

FUZZY SUBGROUPS OF THE DIRECT PRODUCT OF A GENERALIZED QUATERNION GROUP AND A CYCLIC GROUP OF ANY ODD ORDER

J. M. OH

ABSTRACT. Bentea and Tărnăuceanu (An. Ştiinţ. Univ. Al. I. Cuza Iaş, Ser. Nouă, Mat., **54(1)** (2008), 209-220) proposed the following problem: Find an explicit formula for the number of fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_n$ where Q_8 is the quaternion group of order 8 and n is an arbitrary odd integer. In this paper we consider more general group: the direct product of a generalized quaternion group of any even order and a cyclic group of any odd order. For this group we give an explicit formula for the number of fuzzy subgroups.

1. Introduction

Let G be a group with a multiplicative binary operation and identity e , and let $\mu : G \rightarrow [0, 1]$ be a fuzzy subset of G . Then μ is said to be a *fuzzy subgroup* of G if (1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. The set $\{\mu(x) \mid x \in G\}$ is called the *image* of μ and is denoted by $\mu(G)$. For each $a \in \mu(G)$, the set $\mu_a := \{x \in G \mid \mu(x) \geq a\}$ is called a *level subset* of μ . It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G (see [3]).

For given two fuzzy subgroups μ and ν in G , μ and ν are *equivalent*, written as $\mu \sim \nu$, if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ for all $x, y \in G$. It follows that $\mu \sim \nu$ if and only if μ and ν have the same set of level subgroups (see [13]). Hence there exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G and the collection of chains of subgroups of G which end in G . This notion of equivalence relation was used in [2, 11, 13] in order to enumerate fuzzy subgroups of certain families of finite groups. We call a chain of subgroups which ends in G *rooted* (or more exactly *G-rooted*). Otherwise we call it *unrooted*. There is another equivalence relation on the set of fuzzy subgroups used by Murali and Makamba [4, 5, 6, 7] in order to enumerate fuzzy subgroups of certain families of finite abelian groups. In this paper we follow the notion of the equivalence relation used in [2, 11, 13].

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It is still an open problem to determine the numbers of fuzzy subgroups of arbitrary finite abelian groups. As a partial result about the numbers of fuzzy subgroups of abelian groups Tărnăuceanu and Bentea [11] gave an explicit formula for the number of fuzzy subgroups of a finite cyclic group of any order by finding its generating function of one variable. The author [8] did a similar work by finding its generating function of multi-variables. As this problem to non-abelian groups Bentea and Tărnăuceanu [2] gave an explicit formula for the number of fuzzy subgroups of a hamiltonian group of type $Q_8 \times \mathbb{Z}_p^n$ where Q_8 is the quaternion group of order 8 and p is an odd prime number, and then propose the following problem: *Find an explicit formula for the number of fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_n$ where n is an arbitrary odd integer.* In this paper we consider more general group: the direct product of a generalized quaternion group of any even order and a cyclic group of any odd order. For this group we give an explicit formula for the number of fuzzy subgroups.

This paper is organized as follows. In section 2 we present some definitions and results. In section 3 we find a generating function of multi-variables for the number of fuzzy subgroups of the direct product of a generalized quaternion group and a cyclic group of any odd order, and then give an explicit formula for that number.

2. Preliminaries

Given a group G let $\mathcal{C}(G)$, $\mathcal{U}(G)$ and $\mathcal{R}(G)$ be the collection of chains of subgroups of G , of unrooted chains of subgroups of G and of rooted chains of subgroups of G , respectively. Let $C(G) := |\mathcal{C}(G)|$, $U(G) := |\mathcal{U}(G)|$ and $R(G) := |\mathcal{R}(G)|$.

The following simple observation is useful for enumerating chains of subgroups in the lattice of subgroups of a given finite group.

Proposition 2.1. *Let G be a finite group. Then $R(G) = U(G) + 1$ and $C(G) = R(G) + U(G) = 2R(G) - 1$.*

For a fixed positive integer k we define a function λ as follows.

$$\lambda(x_k) := 1 - 2x_k,$$

$$\lambda(x_k, x_{k-1}, \dots, x_j) := \lambda(x_k, x_{k-1}, \dots, x_{j+1}) - (1 + \lambda(x_k, x_{k-1}, \dots, x_{j+1}))x_j$$

for any $j = k - 1, k - 2, \dots, 1$.

