THE RELATIONSHIP BETWEEN L-FUZZY PROXIMITIES AND L-FUZZY QUASI-UNIFORMITIES

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Abstract. In this paper, we investigate the L-fuzzy proximities and the relationships between L-fuzzy topologies, L-fuzzy topogenous order and L-fuzzy uniformity. First, we show that the category of L-fuzzy topological spaces can be embedded in the category of L-fuzzy quasi-proximity spaces as a coreflective full subcategory. Second, we show that the category of L-fuzzy proximity spaces is isomorphic to the category of L-fuzzy topogenous order spaces. Finally, we obtain that the category of L-fuzzy proximity spaces can be embedded in the category of L-fuzzy uniform spaces as a bireflective full subcategory.

1. Introduction

Proximity spaces were introduced in 1936 by V. A. Efremovic. Proximity is an important concept in topology and it can be considered either as axiomatizations of geometric notions, close to but quite independent of topology, or as convenient tools for an investigation of topological spaces. Hence proximity has close relations with topology, uniformity and metric. With the development of topology, the theory of proximity makes a massive progress. In the framework of L-topology, many authors generalized the crisp proximity to L-fuzzy setting. For example, in [8], Ghanim et al. introduced the concept of S-quasi-proximities on $[0,1]^X$ and in [20], Shi studied S-quasi-proximities on L^X and pointwise S-quasi-proximities. Katsaras [14] introduced quasi-proximity in [0,1]-fuzzy set theory. Subsequently, Liu [16], Artico and Moresco [2] extended it into L-fuzzy set theory. In recently Yue extended the proximity theory of L-topology to L-fuzzy topology, see [25]. Motivated by that, in this paper we make a further research on the relationship between L-fuzzy proximities and L-fuzzy quasi-uniformities.

The outline of this paper is as follows. In section 2, we give some concepts and results which will be used in the sequel. In section 3, we show that the category of L-fuzzy topological spaces can be embedded in the category of L-fuzzy quasi-proximity spaces as a coreflective full subcategory. In section 4, we study the relationship between L-fuzzy proximities and L-fuzzy topogenous orders, and show that the category of L-fuzzy proximity spaces is isomorphic to the category of L-fuzzy topogenous order spaces. In section 5, we study the relationship between

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L-fuzzy proximities and L-fuzzy uniformities, and we obtain that the category of L-fuzzy proximity spaces can be embedded in the category of L-fuzzy uniform spaces as a bireflective full subcategory.

2. Preliminaries

In this section, we give some basic concepts and useful results which will be used in the sequel.

An element a in a complete lattice L is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $c(L)$. We say a is wedge below b, in symbols, $a \lhd b$ or $b \rhd a$, if for every arbitrary subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. The lattice L is completely distributive if every element $a \in L$ is the supremum of all elements wedge below it. We know that every element $a \in L$ is the supremum of all coprimes wedge below it if L is completely distributive. It is well known that both the way below relation and the wedge below relation have the interpolation property, hence if $a \lhd b$ in a completely distributive lattice L and a is a coprime, there is some coprime $c \in c(L)$ such that $a \lhd c \lhd b$.

Throughout this paper, L is a completely distributive lattice with an order reversing involution $'$. L^X is the set of all L-fuzzy sets on X. The elements 1_X and 0_X are the top and bottom one of L^X, respectively. The set of all coprimes in L^X is denoted by $c(L^X)$. In this paper, we always assume $e \in c(L^X)$ when we take $e \in L^X$ such that $e \lhd A$ for $A \in L^X$. Let $e|A$ denote the set $\{B \in L^X | e \not\leq B \geq A\}$ for $e \in c(L^X)$ and $A \in L^X$. For undefined notions and results about complete lattices, please refer to [9].

Proposition 2.1. [9] *Let L be a complete lattice. The following conditions are equivalent:*
(1) *L is completely distributive;*
(2) *L is distributive continuous lattice with enough coprimes;*
(3) *L is distributive and both L and L^{op} are continuous;*

Now we present some definition of topology:

Definition 2.2. [15, 21] An L-fuzzy topology on a set X is defined to be a mapping $\tau : L^X \to L$ satisfying
($FT1$) $\tau(1_X) = \tau(0_X) = 1$;
($FT2$) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ for all $A, B \in L^X$;
($FT3$) $\tau(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \tau(A_t)$ for every family $\{A_t |\ t \in T\} \subseteq L^X$.

If τ is an L-fuzzy topology on X, the pair (L^X, τ) is called an L-fuzzy topological space. An L-fuzzy continuous mapping between L-fuzzy topological spaces (L^X, τ) and (L^Y, τ_1) is a mapping $F : X \to Y$ such that $\tau(F_L^{\leftarrow}(B)) \geq \tau_1(B)$ for all $B \in L^Y$.

The category of L-fuzzy topological spaces and L-fuzzy continuous mappings are denoted by L-**FTOP** and let L-**FTOP**(X) denotes all the L-fuzzy topologies on X.

