}\|u_{n-1} - u_n\| + \frac{\tau \rho}{r - \rho m}\|N(x^n_1, x^n_2, \ldots) \]
\[ - N(x^{n-1}_1, x^{n-1}_2, \ldots))(\|u_{n-1} - u_n\| \]
\[ \leq \frac{\lambda A}{r - \rho m}\|u_{n-1} - u_n\| + \frac{\mu}{r - \rho m}\|u_{n-1} - u_n\| \]
\[ + \frac{\tau \rho}{r - \rho m}\|N(x^n_1, x^n_2, \ldots) - N(x^{n-1}_1, x^{n-1}_2, \ldots)\| \].

(10)

Using relaxed Lipschitz continuity of \( g \) and Proposition 3.2, it follows that
\[ \|u_n - u_{n-1} + g(u_n) - g(u_{n-1})\|^2 \]
\[ \leq \|u_n - u_{n-1}\|^2 + 2\|g(u_n) - g(u_{n-1})\| + \|j(u_{n-1} - u_{n-2} + g(u_{n-1}))\| \]
\[ = \|u_n - u_{n-1}\|^2 + 2\|g(u_n) - g(u_{n-1})\| + \|j(u_{n-1} - u_{n-2} + g(u_{n-1}))\| \]
\[ + 2\|g(u_n) - g(u_{n-1})\| + \|j(u_{n-1} - u_{n-2} + g(u_{n-1}))\| \]
\[ - \|u_n - u_{n-1}\|^2 - 2\alpha\|u_n - u_{n-1}\|^2 + 4D^2\gamma_{l^2}(4\|g(u_n) - g(u_{n-1})\|/D) \]
\[ \leq \|u_n - u_{n-1}\|^2 - 2\alpha\|u_n - u_{n-1}\|^2 + 64C\|g(u_n) - g(u_{n-1})\|^2 \]
\[ \leq (1 - 2\alpha + 64C\lambda^2)\|u_n - u_{n-1}\|^2. \]

(11)
Using Lipschitz continuity of $N$ in all arguments with constant $\beta_i$, $i = 1, 2, \ldots$ respectively and using $D$-Lipschitz continuity of $\tilde{T}_i$ with constants $\lambda_{\tilde{T}_i}$, $i = 1, 2, \ldots$ respectively, we have

\[\|N(x^n_1, x^n_2, \ldots) - N(x^{n-1}_1, x^{n-1}_2, \ldots)\| \leq \sum_{i=1}^{\infty} \beta_i \lambda_{\tilde{T}_i} \|u_n - u_{n-1}\|. \quad (12)\]

Combining (11) and (12), (10) becomes

\[\|g(u_n) - g(u_{n-1})\| \leq \left[\frac{\tau \lambda_A}{r - \rho m} \sqrt{1 - 2\rho + 64C_\lambda^2} + \left(\frac{\tau \lambda_A}{r - \rho m} + \mu\right)\right]
+ \frac{\tau}{r - \rho m} \sum_{i=1}^{\infty} \beta_i \lambda_{\tilde{T}_i} \|u_n - u_{n-1}\|. \quad (13)\]

Using (13), (9) becomes

\[\|u_{n+1} - u_n\| \leq \theta \|u_n - u_{n-1}\|, \quad (14)\]

where $\theta = \frac{1}{\sqrt{1 + 2\delta}} \left(\frac{\tau}{r - \rho m} \left(\lambda_A \sqrt{1 - 2\rho + 64C_\lambda^2} + \lambda_A + \rho \sum_{i=1}^{\infty} \beta_i \lambda_{\tilde{T}_i}\right) + \mu\right)$.

It follows from (8) that $\theta < 1$. Hence (14) implies that $\{u_n\}$ is a Cauchy sequence in $E$, and thus there exists $u \in E$ such that $u_n \to u$, as $n \to \infty$. Using Lipschitz continuity of $\tilde{T}_i$ and Algorithm 3.1, we have

\[\|x^n_1 - x^n_1\| \leq \tilde{D}(\tilde{T}_1(u_0), \tilde{T}_1(u_1)) \leq \lambda_{\tilde{T}_1} \|u_0 - u_1\|;\]
\[\|x^n_2 - x^n_2\| \leq \tilde{D}(\tilde{T}_2(u_0), \tilde{T}_2(u_1)) \leq \lambda_{\tilde{T}_2} \|u_0 - u_1\|;\]

\[\vdots \]

It follows that $\{x^n_1\}, \{x^n_2\}, \ldots$ are all Cauchy sequences in $E$. Thus there exist $x_1, x_2, \ldots$ in $E$ such that $x^n_1 \to x_1, x^n_2 \to x_2, \ldots$ as $n \to \infty$. Further, since $x^n_i \in \tilde{T}_i(u_0)$, we have

\[d(x_i, \tilde{T}_i(u)) \leq \|x_i - x^n_i\| + d(x^n_i, \tilde{T}_i(u)) \leq \|x_i - x^n_i\| + \tilde{D}(\tilde{T}_i(u_n), \tilde{T}_i(u)) \leq \|x_i - x^n_i\| + \lambda_{\tilde{T}_i} \|u_n - u\| \to 0, \text{ as } n \to \infty.\]

Hence $d(x_i, \tilde{T}_i(u)) = 0$. Since $\tilde{T}_i(u) \in CB(E)$, it follows that $x_i \in \tilde{T}_i(u)$, for $i = 1, 2, \ldots$. Thus, we have $(T_{1u})(x_1) \geq a_1(u), (T_{2u})(x_2) \geq a_2(u), \ldots$. It follows from Algorithm 3.5, that

\[g(u) = J_{\rho,A}^{\mu}(u,A(g(u)) - \rho N(x_1, x_2, \cdots)).\]

By Lemma 3.2, we conclude that $(u, x_1, x_2, \cdots)$ is a solution of problem (1). \qed
5. Conclusion

The objective of this paper is to establish an existence and convergence result for a mixed variational inclusion problem involving infinite family of fuzzy mappings in real uniformly smooth Banach spaces. It is shown that problem (1) is equivalent to a fixed point problem by using the definition of resolvent operator and an iterative algorithm is constructed for solving problem (1).

We remark that there is a need to develop relationship between fuzzy sets and fuzzy variational inclusions involving infinite family of fuzzy mappings which may be useful for finding the solution of problems occurring in mathematical sciences etc..

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