FINITE DIMENSIONAL GENERATING SPACES OF QUASI-NORM FAMILY

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Abstract. In this paper, some results on finite dimensional generating spaces of quasi-norm family are established. The idea of equivalent quasi-norm families is introduced. Riesz lemma is established in this space. Finally, we re-define B-S fuzzy norm and prove that it induces a generating space of quasi-norm family.

1. **Introduction**

Chang et al.[2] dealt with the generating spaces of quasi-metric family and by studying the common characteristic properties of both fuzzy metric spaces in the sense of Kaleva & Seikkala [6] and Menger probabilistic metric spaces[11] established that the generating spaces of quasi-metric family possess the unification characterization of the fuzzy and probabilistic situations. They also proved several fixed point theorems in generating spaces of quasi-metric family. Jung et al.[5] established some fixed point theorems in generating spaces of quasi-metric family. Many authors introduced different ideas of fuzzy norm ([9], [10], [13]), fuzzy n-norm [3], fuzzy Hilbert space[4] etc. in different ways. In 2006, Xiao & Zhu[12] introduced a concept of a generating space of quasi-norm family (G.S.Q.N.F) and studied its properties in a linear topological space setting. They also introduced the concept of convergent sequences, Cauchyness, completeness, compactness, etc. and extended some fixed point theorems, especially Schauder-type fixed point theorem in such spaces.

In this paper, we study certain results in finite dimensional generating spaces of quasi-norm family (G.S.Q.N.Fs). We observe that most of these results more or less differ from the results in finite dimensional normed linear spaces. We generalize the definition of fuzzy norm as introduced by Bag & Samanta[1] (B-S fuzzy norm), and we deduce a G.S.Q.N.F from it.

The organization of the paper is as follows:

Section 1 comprises of some preliminary results.

In section 2, we study some results in finite dimensional G.S.Q.N.Fs.

In section 3, a generalized form of B-S fuzzy norm is advanced and a G.S.Q.N.F is deduced from it.

Throughout this paper straightforward proofs are omitted.

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2. **Preliminary Results**

In this section, we give some preliminary results which will use in this paper.

Definition 2.1. [12] Let X be a linear space over E and θ the origin of X. Let

$$Q = \{|.|_\alpha : \alpha \in (0,1]\}$$

be a family of mappings from X into $[0,\ \infty)$. $(X,\ Q)$ is called a generating space of quasi-norm family and Q a quasi-norm family if the following conditions are satisfied:
(QN-1) $|x|_\alpha = 0\ \forall \alpha \in (0,\ 1]$ iff $x = \theta$;
(QN-2) $|ex|_\alpha = |e||x|_\alpha$ for $x \in X$ and $e \in E$;
(QN-3) for any $\alpha \in (0,\ 1]$ there exists a $\beta \in (0,\ \alpha]$ such that
$|x + y|_\alpha \leq |x|_\beta + |y|_\beta$ for $x,\ y \in X$;
(QN-4) for any $x \in X$, $|x|_\alpha$ is non-increasing and left-continuous for $\alpha \in (0,\ 1]$.
$(X,\ Q)$ is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where
(QN-3u) for any $\alpha \in (0,\ 1]$ there exists $\beta \in (0,\ \alpha]$ such that

$$|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta$$

for any $n \in Z^+$, $x_i \in X(i = 1,\ 2, ...,\ n)$;
(QN-3t) for any $\alpha \in (0,\ 1]$ there exists a $\beta \in (0,\ \alpha]$ such that
$|x + y|_\alpha \leq |x|_\alpha + |y|_\beta$ for $x,\ y \in X$;
(QN-3e) for any $\alpha \in (0,\ 1]$, it holds that $|x + y|_\alpha \leq |x|_\alpha + |y|_\alpha$ for $x,\ y \in X$.

Remark 2.2. If Q be a family of norms on X, then $(X,\ Q)$ is called a generating space of norm family.

Remark 2.3. [12] Clearly, by Definition 2.1 we obtain the following assertions: (QN-3e) implies (QN-3t); (QN-3t) and (QN-4) imply (QN-3u); (QN-3u) implies (QN-3).

Note 2.4. Clearly from (QN-3) and (QN-4) we get, for any $\alpha \in (0,\ 1]$ and for any $n \in Z^+$, there exists $\beta \in (0,\ \alpha]$ such that $|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta\ \forall\ x_i \in\ X(i = 1,\ 2, ...,\ n)$

Definition 2.5. [12] Let $(X,\ Q)$ be a generating space of quasi-norm family.
(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_\alpha = 0$ for each $\alpha \in (0,\ 1]$;
(b) to be a Cauchy sequence if $\lim_{m,n\to\infty} |x_n - x_m|_\alpha = 0$ for each $\alpha \in (0,\ 1]$.
(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

Note 2.6. In a generating space of quasi-norm family,
(a) every convergent sequence is a Cauchy sequence;
(b) limit of a sequence if it exists is unique;
(c) if $\{x_n\}$ is a convergent sequence which converges to x in X, then any subsequence $\{x_{n_k}\}$ also converges to x.