Similarly, if $\eta : L^X \to L$ satisfying

$(FCT1)$ $\eta(1_X) = \eta(0_X) = 1$;
$(FCT2)$ $\eta(A \vee B) \geq \eta(A) \wedge \eta(B)$ for all $A, B \in L^X$;
$(FCT3)$ $\eta(\bigwedge_{j\in J} A_j) \geq \bigwedge_{j\in J} \eta(A_j)$ for every family $\{A_j | j \in J\} \subseteq L^X$;

η is called an L-fuzzy co-topology. The category of L-fuzzy co-topological spaces and continuous mappings is denoted by L-**FCTOP**.

In [7], Fang introduced the concept of base and subbase in $[0,1]$-fuzzy topology and it is easy to generalize them in L-fuzzy setting as follows:

Definition 2.3. (1) Let τ be an L-fuzzy topology on X, $\mathcal{B} : L^X \to L$ such that $\mathcal{B} \leq \tau$ (in pointwise sense). Then $\mathcal{B}$ is called a base of τ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in L^X, \tau(A) = \bigvee_{\bigvee_{\lambda\in\wedge} B_\lambda = A} \bigwedge_{\lambda\in\wedge} \mathcal{B}(B_\lambda),$$

where the expression $\bigvee_{\bigvee_{\lambda\in\wedge} B_\lambda = A} \bigwedge_{\lambda\in\wedge} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.

(2) Let $\phi : L^X \to L$ be a mapping. Then ϕ is called a subbase of τ if $\phi^{(\sqcap)} : L^X \to L$ is a base, where $\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\lambda\in J} B_\lambda = A} \bigwedge_{\lambda\in J} \phi(B_\lambda)$ for all $A \in L^X$ with $(\sqcap)$ standing for "finite intersection". A mapping $\phi : L^X \to L$ is a subbase if and only if $\phi^{(\sqcup)}(1_X) = 1$.

(3) Let $\{(L^{X_t}, \tau_t)\}_{t\in T}$ be a family of L-fuzzy topological spaces and $P_t : \prod_{t\in T} X_t \to X_t$ be the projection. Then the L-fuzzy topology whose subbase is defined by

$$\forall A \in L^{\prod_{t\in T} X_t}, \phi(A) = \bigvee_{t\in T} \bigvee_{(P_t)_L^{\leftarrow}(B) = A} \tau_t(B)$$

is called the product topology of $\{\tau_t |\ t \in T\}$, denoted $\prod_{t\in T} \tau_t$. $(L^{\prod_{t\in T} X_t}, \prod_{t\in T} \tau_t)$ is called the product space of $\{(L^{X_t}, \tau_t)\}_{t\in T}$.

Definition 2.4. [24] A fuzzy remote neighborhood system is a set $\mathcal{R} = \{R_e | e \in c(L^X)\}$ of maps $R_e : L^X \to L$ such that:

$(FRN1)$ $R_e(1) = 0, R_e(0) = 1$;
$(FRN2)$ $R_e(u) > 0 \Rightarrow e \not\leq u$;
$(FRN3)$ $R_e(u \vee v) = R_e(u) \wedge R_e(v)$.

The pair $(L^X, \mathcal{R})$ is called a fuzzy remote neighborhood space (FRNS, in short), if it also satisfies the following equation:

$(FRN4)$ $R_e(u) = \bigvee_{v\in e|u} \bigwedge_{a\not\leq v} R_a(v)$.

We call it topological fuzzy remote neighborhood space, in short TFRNS. Let $(L^X, \mathcal{R}^1)$ and $(L^Y, \mathcal{R}^2)$ be two fuzzy remote neighborhood spaces $F : (L^X, \mathcal{R}^1) \to (L^Y, \mathcal{R}^2)$ is called continuous if $R^2_{F_L^{\to}(e)}(A) \leq R^1_e(F_L^{\leftarrow}(A))$ for $e \in c(L^X)$ and $A \in L^Y$. The category of topological remote neighborhood spaces is denoted by L-**TFRNS**.

Let $\eta : L^X \to L$ be an L-fuzzy co-topology and define $R_e : L^X \to L^X$ as follows:

$$R_e^\eta(A) = \begin{cases} \bigvee_{B\in e|A} \eta(B), & e \not\leq A, \\ 0, & e \leq A. \end{cases}$$

Then $\mathcal{R}^\eta = \{R_e^\eta | e \in c(L^X)\}$ is topological fuzzy remote neighborhood system. Conversely, if $\mathcal{R} = \{R_e | e \in c(L^X)\}$ is fuzzy remote neighborhood system then $\eta^{\mathcal{R}}$ is a L-fuzzy co-topology, where $\eta^{\mathcal{R}} : L^X \to M$ is defined by $\eta^{\mathcal{R}}(A) = \bigwedge_{e\not\leq A} R_e(A)$.

In [24], we know that L-**TFRNS** is isomorphic to L-**FCTOP**.

Lemma 2.5. *Let $\mathcal{R} = \{R_e | e \in c(L^X)\}$ be a set satisfying (FRN1)–(FRN3). Then the following two statements are equivalent*

$(FRN4)$ $R_e(u) = \bigvee_{v\in e|u} \bigwedge_{a\not\leq v} R_a(v)$;

$(FRN4^*)$ $R_e(u) = \bigvee_{v\in e|u}(R_e(v) \wedge \bigwedge_{a\not\leq v} R_a(u))$.