Proposition 2.2. [8] *Let \mathbb{Z}_n be the cyclic group of order*

$$n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where p_1, \dots, p_k are distinct prime numbers and β_1, \dots, β_k are positive integers. Then the number $R(\mathbb{Z}_n)$ of fuzzy subgroups of \mathbb{Z}_n is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$\chi_{\beta_1, \dots, \beta_k}(x_k, \dots, x_1) = \frac{1}{\lambda(x_k, \dots, x_1)}.$$

Let \mathbb{Z} be the set of all integer numbers. Given distinct positive integers i_1, \dots, i_t we define a function

$$\pi_{i_1 \dots i_t} : \mathbb{Z}^k \rightarrow \mathbb{Z}^k, (x_1, \dots, x_k) \mapsto (y_1, \dots, y_k)$$

where

$$y_\ell = \begin{cases} x_\ell & \text{if } \ell \neq i_j \text{ for all } j = 1, \dots, t \\ x_\ell - 1 & \text{if } \ell = i_j \text{ for some } j \text{ such that } j = 1, \dots, t. \end{cases}$$

Given a positive integer α , the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ has the following presentation:

$$Q_{2^{\alpha+2}} = \langle a, b \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1} \rangle.$$

Most of our notations are standard and for undefined group theoretical terminologies we refer the reader to [9, 10]. For a general theory of solving a recurrence relation using a generating function we refer the reader to [1, 12].

3. The Number of Chains of Subgroups of $Q_{2^{\alpha+2}} \times \mathbb{Z}_n$

Throughout the section we assume that

$$n := p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where p_1, \dots, p_k are distinct odd prime numbers and β_1, \dots, β_k are non-negative integers.

Let $\mathbb{Z}_n := \langle c \mid c^n = e \rangle$ be the cyclic group of order n . For any prime factors $p_{i_1}, p_{i_2}, \dots, p_{i_t}$ of n let

$$\mathbb{Z}_{n/p_{i_1} \cdots p_{i_t}} := \langle c^{p_{i_1} \cdots p_{i_t}} \rangle$$

be the subgroup of \mathbb{Z}_n of order $n/p_{i_1} \cdots p_{i_t}$.

Let

$$Q_{2^{\alpha+2}} \times \mathbb{Z}_n := \langle a, b, c \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1}, c^n = e \rangle$$

be the direct product of the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ and the cyclic group \mathbb{Z}_n of order n .

Lemma 3.1. *Assume that β_1, \dots, β_k are positive integers. The group $Q_{2^{\alpha+2}} \times \mathbb{Z}_n$ has three index 2 subgroups $\langle a, c \rangle \cong \mathbb{Z}_{2^{\alpha+1}} \times \mathbb{Z}_n$, $\langle a^2, b, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$ and $\langle a^2, ab, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$, and one index p_i subgroup $\langle a, b, c^{p_i} \rangle \cong Q_{2^{\alpha+2}} \times \mathbb{Z}_{n/p_i}$ where $i = 1, 2, \dots, k$.*

Proof. Let $G := Q_{2^{\alpha+2}} \times \mathbb{Z}_n$. Clearly the subgroup $H := \langle a, c \rangle \cong \mathbb{Z}_{2^{\alpha+1}} \times \mathbb{Z}_n$ is index 2 subgroup of G . Let K be an index 2 subgroup of G with $K \neq H$. Then $G = HK$ and

$$2^{\alpha+2}n = |HK| = \frac{|H||K|}{|H \cap K|} = \frac{2^{\alpha+1}n2^{\alpha+1}n}{|H \cap K|}.$$

Thus $|H \cap K| = 2^\alpha n$ and so $H \cap K = \langle a^2, c \rangle$. Therefore one can see that $H = \langle a^2, b, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$ or $H = \langle a^2, ab, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$.

Let M be an index p_i subgroup of G where $i = 1, 2, \dots, k$. Then $|M| = 2^{\alpha+2}n/p_i$. Since

$$|G| = |MQ_{2^{\alpha+2}}| = \frac{|M||Q_{2^{\alpha+2}}|}{|M \cap Q_{2^{\alpha+2}}|},$$

we have $\langle a, b \rangle \leq M$. Now it is easy to see that $M = \langle a, b, c^{p_i} \rangle \cong Q_{2^{\alpha+2}} \times \mathbb{Z}_{n/p_i}$. \square

By Lemma 3.1

$$\begin{aligned} \mathcal{U}(Q_{2\alpha+2} \times \mathbb{Z}_n) &= \mathcal{C}(\langle a, c \rangle \cong \mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_n) \cup \mathcal{C}(\langle a^2, b, c \rangle \cong Q_{2\alpha+1} \times \mathbb{Z}_n) \\ &\quad \cup \mathcal{C}(\langle a^2, ab, c \rangle \cong Q_{2\alpha+1} \times \mathbb{Z}_n) \cup \bigcup_{i=1}^k \mathcal{C}(\langle a, b, c^{p_i} \rangle \cong Q_{2\alpha+2} \times \mathbb{Z}_{n/p_i}). \end{aligned}$$

Using the inclusion-exclusion principle one can see that

$$\begin{aligned} U(Q_{2\alpha+2} \times \mathbb{Z}_n) &= 2C(Q_{2\alpha+1} \times \mathbb{Z}_n) + C(\mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_n) - 2C(\mathbb{Z}_{2\alpha} \times \mathbb{Z}_n) \\ &\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} C(Q_{2\alpha+2} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t C(Q_{2\alpha+1} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t C(\mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} C(\mathbb{Z}_{2\alpha} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}). \end{aligned}$$