Lemma 2.7. [12] *Let $(X,\ Q)$ be a generating space of strong quasi-norm family. Then for each $\alpha \in (0,\ 1]$, $|x|_\alpha$ is a continuous function on X.*

Lemma 2.8. [8](Riesz) *Let Y and Z be subspaces of a normed linear space $(X,\ ||.||)$ and suppose that Y is a closed and proper subset of Z. Then for every real number $\theta \in (0,\ 1)$ there exist a $z \in\ Z$ such that $||z|| = 1$ and $||z-y|| > \theta\ \forall\ y \in\ Y$.*

Definition 2.9. [1] Let X be a linear space over the field F(real or complex). A fuzzy subset N on $X \times R$ (R-set of all real numbers) is called a fuzzy norm on X if and only if for $x,\ y \in X$ and $c\ \in F$
(N1) $\forall t \in R$ with $t \leq 0,\ N(x,\ t) = 0$
(N2) $(\forall t \in R,\ t > 0,\ N(x,\ t) = 1)$ iff $x = \theta$.
(N3) $\forall t \in R,\ t > 0,\ N(cx,\ t) = N(x,\ \frac{t}{|c|})$ if $c\ \neq 0$.
(N4) $\forall s,\ t \in R,\ x,\ y\ \in X,\ N(x+y,\ t+s\ ) \geq\ min\{N(x,\ t),\ N(y,\ s)\}$
(N5) $N(x,\ .)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,\ t) = 1$.
The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.10. [1] Let $(X,\ N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if $\exists\ x\ \in X$ such that
$\lim_{n\to\infty} N(x_n - x,\ t) = 1\ \forall t > 0$.
Then x is called the limit of the sequence $\{x_n\}$ and we denote it by $lim\ x_n$.

Definition 2.11. [1] A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if $\lim_{n\to\infty} N(x_{n+p} - x_n,\ t) = 1\ \forall t > 0$ and p=1, 2, 3,.............

Definition 2.12. [1] A fuzzy normed linear spacs $(X,\ N)$ is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.13. [7] A binary operation $* : [0,\ 1] \times [0,\ 1] \to\ [0,\ 1]$ is called t-norm if the following axioms are satisfied for all $a,\ b,\ d \in [0,\ 1]$:
(T1) $a * 1 = a$ (boundary condition);
(T2) $b \leq\ d$ implies $a * b \leq\ a * d$ (monotonicity);
(T3) $a * b = b * a$ (commutativity);
(T4) $a * (b * d) = (a * b) * d$ (associativity).

Definition 2.14. The binary operation $*$ is said to be continuous if for any sequences $\{a_n\},\ \{b_n\}$ in [0, 1] with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ implies $\lim_{n\to\infty} (a_n * b_n) = (a * b)$.

If $*$ is continuous then it is called continuous t-norm. The following are examples of some t-norms that are frequently used as fuzzy intersections defined for all $a,\ b \in\ [0,\ 1]$.
(i) Standard intersection: $a * b = min(a,\ b)$.

(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = max(0,\ a + b - 1)$.
(iv) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

The relations among the t-norms are $a{*}b$ (Drastic) $\leq\ max(0,\ a{+}b{-}1) \leq\ ab \leq\ min(a,\ b)$ and Drastic t-norm $\leq$ any t-norm $\leq$ minimum t-norm.

3. Some Results on Finite Dimensional Generating Spaces of Quasi-Norm Family

In this section, we give some examples and study some results on finite dimensional generating spaces of quasi-norm family.

Example 3.1. Let $(X,\ ||.||)$ be a normed linear space over a field F(real or complex). Define

$$|x|_\alpha = \frac{(1-\alpha)}{(1+\alpha)}||x||\ \forall \alpha \in (0,\ 1].$$

Then

$$Q = \{|.|_\alpha : \alpha \in (0,\ 1]\}$$

is a quasi-norm family and $(X,\ Q)$ is a G.S.Q.N.F.

Example 3.2. Let $X = R^2$ be a linear space. For $x = (x_1,\ x_2) \in\ X$ define

$$|x|_\alpha = \frac{1}{\alpha}(\sqrt{|x_1|} + \sqrt{|x_2|})^2\ \forall \alpha \in (0,\ 1].$$

Then

$$Q = \{|.|_\alpha : \alpha \in (0,\ 1]\}$$

is a quasi-norm family and $(X,\ Q)$ is a G.S.Q.N.F.

Proof. Conditions (QN-1), (QN-2) and (QN-4) directly follow from definition. For (QN-3), let $x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in\ X,\ \alpha \in (0,\ 1]$ then