Next, we recall some definitions of uniformities. Let $H(L^X)$ denote the set of all mappings $f : L^X \to L^X$ satisfying the following conditions:

(1) $A \leq f(A)$ for all $A \in L^X$;

(2) for all $\{A_t\}_{t\in T}$, $f(\bigvee_{t\in T} A_t) = \bigvee_{t\in T} f(A_t)$, i.e., f preserves sups.

f_1 is the biggest element of $H(L^X)$, i.e., $f_1(A) = 0_X$ when $A = 0_X$ and $f_1(A) = 1_X$ otherwise.

For all $f, g \in H(L^X)$, we define

(1) $f \leq g$ if and only if $f(A) \leq g(A)$ for all $A \in L^X$;

(2) $(f \wedge g)(A) = \bigvee_{e\in\beta^*(A)}(f(e) \wedge g(e))$ for all $A \in L^X$ and $f \wedge g \in H(L^X)$ obviously, where $\beta^*(A) = \{e \in c(L^X) | e \lhd A\}$.

Definition 2.6. [12] A Hutton quasi-uniformity on X is a nonempty subset $\mathcal{H}U$ of $H(L^X)$ such that

$(U1)$ if $f \in \mathcal{H}U$, $g \in H(L^X)$, and $g \geq f$ then $g \in \mathcal{H}U$;

$(U2)$ if $f, g \in \mathcal{H}U$ then $f \wedge g \in \mathcal{H}U$;

$(U3)$ if $f \in \mathcal{H}U$ then there exist $g \in \mathcal{H}U$ such that $g \circ g \leq f$.

$(L^X, \mathcal{H}U)$ is called Hutton quasi-uniform space (or HQUS, for short).

$F : (L^X, \mathcal{H}U) \to (L^Y, \mathcal{H}U_1)$ is said to be quasi-uniformly continuous with respect to $\mathcal{H}U$ and $\mathcal{H}U_1$ if $F_L^{\leftarrow} \circ g \circ F_L^{\rightarrow} \in \mathcal{H}U$ for all $g \in \mathcal{H}U_1$. Let **HuQUnif** denote the category of HQUSs and quasi-uniformly continuous mappings.

Definition 2.7. [6, 10, 18, 22, 25] An L-fuzzy quasi-uniformity on X is a mapping $\mathcal{U} : H(L^X) \to L$ such that

$(FQU1)$ $\mathcal{U}(f_1) = 1$;

$(FQU2)$ $\mathcal{U}(f \wedge g) = \mathcal{U}(f) \wedge \mathcal{U}(g)$ for all $f, g \in H(L^X)$;

$(FQU3)$ $\mathcal{U}(f) = \bigvee_{g\circ g\leq f} \mathcal{U}(g)$ for each $f \in H(L^X)$.

If $\mathcal{U}$ is an L-fuzzy quasi-uniformity on X, we say $(L^X,\mathcal{U})$ is an L-fuzzy quasi-uniform space (or L-FQUS, for short). Let $(L^X,\mathcal{U})$ and $(L^Y,\mathcal{U}_1)$ be two L-FQUSs, $F : (L^X,\mathcal{U}) \to (L^Y,\mathcal{U}_1)$ is called L-fuzzy quasi-uniformly continuous if $\mathcal{U}(F_L^{\leftarrow} \circ$

$g \circ F_L^{\rightarrow}) \geq \mathcal{U}_1(g)$ for each $g \in \mathcal{H}(L^Y)$. The category of L-FQUSs and L-fuzzy quasi-uniformly continuous mappings is denoted by L-**HuQUnif**.

3. L-fuzzy Proximity

In this section, we show that the category of L-fuzzy topological spaces can be embedded in the category of L-fuzzy quasi-proximity spaces as a coreflective full subcategory.

Firstly, we give the definition of L-fuzzy quasi-proximity as follows:

Definition 3.1. [25] An L-fuzzy quasi-proximity on L^X is a mapping $\delta : L^X \times L^X \to L$ satisfying the following conditions:

$(FP1)$ $\delta(1_X, 0_X) = \delta(0_X, 1_X) = 0$;
$(FP2)$ $\delta(A \vee B, C) = \delta(A, C) \vee \delta(B, C)$; $\delta(A, B \vee C) = \delta(A, B) \vee \delta(A, C)$;
$(FP3)$ $\delta(A, B) \geq \bigwedge_{C \in L^X}(\delta(A, C) \vee \delta(C', B))$;
$(FP4)$ $\delta(A, B) < 1$ implies $A \leq B'$.

δ is called an L-fuzzy proximity on L^X if it also satisfies the following condition:

$(FP5)$ $\delta(A, B) = \delta(B, A)$.

The pair (L^X, δ) is called an L-fuzzy (quasi-)proximity space. A mapping $F : (L^X, \delta) \to (L^Y, \delta_1)$ is called continuous if $\delta(A, B) \leq \delta_1(F_L^{\rightarrow}(A), F_L^{\rightarrow}(B))$ holds for all $A, B \in L^X$. The categories of L-fuzzy quasi-proximity spaces and L-fuzzy proximity spaces are denoted by L-**FQProX** and L-**FProX**, respectively.

Note that, Yue showed that $\mathcal{R}^{(\delta)} = \{R_e^{(\delta)} | \ e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system, where $R_e^{(\delta)} : L^X \to L$ is defined by $R_e^{(\delta)}(A) = \bigvee_{e \not\leq B} \delta(B', A)'$ for the given L-fuzzy quasi-proximity δ, see [25]. In fact, we can also prove the following similarly results.