Further, by Proposition 2.1 we have

$$\begin{aligned} R(Q_{2\alpha+2} \times \mathbb{Z}_n) &= 4R(Q_{2\alpha+1} \times \mathbb{Z}_n) + 2R(\mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_n) - 4R(\mathbb{Z}_{2\alpha} \times \mathbb{Z}_n) \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} R(Q_{2\alpha+2} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t R(Q_{2\alpha+1} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t R(\mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}) \\ &\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} R(\mathbb{Z}_{2\alpha} \times \mathbb{Z}_{n/p_{i_1} \dots p_{i_t}}). \quad (1) \end{aligned}$$

Let $a_{\alpha, \beta_1, \beta_2, \dots, \beta_k} := R(Q_{2\alpha+2} \times \mathbb{Z}_n)$ and $b_{\alpha, \beta_1, \beta_2, \dots, \beta_k} := R(\mathbb{Z}_{2\alpha+1} \times \mathbb{Z}_n)$. Then equation (1) becomes

$$\begin{aligned} a_{\alpha, \beta_1, \dots, \beta_k} &= 4a_{\alpha-1, \beta_1, \dots, \beta_k} + 2b_{\alpha, \beta_1, \dots, \beta_k} - 4b_{\alpha-1, \beta_1, \dots, \beta_k} \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} a_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k) \\ &\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t a_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k) \\ &\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t b_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k) \\ &\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k). \quad (2) \end{aligned}$$

Let k be a positive integer. We define

$$\begin{aligned} \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, x_{k-1}, \dots, x_j) &:= \sum_{\beta_j=0}^{\infty} \cdots \sum_{\beta_{k-1}=0}^{\infty} \sum_{\beta_k=0}^{\infty} a_{\alpha, \beta_1, \dots, \beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}} \cdots x_j^{\beta_j}, \\ \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, x_{k-1}, \dots, x_j) &:= \sum_{\beta_j=0}^{\infty} \cdots \sum_{\beta_{k-1}=0}^{\infty} \sum_{\beta_k=0}^{\infty} b_{\alpha, \beta_1, \dots, \beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}} \cdots x_j^{\beta_j} \end{aligned}$$

where $j = k, k - 1, \dots, 1$ and

$$\begin{aligned} \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, x_{k-1}, \dots, x_1, y) &:= \sum_{\alpha=0}^{\infty} \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_{k-1}=0}^{\infty} \sum_{\beta_k=0}^{\infty} a_{\alpha, \beta_1, \dots, \beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}} \cdots x_1^{\beta_1} y^{\alpha}, \\ \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, x_{k-1}, \dots, x_1, y) &:= \sum_{\alpha=0}^{\infty} \sum_{\beta_1=0}^{\infty} \cdots \sum_{\beta_{k-1}=0}^{\infty} \sum_{\beta_k=0}^{\infty} b_{\alpha, \beta_1, \dots, \beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}} \cdots x_1^{\beta_1} y^{\alpha}. \end{aligned}$$

From now on we explicitly find the function $\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, x_{k-1}, \dots, x_1, y)$ through several lemmas.

Lemma 3.2. *Let k be a non-negative integer. If $k = 0$, then*

$$\psi_{\alpha}(y) = \frac{4}{1 - 4y} \text{ and } a_{\alpha} = 2^{2\alpha+2}.$$

If $k = 1$, then

$$\lambda(x_1)\psi_{\alpha, \beta_1}(x_1) = (1 + \lambda(x_1))[2\psi_{\alpha-1, \beta_1}(x_1) + \phi_{\alpha, \beta_1}(x_1) - 2\phi_{\alpha-1, \beta_1}(x_1)]. \quad (3)$$

If $k \geq 2$, then

$$\begin{aligned} &\lambda(x_k, \dots, x_j)\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_j) \\ &= (1 + \lambda(x_k, \dots, x_j)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_j) \\ &+ \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_j) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_j) \\ &+ \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k, \dots, x_j) \\ &+ 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^t \psi_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k, \dots, x_j) \\ &+ \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^t \phi_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k, \dots, x_j) \\ &+ 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k, \dots, x_j) \end{aligned} \quad (4)$$

for any $j = k, k-1, \dots, 2$.