Since the proofs are similar to those in [25], we omit the proofs of Theorem 3.2, 3.3, 3.5 and Theorem 3.6, and just list the results.

Theorem 3.2. *Let (L^X, δ) be an L-fuzzy quasi-proximity space and define $R_e^\delta : L^X \to L$ by $R_e^\delta(A) = \bigvee_{e \not\leq B} \delta(A, B')'$. Then $\mathcal{R}^\delta = \{R_e^\delta | e \in c(L^X)\}$ is a topological fuzzy remote neighborhood system.*

Theorem 3.3. *Let $(L^X, \mathcal{R})$ be a topological remote neighborhood space and define $\delta^{\mathcal{R}} : L^X \times L^X \to L$ by $\delta^{\mathcal{R}}(A, B) = \bigvee_{a \not\leq B'} R_a(A)'$. Then $\delta^{\mathcal{R}}$ is an L-fuzzy quasi-proximity.*

Remark 3.4. Let $(L^X, \mathcal{R})$ be a topological remote neighborhood space and define $\delta^{(\mathcal{R})} : L^X \times L^X \to L$ by $\delta^{(\mathcal{R})}(A, B) = \bigvee_{a \not\leq A'} R_a(B)'$. Then $\delta^{(\mathcal{R})}$ is also an L-fuzzy quasi-proximity. If $\mathcal{R}$ satisfies the following propasitionerties:

$$\forall A, B \in L^X, \quad \bigwedge_{a \not\leq A'} R_a(B) = \bigwedge_{a \not\leq B'} R_a(A),$$

then $\delta^{(\mathcal{R})} = \delta^{\mathcal{R}}$.

Theorem 3.5. (1) *Let $(L^X, \mathcal{R})$ be a topological remote neighborhood space, then $\mathcal{R}^{\delta^{\mathcal{R}}} = \mathcal{R}$.*

(2) *Let (L^X, δ) be an L-fuzzy quasi-proximity space, then $\delta^{\mathcal{R}^\delta} \leq \delta$.*

Theorem 3.6. (1) *If $F : (L^X, \mathcal{R}) \to (L^Y, \mathcal{R}^1)$ is continuous, then $F : (L^X, \delta^{\mathcal{R}}) \to (L^Y, \delta^{\mathcal{R}_1})$ is continuous.*

(2) *If $F : (L^X, \delta) \to (L^Y, \delta_1)$ is continuous, then $F : (L^X, \mathcal{R}^\delta) \to (L^Y, \mathcal{R}^{\delta_1})$ is continuous.*

From Theorem 3.5 and Theorem 3.6, we also can get the following result.

Theorem 3.7. FRNS *can be embedded in L-**FQProX** as a bicoreflective full subcategory. Hence, L-**FCTOP** can be embedded in L-**FQProX** as a bicoreflective full subcategory.*

4. L-fuzzy Topogenous Orders

In [5], Csaszar gave a new method for the foundation of general topology based on the theory of topogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. In [13], Katsaras studied topogenous structures in fuzzy setting. Moreover, in $[0,1]$-fuzzy topology the theory of topogeneous orders was developed in [23] . In this section, we will generalized the concept of topogenous orders in L-fuzzy topology, and show that the category of L-fuzzy proximity spaces is isomorphic to the category of L-fuzzy topogenous order spaces.

Definition 4.1. An L-fuzzy topogenous order on L^X is a mapping $T : L^X \times L^X \to L$ satisfying the following conditions:
$(FTO1)$ $T(0_X, 0_X) = T(1_X, 1_X) = 1$;
$(FTO2)$ $T(A_1 \vee A_2, B_1 \vee B_2) \geq T(A_1, B_1) \wedge T(A_2, B_2)$ and $T(A_1 \wedge A_2, B_1 \wedge B_2) \geq T(A_1, B_1) \wedge T(A_2, B_2)$;
$(FTO3)$ $T(A, B) \leq \bigvee_{C \in L^X} (T(A, C) \wedge T(C, B))$;
$(FTO4)$ If $A_1 \leq A$ and $B \leq B_1$, then $T(A_1, B_1) \geq T(A, B)$;
$(FTO5)$ $T(A, B) > 0$ implies $A \leq B$;
$(FTO6)$ $T(A, B) = T(B', A')$.

The pair (L^X, T) is called an L-fuzzy topogenous order space. A mapping $F : (L^X, T) \to (L^Y, T_1)$ is called continuous if $T_1(C, D) \leq T(F_L^{\leftarrow}(C), F_L^{\leftarrow}(D))$ holds for all $C, D \in L^Y$. The category of L-fuzzy topogenous order spaces and continuous mappings is denoted by L-**FTopO**.

The readers can easily check the following results.

Theorem 4.2. *Let δ be an L-fuzzy proximity on L^X. Then $T^\delta : L^X \times L^X \to L$ defined by $T^\delta(A, B) = \delta(A, B')'$ is an L-fuzzy topogenous order on L^X.*

Theorem 4.3. *Let T be an L-fuzzy topogenous order on L^X. Then $\delta^T : L^X \times L^X \to L$ defined by $\delta^T(A, B) = T(A, B')'$ is an L-fuzzy proximity on L^X.*

Theorem 4.4. *Let T be an L-fuzzy topological order on L^X and δ be an L-fuzzy proximity on L^X. Then we have $T^{\delta^T} = T$ and $\delta^{T^\delta} = \delta$.*

Theorem 4.5. (1) *If $F:(L^X,T)\to(L^Y,T_1)$ is continuous, then $F:(L^X,\delta^T)\to(L^Y,\delta^{T_1})$ is continuous.*

(2) *If $F:(L^X,\delta)\to(L^Y,\delta_1)$ is continuous, then $F:(L^X,T^\delta)\to(L^Y,T^{\delta_1})$ is continuous.*

Hence, we have the following result.

Theorem 4.6. *L-**FProX** is isomorphic to L-**FTopO**.*

5. Relationships Between L-fuzzy Proximity and L-fuzzy Uniformity

In this section, we study the relationship between L-fuzzy proximities and L-fuzzy uniformities, and we obtain that the category of L-fuzzy proximity spaces can be embedded in the category of L-fuzzy uniform spaces as a bireflective full subcategory.

Theorem 5.1. *Let $\mathcal{U}$ be an L-fuzzy quasi-uniformity on L^X, and define $\delta^{\mathcal{U}}:L^X\times L^X\to L$ as follows:*

$$\delta^{\mathcal{U}}(A,B)=\bigwedge_{f(A)\le B'}\mathcal{U}(f)'.$$

Then $\delta^{\mathcal{U}}$ is an L-fuzzy quasi-proximity. Furthermore, if $\mathcal{U}$ is an L-fuzzy uniformity, then $\delta^{\mathcal{U}}$ is an L-fuzzy proximity.

Proof. (FP1) and (FP4) are trivial, and the proof of (FP2) is routine. We only prove (FP3).

(FP3): In order to prove $\delta^{\mathcal{U}}(A,B)\ge\bigwedge_{C\in L^X}(\delta^{\mathcal{U}}(A,C)\vee\delta^{\mathcal{U}}(C',B))$, it suffices to show that

$$\bigvee_{f(A)\le B'}\mathcal{U}(f)\le\bigvee_{C\in L^X}(\bigvee_{g(A)\le C}\mathcal{U}(g)\wedge\bigvee_{h(C')\le B'}\mathcal{U}(h)).$$

Let $t\lhd\bigvee_{f(A)\le B'}\mathcal{U}(f)$, i.e.,

$$t\lhd\bigvee_{f(A)\le B'}\mathcal{U}(f)=\bigvee_{f(A)\le B'}\bigvee_{g\circ g\le f}\mathcal{U}(g).$$

Then there exist $f_t,g_t\in H(L^X)$ such that $f_t(A)\le B'$, $g_t\circ g_t\le f_t$ and $t\le\mathcal{U}(g_t)$. Let $C=(g_t(A))'$. Then $t\le\mathcal{U}(g_t)\le\bigvee_{g(A)\le C'}\mathcal{U}(g)$. Furthermore, we have $g_t(C')=g_t(g_t(A))\le f_t(A)\le B'$ and $t\le\mathcal{U}(g_t)\le\bigvee_{h(C')\le B'}\mathcal{U}(h)$. From the arbitrariness of t, we have the conclusion as desired.

Furthermore, when $\mathcal{U}$ is an L-fuzzy uniformity, it can be easily checked that $\delta^{\mathcal{U}}$ satisfies (FP5). □

Theorem 5.2. *Let $(L^X,\mathcal{U})$ be an L-fuzzy quasi-uniformity. Then $\mathcal{R}^{\mathcal{U}}=\mathcal{R}^{\delta^{\mathcal{U}}}$, this is to say that $\mathcal{U}$ and $\delta^{\mathcal{U}}$ induce the same topological fuzzy remote neighborhood system.*

Proof. It needs to prove that $R_e^{\mathcal{U}}(A)=R_e^{\delta^{\mathcal{U}}}(A)$ for all $e\in c(L^X)$ and $A\in L^X$, i.e.,

$$\bigvee_{e\not\le f(A)}\mathcal{U}(f)=\bigvee_{e\not\le B}\bigvee_{f(A)\le B}\mathcal{U}(f).$$

First, $\bigvee_{e\not\leq f(A)}\mathcal{U}(f) \geq \bigvee_{e\not\leq B}\bigvee_{f(A)\leq B}\mathcal{U}(f)$ is obvious. Conversely, let

$$t \lhd \bigvee_{e\not\leq f(A)} \mathcal{U}(f) = \bigvee_{e\not\leq f(A)} \bigvee_{g\circ g\leq f} \mathcal{U}(g),$$

then there exist $f, g \in H(L^X)$ such that $e \not\leq f(A)$, $g \circ g \leq f$ and $t \leq \mathcal{U}(g)$. Now let $B = g(A)$. Then $e \not\leq f(A) \geq g(A) = B$. Thus $t \leq \mathcal{U}(g) \leq \bigvee_{e\not\leq B}\bigvee_{f(A)\leq B}\mathcal{U}(f)$, as desired. □