Proof. We first assume that $k = 0$. Equation (2) with $k = 0$ gives us that

$$a_\alpha = 4a_{\alpha-1} + 2b_\alpha - 4b_{\alpha-1}. \quad (5)$$

Taking $\sum_{\alpha=1}^{\infty} y^\alpha$ to both sides of equation (5) we have

$$(1-4y)\psi_\alpha(y) = 2(1-2y)\phi_\alpha(y)$$

because $a_0 = R(Q_{2^2}) = R(\mathbb{Z}_{2^2}) = 2^2$, $b_0 = R(\mathbb{Z}_2) = 2$. Since

$$\phi_\alpha(y) = \sum_{\alpha=0}^{\infty} 2^{\alpha+1} y^\alpha = \frac{2}{1-2y},$$

we get that

$$\psi_\alpha(y) = \frac{4}{1-4y},$$

and hence $a_\alpha = 2^{2\alpha+2}$.

Second we assume that $k = 1$. Equation (2) with $k = 1$ gives us that

$$a_{\alpha,\beta_1} = 4a_{\alpha-1,\beta_1} + 2b_{\alpha,\beta_1} - 4b_{\alpha-1,\beta_1} + 2a_{\alpha,\beta_1-1} - 4a_{\alpha-1,\beta_1-1} - 2b_{\alpha,\beta_1-1} + 4b_{\alpha-1,\beta_1-1}. \quad (6)$$

Taking $\sum_{\beta_1=1}^{\infty} x_1^{\beta_1}$ to both sides of equation (6) one can see that

$$(1-2x_1)\psi_{\alpha,\beta_1}(x_1) = 2(1-x_1)[2\psi_{\alpha-1,\beta_1}(x_1) + \phi_{\alpha,\beta_1}(x_1) - 2\phi_{\alpha-1,\beta_1}(x_1)].$$

because

$$a_{\alpha,0} - 4a_{\alpha-1,0} - 2b_{\alpha,0} + 4b_{\alpha-1,0} = 0$$

by equation (5). Thus equation (3) holds.

From now on we assume that $k \geq 2$. We prove equation (4) by double induction on k and j . If $k = 2$, then equation (2) with $k = 2$ gives us that

$$\begin{aligned} a_{\alpha,\beta_1,\beta_2} &= 4a_{\alpha-1,\beta_1,\beta_2} + 2b_{\alpha,\beta_1,\beta_2} - 4b_{\alpha-1,\beta_1,\beta_2} \\ &\quad + 2a_{\alpha,\beta_1-1,\beta_2} + 2a_{\alpha,\beta_1,\beta_2-1} - 2a_{\alpha,\beta_1-1,\beta_2-1} - 4a_{\alpha-1,\beta_1-1,\beta_2} \\ &\quad - 4a_{\alpha-1,\beta_1,\beta_2-1} + 4a_{\alpha-1,\beta_1-1,\beta_2-1} - 2b_{\alpha,\beta_1-1,\beta_2} - 2b_{\alpha,\beta_1,\beta_2-1} \\ &\quad + 2b_{\alpha,\beta_1-1,\beta_2-1} + 4b_{\alpha-1,\beta_1-1,\beta_2} + 4b_{\alpha-1,\beta_1,\beta_2-1} - 4b_{\alpha-1,\beta_1-1,\beta_2-1}. \end{aligned} \quad (7)$$

Taking $\sum_{\beta_2=1}^{\infty} x_2^{\beta_2}$ of both sides of equation (2) one can see that

$$\begin{aligned} (1-2x_2)\psi_{\alpha,\beta_1,\beta_2}(x_2) &= 2(1-x_2)[2\psi_{\alpha-1,\beta_1,\beta_2}(x_2) + \phi_{\alpha,\beta_1,\beta_2}(x_2) \\ &\quad - 2\phi_{\alpha-1,\beta_1,\beta_2}(x_2) + \psi_{\alpha,\beta_1-1,\beta_2}(x_2) - 2\psi_{\alpha-1,\beta_1-1,\beta_2}(x_2) \\ &\quad - \phi_{\alpha,\beta_1-1,\beta_2}(x_2) + 2\phi_{\alpha-1,\beta_1-1,\beta_2}(x_2)] \end{aligned}$$

because

$$a_{\alpha,\beta_1,0} - [4a_{\alpha-1,\beta_1,0} + 2b_{\alpha,\beta_1,0} - 4b_{\alpha-1,\beta_1,0} + 2a_{\alpha,\beta_1-1,0} - 4a_{\alpha-1,\beta_1-1,0} - 2b_{\alpha,\beta_1-1,0} + 4b_{\alpha-1,\beta_1-1,0}] = 0$$

by equation (2). Thus equation (4) holds for $k = 2$.