Theorem 5.3. *Let (L^X, δ) be an L-fuzzy quasi-proximity and put*

$$\mathcal{A}_\delta = \{(A,B) \in L^X \times L^X | \delta(A,B) < 1\}.$$

For all $(A,B) \in \mathcal{A}$, define $f_{BA} : L^X \to L^X$ as follows:

$$f_{BA}(D) = \begin{cases} B', & D \leq A, \\ 1_X, & \textit{others.} \end{cases}$$

Then $\mathcal{U}^\delta$ is an L-fuzzy quasi-uniformity on L^X, where $\mathcal{U}^\delta : H(L^X) \to L$ is defined by

$$\forall f \in H(L^X), \quad \mathcal{U}^\delta(f) = \bigvee\{\wedge_{i=1}^{i=n}(\delta(A_i,B_i))'|f \geq \wedge_{i=1}^{i=n} f_{B_iA_i}, n \in N\}.$$

Furthermore, if δ is an L-fuzzy proximity, then $\mathcal{U}^\delta$ is an L-fuzzy uniformity.

Proof. First of all, it can be easily checked that $f_{BC'} \circ f_{CA} = f_{BA}$. Now we show that $\mathcal{U}^\delta$ is an L-fuzzy quasi-uniformity and we only check (FQU3), i.e., $\mathcal{U}^\delta(f) = \bigvee_{g\circ g\leq f}\mathcal{U}^\delta(g)$. Let

$$t \lhd \mathcal{U}^\delta(f) = \bigvee\{\wedge_{i=1}^{i=n}(\delta(A_i,B_i))'|f \geq \wedge_{i=1}^{i=n} f_{B_iA_i}, n \in N\},$$

then there exists $\{(A_i,B_i)\}_{i=1}^n$ such that $f \geq \wedge_{i=1}^{i=n} f_{B_iA_i}$ and $t \lhd \wedge_{i=1}^{i=n}(\delta(A_i,B_i))'$. From (FP3), there exist $\{C_i\}_{i=1}^n$ such that $t \leq \delta(A_i,C_i)' \wedge \delta(C_i', B_i)'$ for all $i = 1, \cdots, n$. Let $g = \wedge_{i=1}^{i=n} f_{B_iC_i'} \wedge \wedge_{i=1}^{i=n} f_{C_iA_i}$. Then we have

$$g \circ g \leq \wedge_{i=1}^{i=n}(f_{B_iC_i'} \circ f_{C_iA_i}) = \wedge_{i=1}^{i=n} f_{B_iA_i} \leq f.$$

Therefore, $t \leq \wedge_{i=1}^{i=n}\delta(A_i,C_i)' \wedge \delta(C_i',B_i)' \leq \mathcal{U}^\delta(g)$. This completes the proof.

Furthermore, it is easy to check $f_{BA}^{\lhd} = f_{AB}$ for all $(A,B) \in \mathcal{A}$. From this, it is easy to show that $\mathcal{U}^\delta$ is an L-fuzzy uniformity if δ is an L-fuzzy proximity. □

Theorem 5.4. *Let (L^X, δ) be an L-fuzzy quasi-proximity space. Then we have $\delta = \delta^{\mathcal{U}^\delta}$.*

Proof. First of all, we have the interpretation of $\delta^{\mathcal{U}^\delta}$ as follows:

$$\begin{aligned} \delta^{\mathcal{U}^\delta}(A,B) &= \bigwedge_{f(A)\leq B'} \mathcal{U}^\delta(f)' \\ &= \bigwedge_{f(A)\leq B'} \bigwedge_{f\geq\wedge_{i=1}^{i=n} f_{B_iA_i}} \vee_{i=1}^{i=n}\delta(A_i,B_i) \end{aligned}$$

If $\delta(A,B)=1$, then $\delta(A,B)\geq \delta^{\mathcal{U}^{\delta}}(A,B)$. We assume that $\delta(A,B)<1$. From $f_{BA}(A)=B'$, we know that

$$\bigwedge_{f(A)\leq B'}\ \bigwedge_{f\geq \wedge_{i=1}^{i=n} f_{B_iA_i}} \vee_{i=1}^{i=n}\delta(A_i,B_i)\leq \delta(A,B).$$

On the other hand, without loss of generality, we assume that $\delta^{\mathcal{U}^{\delta}}(A,B)<1$. We want to show that $\vee_{i=1}^{i=n}\delta(A_i,B_i)\geq\delta(A,B)$ for all $f\in H(L^X)$ and $\{(A_i,B_i)\}_{i=1}^{n}$ satisfying $f(A)\leq B'$ and $f\geq\wedge_{i=1}^{i=n}f_{B_iA_i}$. Notice that $\bigvee_{e\lhd A}\wedge_{i=1}^{i=n}f_{B_iA_i}(e)=\wedge_{i=1}^{i=n}f_{B_iA_i}(A)\leq f(A)\leq B'$, and $\forall e\lhd A$, there exists $I(e)\subseteq\{1,2,\cdots,n\}$ such that $e\leq\wedge_{j\in I(e)}A_j$ and $e\leq\wedge_{j\in I(e)}B_j'$ when $\wedge_{i=1}^{i=n}f_{B_iA_i}(e)=\wedge_{j\in I(e)}B_j'$ (say), we have $\bigwedge_{e\lhd A}\vee_{j\in I(e)}B_j\geq B$ and $A=\bigvee_{e\lhd A}e\leq\bigvee_{e\lhd A}\wedge_{j\in I(e)}A_j$. It is easy to see $\{\bigwedge_{j\in I(e)}B_j|\ e\lhd A\}$ and $\{\bigwedge_{j\in I(e)}A_j\ |e\lhd A\}$ are two finite sets. Therefore,

$$\begin{aligned}
\delta(A,B) &\leq \delta(\bigvee_{e\lhd A}\wedge_{j\in I(e)}A_j,\ \bigwedge_{e\lhd A}\vee_{j\in I(e)}B_j)\\
&= \bigvee_{e\lhd A}\delta(\wedge_{j\in I(e)}A_j,\ \bigwedge_{e\lhd A}\vee_{j\in I(e)}B_j)\\
&\leq \bigvee_{e\lhd A}\delta(\wedge_{j\in I(e)}A_j,\vee_{j\in I(e)}B_j)\\
&= \bigvee_{e\lhd A}\vee_{j\in I(e)}\delta(\wedge_{j\in I(e)}A_j,B_j)\\
&\leq \bigvee_{e\lhd A}\vee_{j\in I(e)}\delta(A_j,B_j)\\
&\leq \vee_{i=1}^{i=n}\delta(A_i,B_i),
\end{aligned}$$

as desired. □

Theorem 5.5. *Let (L^X,δ) be an L-fuzzy quasi-proximity space. Then $\mathcal{R}^{\delta}=\mathcal{R}^{\mathcal{U}^{\delta}}$.*

Proof. The proof is similar to that of Theorem 5.4. □

Theorem 5.6. *Let $(L^X,\mathcal{U})$ be an L-fuzzy quasi-uniform space. Then $\mathcal{U}^{\delta^{\mathcal{U}}}\leq\mathcal{U}$.*

Proof. $\mathcal{U}^{\delta^{\mathcal{U}}}\leq\mathcal{U}$ can be proved by the following computation.

$$\begin{aligned}
\mathcal{U}^{\delta^{\mathcal{U}}}(f) &= \bigvee\{\wedge_{i=1}^{i=n}(\delta^{\mathcal{U}}(A_i,B_i))'|f\geq\wedge_{i=1}^{i=n}f_{B_iA_i},n\in N\}\\
&= \bigvee_{f\geq\wedge_{i=1}^{i=n}f_{B_iA_i}}\wedge_{i=1}^{i=n}\bigvee_{g(A_i)\leq B_i'}\mathcal{U}(g)\\
&\leq \bigvee_{f\geq\wedge_{i=1}^{i=n}f_{B_iA_i}}\wedge_{i=1}^{i=n}\mathcal{U}(f_{B_iA_i})\\
&= \bigvee_{f\geq\wedge_{i=1}^{i=n}f_{B_iA_i}}\mathcal{U}(\wedge_{i=1}^{i=n}f_{B_iA_i})\\
&\leq \mathcal{U}(f).
\end{aligned}$$

□

Theorem 5.7. (1) *If $F:(L^X,\mathcal{U})\to(L^Y,\mathcal{U}_1)$ is continuous, then $F:(L^X,\delta^{\mathcal{U}})\to(L^Y,\delta^{\mathcal{U}_1})$ is continuous.*

(2) *If $F:(L^X,\delta)\to(L^Y,\delta_1)$ is continuous, then $F:(L^X,\mathcal{U}^{\delta})\to(L^Y,\mathcal{U}^{\delta_1})$ is continuous.*

Proof. (1) Suppose $F : (L^X, \mathcal{U}) \to (L^Y, \mathcal{U}_1)$ is continuous, then $\mathcal{U}_1(g) \leq \mathcal{U}(F^{\Leftarrow}(g))$ for all $g \in H(L^Y)$. For every $A, B \in L^X$, from the definition of $\delta^{\mathcal{U}_1}$, we have

$$\begin{aligned}
\delta^{\mathcal{U}_1}(F_L^{\rightarrow}(A), F_L^{\rightarrow}(B)) &= \bigwedge_{g(F_L^{\rightarrow}(A)) \leq F_L^{\rightarrow}(B)'} \mathcal{U}_1(g)' \\
&\geq \bigwedge_{g(F_L^{\rightarrow}(A)) \leq F_L^{\rightarrow}(B)'} \mathcal{U}(F^{\Leftarrow}(g))' \\
&\geq \bigwedge_{F^{\Leftarrow}(g)(A) \leq B'} \mathcal{U}(F^{\Leftarrow}(g))' \\
&\geq \bigwedge_{f(A) \leq B'} \mathcal{U}(f)' \\
&= \mathcal{U}^{\delta}(A, B),
\end{aligned}$$

as desired.