Assume now that equation (4) holds from 2 to $k - 1$ and consider the case for k . Note that the last four terms of the right side of equation (2) can be divided into three terms respectively as follows.

$$\begin{aligned}
 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} a_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k) &= 2a_{\alpha, \beta_1, \dots, \beta_{k-1}, \beta_k-1} \\
 &- 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} a_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{k-1}, \beta_k-1) \\
 &+ 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} a_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k),
 \end{aligned}$$

$$\begin{aligned}
 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t a_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k) &= -4a_{\alpha-1, \beta_1, \dots, \beta_{k-1}, \beta_k-1} \\
 &- 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t a_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{k-1}, \beta_k-1) \\
 &+ 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t a_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k),
 \end{aligned}$$

$$\begin{aligned}
 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^t b_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k) &= -2b_{\alpha, \beta_1, \dots, \beta_{k-1}, \beta_k-1} \\
 &- 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t b_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{k-1}, \beta_k-1) \\
 &+ 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t b_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)
 \end{aligned}$$

and

$$\begin{aligned}
 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k, \\ 1 \leq t \leq k}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k) &= 4b_{\alpha-1, \beta_1, \dots, \beta_{k-1}, \beta_k-1} \\
 &- 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{k-1}, \beta_k-1) \\
 &+ 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k).
 \end{aligned}$$

Now taking $\sum_{\beta_k=1}^{\infty} x_k^{\beta_k}$ of both sides of equation (2) we get that

$$\begin{aligned}
(1 - 2x_k)\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k) &= 2(1 - x_k) \left[2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k) + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k) \right. \\
&\quad - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k) + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k) \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t \psi_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k) \\
&\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t \phi_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k) \\
&\quad \left. + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_k)}(x_k) \right] \\
&+ a_{\alpha, \beta_1, \dots, \beta_{k-1}, 0} - \left[4a_{\alpha-1, \beta_1, \dots, \beta_{k-1}, 0} + 2b_{\alpha, \beta_1, \dots, \beta_{k-1}, 0} - 4b_{\alpha-1, \beta_1, \dots, \beta_{k-1}, 0} \right. \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} a_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t a_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t b_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad \left. + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \right].
\end{aligned}$$

Note that by equation (2)

$$\begin{aligned}
a_{\alpha, \beta_1, \dots, \beta_{k-1}, 0} - \left[4a_{\alpha-1, \beta_1, \dots, \beta_{k-1}, 0} + 2b_{\alpha, \beta_1, \dots, \beta_{k-1}, 0} - 4b_{\alpha-1, \beta_1, \dots, \beta_{k-1}, 0} \right. \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} a_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t a_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^t b_{\alpha, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \\
&\quad \left. + 4 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq k-1, \\ 1 \leq t \leq k-1}} (-1)^{t+1} b_{\alpha-1, \pi_{i_1 \dots i_t}(\beta_1, \dots, \beta_{k-1}, 0)} \right] = 0.
\end{aligned}$$

Thus equation (4) holds for $j = k$. Assume that equation (4) holds from k to j and consider the case for $j - 1$. Note that the last four terms of the right side of equation (4) can be divided into three terms respectively as follows.

$$\begin{aligned}
 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad = \psi_{\alpha, \beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
 & - \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad \qquad \qquad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 2 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^t \psi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad = -2\psi_{\alpha-1, \beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
 - 2 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \psi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad \qquad \qquad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \psi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^t \phi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad = -\phi_{\alpha, \beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
 - & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \phi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad \qquad \qquad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \phi_{\alpha, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 2 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-1, \\ 1 \leq t \leq j-1}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad = 2\phi_{\alpha-1, \beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
 - 2 & \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_{j-2}, \beta_{j-1}-1, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
 & \qquad \qquad \qquad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1 \dots i_t}}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j)
 \end{aligned}$$

Now taking $\sum_{\beta_{j-1}=1}^{\infty} x_{j-1}^{\beta_{j-1}}$ of both sides of equation (4) we have

$$\begin{aligned}
& [\lambda(x_k, \dots, x_j) - (1 + \lambda(x_k, \dots, x_j))x_{j-1}] \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_j, x_{j-1}) \\
&= [1 + \lambda(x_k, \dots, x_j) - (1 + \lambda(x_k, \dots, x_j))x_{j-1}] [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_j, x_{j-1}) \\
&\quad + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_j, x_{j-1}) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_j, x_{j-1}) \\
&\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j, x_{j-1}) \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \psi_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j, x_{j-1}) \\
&\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \phi_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j, x_{j-1}) \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_k)(x_k, \dots, x_j, x_{j-1})]
\end{aligned}$$

because it holds by induction hypothesis that

$$\begin{aligned}
& \lambda(x_k, \dots, x_j) \psi_{\alpha, \beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
&\quad - (1 + \lambda(x_k, \dots, x_j)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
&\quad + \phi_{\alpha, \beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k}(x_k, \dots, x_j) \\
&\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \psi_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \psi_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
&\quad + \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^t \phi_{\alpha, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_{j-2}, 0, \beta_j, \dots, \beta_k)(x_k, \dots, x_j) \\
&\quad + 2 \sum_{\substack{1 \leq i_1 < \dots < i_t \leq j-2, \\ 1 \leq t \leq j-2}} (-1)^{t+1} \phi_{\alpha-1, \pi_{i_1} \dots i_t}(\beta_1, \dots, \beta_{j-2}, 0, \beta_j, \beta_k)(x_k, \dots, x_j) = 0.
\end{aligned}$$