(2) Let $F : (L^X, \delta) \to (L^Y, \delta_1)$ be continuous, we know $\delta(A, B) \leq \delta_1(F_L^{\rightarrow}(A), F_L^{\rightarrow}(B))$ for all $A, B \in L^X$. For each $g \in H(L^Y)$, from the definition of $\mathcal{U}^{\delta_1}$, we have

$$\begin{aligned}
\mathcal{U}^{\delta_1}(g) &= \bigvee \{\wedge_{i=1}^{i=n} \delta_1(C_i, D_i)' |\ g \geq \wedge_{i=1}^{i=n} f_{D_i C_i}, n \in N\} \\
&\leq \bigvee \{\wedge_{i=1}^{i=n} \delta_1(F_L^{\rightarrow}(F_L^{\leftarrow}(C_i)), F_L^{\rightarrow}(F_L^{\leftarrow}(D_i)))' |\ g \geq \wedge_{i=1}^{i=n} f_{D_i C_i}, n \in N\} \\
&\leq \bigvee \{\wedge_{i=1}^{i=n} \delta(F_L^{\leftarrow}(C_i), F_L^{\leftarrow}(D_i))' |\ F^{\Leftarrow}(g) \geq \wedge_{i=1}^{i=n} f_{F_L^{\leftarrow}(D_i) F_L^{\leftarrow}(C_i)}, n \in N\} \\
&\leq \bigvee \{\wedge_{i=1}^{i=n} \delta(A_i, B_i)' |\ F^{\Leftarrow}(g) \geq \wedge_{i=1}^{i=n} f_{B_i A_i}, n \in N\} \\
&= \mathcal{U}^{\delta}(F^{\Leftarrow}(g)),
\end{aligned}$$

as desired. □

We summarize the results of this paper as follows.

Theorem 5.8. (1) L-**FCTOP** *can be embedded in* L-**FQProX** *as a bicoreflective full subcategory.*

(2) L-**FProX** *is isomorphic to* L-**FTopO**.

(3) L-**FProX** *can be embedded in* L-**FHuUnif** *as a bireflective full subcategory.*

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References

[1] J. Adamek J, H. Herrlich and G. E. Strecker, *Abstract and concrete categories*, J. Wiley and Sons, New York, 1990.

[2] G. Artico and R. Moresco, *Fuzzy proximities and totally bounded fuzzy uniformities*, J. Math. Anal. Appl., **9** (1984), 320-1337.

[3] G. Artico and R. Moresco, *Fuzzy proximities compatible with Lowen uniformities*, Fuzzy Sets and Systems, **21** (1987), 85-99.

[4] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[5] A. Csaszar, *Foundations of general topology*, Pergamon Press, 1963.

[6] D. Dzajanbajev and A. Šostak, *On a fuzzy uniform structure*, Acta Comm. Univ. Tartu, **940** (1992), 31-36.

[7] J. M. Fang and Y. L. Yue, *Base and subbase in I-fuzzy topological spaces*, J. Math. Res. Expositon, **26** (2006), 89-95.
[8] M. H. Ghanim, O. A. Tantawy and F. M. Selim, *On S-fuzzy quasi-proximity spaces*, Fuzzy Sets and Systems, **109** (2000), 285-290.
[9] F. Gierz, et al., *A compendium of continuous lattices*, Berlin: Springer Verlag, 1980.
[10] J. Gutierrez Garcia, M. A. de Prada Vicente and A. Šostak, *A unified approach to the concept of fuzzy L-uniform space*, In: Topological and Algebraic Strutures in Fuzzy Sets, A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, S. E. Rodabaugh and E. P. Klement eds. Kluwer Acad. Publ., Dordrecht, Boston, London, Chapter 3, (2003), 81-114.
[11] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl., **78** (1980), 659-674.
[12] B. Hutton, *Uniformities on fuzzy topological spaces*, J. Math. Anal. Appl., **58** (1977), 557-571.
[13] A. K. Katsaras, *On fuzzy syntopogenous structures*, J. Math. Anal. Appl., **99** (1983), 219-236.
[14] A. K. Katsaras, *On fuzzy uniform spaces*, J. Math. Anal. Appl., **101** (1984), 97-114.
[15] T. Kubiak, *On fuzzy topologies*, PhD Thesis, Adam Mickiewicz, Poznan (Poland), 1985.
[16] W. J. Liu, *Fuzzy proximity spaces redefined*, Fuzzy Sets and Systems, **15** (1985), 241-248.
[17] Y. M. Liu and M. K. Luo, *Fuzzy topology*, Singapore: World Scientific Press, 1997.
[18] S. Markin and A.Šostak , *Another approach to the concept of a fuzzy proximity*, Suppl. Rend. Mat. Palermo, **29** (1992), 530-551.
[19] G. Preuss, *Theory of topological structures: an approch to categorical topology*, D. Reidel Publishing Company, 1987.
[20] F. G. Shi, *The category of pointwise S-proximity spaces*, Fuzzy Sets and Systems, **152** (2005), 349-372.
[21] A. Šostak, *On a fuzzy topological structure*, Rend. Cire. Matem. Palermo, Ser. II, **11** (1985), 89-103.
[22] A. Šostak, *Basic structures of fuzzy topology*, J. Math. Sciences, **78** (1996), 662-697.
[23] A. Šostak, *Fuzzy syntopogenous structures*, Quaestiones Math., **20** (1997), 431-461.
[24] Y. Yue and J. Fang, *Categories isomorphic to the Kubiak-Šostak extension of TML*, Fuzzy Sets and Systems, **157** (2006), 832-842.
[25] Y. Yue and F. G. Shi, *Generalized quasi-proximities*, Fuzzy Sets and Systems, **158** (2007), 386-398.

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