Thus equation (4) holds for $j - 1$, and the lemma is proved. \square

Equation (4) with $j = 2$ gives us that

$$\begin{aligned} &\lambda(x_k, \dots, x_2)\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_2) \\ &= (1 + \lambda(x_k, \dots, x_2)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_2) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad + \psi_{\alpha, \beta_1-1, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) - 2\psi_{\alpha-1, \beta_1-1, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad - \phi_{\alpha, \beta_1-1, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) + 2\phi_{\alpha-1, \beta_1-1, \beta_2, \dots, \beta_k}(x_k, \dots, x_2)]. \end{aligned} \quad (8)$$

Taking $\sum_{\beta_1=1}^{\infty} x_1^{\beta_1}$ of both sides of equation (8) one can see that

$$\begin{aligned} &\lambda(x_k, \dots, x_1)\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ &= (1 + \lambda(x_k, \dots, x_1)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ &\quad + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ &\quad + \lambda(x_k, \dots, x_2)\psi_{\alpha, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad - (1 + \lambda(x_k, \dots, x_2)) [2\psi_{\alpha-1, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad + \phi_{\alpha, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) - 2\phi_{\alpha-1, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2)]. \end{aligned} \quad (9)$$

Lemma 3.3. *If $k \geq 2$, then*

$$\begin{aligned} &\lambda(x_k, \dots, x_2)\psi_{\alpha, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) \\ &= (1 + \lambda(x_k, \dots, x_2)) [2\psi_{\alpha-1, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) \\ &\quad + \phi_{\alpha, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2) - 2\phi_{\alpha-1, 0, \beta_2, \dots, \beta_k}(x_k, \dots, x_2)]. \end{aligned} \quad (10)$$

Proof. If $k = 2$, then since $\psi_{\alpha, 0, \beta_2}(x_2) = \psi_{\alpha, \beta_2}(x_2)$ and $\phi_{\alpha, 0, \beta_2}(x_2) = \phi_{\alpha, \beta_2}(x_2)$, the equation

$$\lambda(x_2)\psi_{\alpha, 0, \beta_2}(x_2) = (1 + \lambda(x_2)) [2\psi_{\alpha-1, 0, \beta_2}(x_2) + \phi_{\alpha, 0, \beta_2}(x_2) - 2\phi_{\alpha-1, 0, \beta_2}(x_2)]$$

holds by equation (3). Assume now that equation (10) holds for k . Then by equation (9) we get that

$$\begin{aligned} &\lambda(x_k, \dots, x_1)\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ &= (1 + \lambda(x_k, \dots, x_1)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ &\quad + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1)] \end{aligned}$$

which implies that

$$\begin{aligned} &\lambda(x_{k+1}, \dots, x_2)\psi_{\alpha, 0, \beta_2, \dots, \beta_{k+1}}(x_{k+1}, \dots, x_2) = (1 + \lambda(x_{k+1}, \dots, x_2)) \times \\ &\quad [2\psi_{\alpha-1, 0, \beta_2, \dots, \beta_{k+1}}(x_{k+1}, \dots, x_2) + \phi_{\alpha, 0, \beta_2, \dots, \beta_{k+1}}(x_{k+1}, \dots, x_2) \\ &\quad - 2\phi_{\alpha-1, 0, \beta_2, \dots, \beta_{k+1}}(x_{k+1}, \dots, x_2)]. \end{aligned}$$

Thus equation (10) holds for $k + 1$. □

By equations (3), (9) and Lemma 3.3 it holds that

$$\begin{aligned} \lambda(x_k, \dots, x_1) \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ = (1 + \lambda(x_k, \dots, x_1)) [2\psi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ + \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) - 2\phi_{\alpha-1, \beta_1, \dots, \beta_k}(x_k, \dots, x_1)] \end{aligned} \quad (11)$$

for any positive integer k . Taking $\sum_{\alpha=1}^{\infty} y^\alpha$ of both sides of equation (11) one can see that

$$\begin{aligned} [\lambda(x_k, \dots, x_1) - 2(1 + \lambda(x_k, \dots, x_1))y] \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ = (1 + \lambda(x_k, \dots, x_1))(1 - 2y)\phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ + \lambda(x_k, \dots, x_1)\psi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ - (1 + \lambda(x_k, \dots, x_1))\phi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1). \end{aligned} \quad (12)$$

Lemma 3.4. *For any positive integer k ,*

$$\begin{aligned} \lambda(x_k, \dots, x_1) \psi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) \\ = (1 + \lambda(x_k, \dots, x_1))\phi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1). \end{aligned} \quad (13)$$

Proof. Note that $a_{0, \beta_1, \beta_2, \dots, \beta_k} = R(Q_{2^2} \times \mathbb{Z}_n) = R(\mathbb{Z}_{2^2} \times \mathbb{Z}_n)$ and $b_{0, \beta_1, \beta_2, \dots, \beta_k} := R(\mathbb{Z}_2 \times \mathbb{Z}_n)$. Since n is a product of odd prime numbers, by Proposition 2.2 one can see that

$$\begin{aligned} \psi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) &= \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right]^2, \\ \phi_{0, \beta_1, \dots, \beta_k}(x_k, \dots, x_1) &= \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right]. \end{aligned}$$

Now it is easy to see that equation (13) holds. \square

By Lemma 3.4 equation (12) becomes

$$\begin{aligned} \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ = \left[1 + \frac{1}{\lambda(x_k, \dots, x_1) - 2(1 + \lambda(x_k, \dots, x_1))y} \right] \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y). \end{aligned} \quad (14)$$

Since $b_{\alpha, \beta_1, \beta_2, \dots, \beta_k} = R(\mathbb{Z}_{2^{\alpha+1}} \times \mathbb{Z}_n)$ by the definition and n is a product of odd prime numbers, by Proposition 2.2 one can see that

$$\phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) = \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y}.$$

In summary we have proved the following.

Theorem 3.5. *Let*

$$n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where p_1, \dots, p_k are distinct odd prime numbers and β_1, \dots, β_k are positive integers. Let α be a non-negative integer. Let

$$G := Q_{2^{\alpha+2}} \times \mathbb{Z}_n = \langle a, b, c \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1}, c^n = e \rangle$$

be the direct product of the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ and the cyclic group \mathbb{Z}_n of order n . Then the number $R(G)$ of fuzzy subgroups of G is the coefficient of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \dots x_k^{\beta_k}$ of

$$\begin{aligned} & \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ &= \left[1 + \frac{1}{\lambda(x_k, \dots, x_1) - 2(1 + \lambda(x_k, \dots, x_1))y} \right] \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \quad (15) \end{aligned}$$

where

$$\begin{aligned} & \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ &= \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y}. \end{aligned}$$

The following is useful for specifically given integers n and α .

Corollary 3.6. Let G and $\psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y)$ be the group and the function defined in Theorem 3.5. Then

$$R(G) = \frac{\partial^{\beta_1 + \beta_2 + \dots + \beta_k + \alpha} \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y)}{\beta_k! \dots \beta_2! \beta_1! \alpha! (\partial x_k)^{\beta_k} \dots (\partial x_2)^{\beta_2} (\partial x_1)^{\beta_1} (\partial y)^\alpha} \Big|_{\substack{x_i=0, y=0, \\ 1 \leq i \leq k}}.$$

We now want to find the coefficients of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \dots x_k^{\beta_k}$ of

$$\begin{aligned} & \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) = \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ &+ \frac{1}{\lambda(x_k, \dots, x_1) - 2(1 + \lambda(x_k, \dots, x_1))y} \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \end{aligned}$$

explicitly. Note that

$$\begin{aligned} & \psi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ &= \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y} \\ &- \frac{1}{\lambda^2(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y} \\ &+ \frac{1}{\lambda^2(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{2}{1 - 2 \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y} \end{aligned}$$

Since

$$\begin{aligned} & \phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y) \\ &= \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y}, \end{aligned}$$

the coefficient of y^α of $\phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y)$ is

$$\frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right]^{\alpha+1}.$$

Since

$$\begin{aligned} \frac{1}{\lambda(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right]^{\alpha+1} &= \sum_{i_0=0}^{\alpha+1} \binom{\alpha+1}{i_0} \left[\frac{1}{\lambda(x_k, \dots, x_1)} \right]^{i_0+1} \\ &= \sum_{i_0=0}^{\alpha+1} \binom{\alpha+1}{i_0} \left[\frac{1}{\lambda(x_k, \dots, x_2)} \right]^{i_0+1} \left[\frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_2)} \right] x_1} \right]^{i_0+1}, \end{aligned}$$

the coefficient of $y^\alpha x_1^{\beta_1}$ of $\phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y)$ is

$$\sum_{i_0=0}^{\alpha+1} \binom{\alpha+1}{i_0} \binom{\beta_1+i_0}{\beta_1} \left[\frac{1}{\lambda(x_k, \dots, x_2)} \right]^{i_0+1} \left[1 + \frac{1}{\lambda(x_k, \dots, x_2)} \right]^{\beta_1},$$

which is equal to

$$\sum_{i_0=0}^{\alpha+1} \sum_{i_1=0}^{\beta_1} \binom{\alpha+1}{i_0} \binom{\beta_1}{i_1} \binom{\beta_1+i_0}{\beta_1} \left[\frac{1}{\lambda(x_k, \dots, x_2)} \right]^{i_0+i_1+1}.$$

By continuing this process, one can see that the coefficient of $y^\alpha x_1^{\beta_1} \dots x_k^{\beta_k}$ of $\phi_{\alpha, \beta_1, \dots, \beta_k}(x_k, \dots, x_1, y)$ is

$$2^{\beta_k} \sum_{i_0=0}^{\alpha+1} \sum_{i_1=0}^{\beta_1} \sum_{i_2=0}^{\beta_2} \dots \sum_{i_{k-1}=0}^{\beta_{k-1}} \binom{\alpha+1}{i_0} \binom{\beta_1+i_0}{\beta_1} \prod_{r=1}^{k-1} \binom{\beta_r}{i_r} \binom{\beta_{r+1} + \sum_{m=0}^r i_m}{\beta_{r+1}}.$$

Similarly one can see that the coefficient of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \dots x_{k-1}^{\beta_{k-1}} x_k^{\beta_k}$ of

$$\frac{1}{\lambda^2(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{1}{1 - \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y}$$

is

$$2^{\beta_k} \sum_{i_0=0}^{\alpha+1} \sum_{i_1=0}^{\beta_1} \sum_{i_2=0}^{\beta_2} \dots \sum_{i_{k-1}=0}^{\beta_{k-1}} \binom{\alpha+1}{i_0} \binom{\beta_1+1+i_0}{\beta_1} \prod_{r=1}^{k-1} \binom{\beta_r}{i_r} \binom{\beta_{r+1}+1+\sum_{m=0}^r i_m}{\beta_{r+1}}$$

and the coefficient of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \dots x_{k-1}^{\beta_{k-1}} x_k^{\beta_k}$ of

$$\frac{1}{\lambda^2(x_k, \dots, x_1)} \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] \frac{2}{1 - 2 \left[1 + \frac{1}{\lambda(x_k, \dots, x_1)} \right] y}$$

is

$$2^{\alpha+1+\beta_k} \sum_{i_0=0}^{\alpha+1} \sum_{i_1=0}^{\beta_1} \sum_{i_2=0}^{\beta_2} \cdots \sum_{i_{k-1}=0}^{\beta_{k-1}} \left[\binom{\alpha+1}{i_0} \times \binom{\beta_1+1+i_0}{\beta_1} \prod_{r=1}^{k-1} \binom{\beta_r}{i_r} \binom{\beta_{r+1}+1+\sum_{m=0}^r i_m}{\beta_{r+1}} \right].$$

Therefore we get the following.

Corollary 3.7. *Let G be the group defined in Theorem 3.5. Then the number $R(G)$ of fuzzy subgroups of G is*

$$R(G) = 2^{\beta_k} \sum_{i_0=0}^{\alpha+1} \sum_{i_1=0}^{\beta_1} \sum_{i_2=0}^{\beta_2} \cdots \sum_{i_{k-1}=0}^{\beta_{k-1}} \left[\binom{\alpha+1}{i_0} \binom{\beta_1+i_0}{\beta_1} \prod_{r=1}^{k-1} \binom{\beta_r}{i_r} \binom{\beta_{r+1}+\sum_{m=0}^r i_m}{\beta_{r+1}} + (2^{\alpha+1}-1) \binom{\alpha+1}{i_0} \binom{\beta_1+1+i_0}{\beta_1} \prod_{r=1}^{k-1} \binom{\beta_r}{i_r} \binom{\beta_{r+1}+1+\sum_{m=0}^r i_m}{\beta_{r+1}} \right] \quad (16)$$

where if $k = 0$, then $R(Q_{2^{\alpha+2}}) = 2^{2^{\alpha+2}}$ and if $k = 1$, then

$$R(Q_{2^{\alpha+2}} \times \mathbb{Z}_{p_1^{\beta_1}}) = 2^{\beta_1} \sum_{i_0=0}^{\alpha+1} \left[\binom{\alpha+1}{i_0} \binom{\beta_1+i_0}{\beta_1} + (2^{\alpha+1}-1) \binom{\alpha+1}{i_0} \binom{\beta_1+1+i_0}{\beta_1} \right].$$

Remark 3.8. If one adopts the equivalence relation on the set of fuzzy subgroups of a given finite group G used by Murali and Makamba [4, 5, 6, 7], then one can see that the number of fuzzy subgroups of G is equal to the number of chains of subgroups of G . Thus by Proposition 2.1 the result of this paper leads us to the number of fuzzy subgroups of $Q_{2^{\alpha+2}} \times \mathbb{Z}_n$ under the equivalence relation by Murali and Makamba.

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JU-MOK OH, DEPARTMENT OF MATHEMATICS, GANGNEUNG-WONJU NATIONAL UNIVERSITY, 7,
JUKHEON-GIL, GANGNEUNG-SI, GANGWON-DO 210-702, REPUBLIC OF KOREA
E-mail address: jumokoh@gwnu.ac.kr