$2(|x|_\alpha + |y|_\alpha) - |x+y|_\alpha\ = \frac{2}{\alpha}(\sqrt{|x_1|} + \sqrt{|x_2|})^2 + \frac{2}{\alpha}(\sqrt{|y_1|} + \sqrt{|y_2|})^2 - \frac{1}{\alpha}(\sqrt{|x_1+y_1|} + \sqrt{|x_2+y_2|})^2\ = \frac{2}{\alpha}\{|x_1| + |x_2| + |y_1| + |y_2| + 2\sqrt{|x_1||x_2|} + 2\sqrt{|y_1||y_2|}\} - \frac{1}{\alpha}\{|x_1+y_1| + |x_2+y_2| + 2\sqrt{|x_1+y_1||x_2+y_2|}\}\ \geq \frac{1}{\alpha}\{2|x_1| + 2|x_2| + 2|y_1| + 2|y_2| + 4\sqrt{|x_1||x_2|} + 4\sqrt{|y_1||y_2|} - |x_1| - |y_1| - |x_2| - |y_2| - 2\sqrt{(|x_1| + |x_2|)(|y_1| + |y_2|)}\}\ = \frac{1}{\alpha}\{(|x_1| + |x_2|) + (|y_1| + |y_2|) - 2\sqrt{(|x_1| + |x_2|)(|y_1| + |y_2|)} + 4\sqrt{|x_1||x_2|} + 4\sqrt{|y_1||y_2|}\}\ = \frac{1}{\alpha}\{(\sqrt{(|x_1| + |x_2|)} - \sqrt{(|y_1| + |y_2|)})^2 + 4\sqrt{|x_1||x_2|} + 4\sqrt{|y_1||y_2|}\}\ \geq\ 0.$

Thus $|x+y|_\alpha \leq\ 2\{|x|_\alpha + |y|_\alpha\}\ \forall \alpha \in (0,\ 1]$.

Now if we put $\beta = \frac{\alpha}{2} \in (0,\ \alpha)$ then $|x+y|_\alpha \leq\ |x|_\beta + |y|_\beta$. Hence the results are proved. □

Remark 3.3. In Example 3.2, $Q = \{|.|_\alpha : \alpha \in (0,\ 1]\}$ is a quasi-norm family but $|.|_\alpha$ is not a norm for any $\alpha \in (0,\ 1]$, because if we take $x = (1,\ 0),\ y = (0,\ 1)$ then

$$|x|_\alpha = \frac{1}{\alpha},\ |y|_\alpha = \frac{1}{\alpha}$$

but $|x + y|_\alpha = \frac{4}{\alpha}$. So the triangle inequality

$$|x + y|_\alpha \leq\ |x|_\alpha + |y|_\alpha\ \forall\ x,\ y \in\ X$$

is not satisfied for any $\alpha \in (0,\ 1]$.

Definition 3.4. Let (X, Q) be a generating space of quasi-norm family.
(a) A subset F of X is said to be closed if for any sequence $\{x_n\}$ in F with $lim\ x_n = x$ implies $x \in\ F$;

(b) A subset A of X is said to be compact if any sequence $\{x_n\}$ in A has a subsequence converging to some point in A;

(c) A subset A of X is said to be bounded if for each $\alpha \in (0,\ 1]$ there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha)\ \forall\ x \in\ A$.

Note 3.5. The underlying topology of the generating space of quasi-norm family (X, Q) is the topology τ_Q mentioned by Xiao & Zhu in Lemma 2.3[13], with respect to which the family $\{U(\epsilon,\ \alpha) :\ \epsilon > 0,\ \alpha \in\ (0,\ 1]\}$, where $U((\epsilon,\ \alpha) = \{x :\ |x|_\alpha < \epsilon\}$, forms a neighborhood base of θ. The topological space $(X,\ \tau_Q)$ is first countable Hausdorff and linear. Therefore it is metrizable so that the topology τ_Q can be derived using sequences instead of nets or filters.

Lemma 3.6. *Let $\{x_1,\ x_2,\ x_3, ...,\ x_n\}$ be a linearly independent set of vectors in a generating space (X, Q) of quasi-norm family. Then $\exists\ C > 0$ and $\exists\ \alpha \in (0,\ 1]$ such that for any choice of scalars $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ we have*

$$|\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n|_\alpha \geq\ C(|\lambda_1| + |\lambda_2| + ... + |\lambda_n|)$$

Proof. Let $s = |\lambda_1| + |\lambda_2| + ... + |\lambda_n|$. If $s = 0$, then the above condition is holds for every $C > 0$.
Let $s > 0$. Then the problem is equivalent to the existence of some $C > 0$ and some $\alpha \in (0,\ 1]$ such that for any scalars $\mu_1,\ \mu_2, ...,\ \mu_n$ with $\sum_{i=1}^{n} |\mu_i| = 1$
such that $|\mu_1 x_1 + \mu_2 x_2 + ...\mu_n x_n|_\alpha \geq\ C$ (1)

If possible, suppose (1) is not true. Then there exists a sequence $\{y_m = \sum_{i=1}^{n} \mu_i^{(m)} x_i\}$
with $\sum_{i=1}^{n} |\mu_i^{(m)}| = 1$ such that $|y_m|_\alpha \to\ 0$ as $m \to \infty\ \forall \alpha \in (0,\ 1]$
i.e. $\{y_m\} \to\ \theta$ (2)

Since $\sum_{i=1}^{n} |\mu_i^{(m)}| = 1$ we have $|\mu_i^{(m)}| \leq\ 1$ for $i = 1,\ 2,\ 3, ...,\ n$.

Thus for each $i(= 1,\ 2, ...,\ n)$, the sequence $\{\mu_i^{(m)}\}$ is bounded and by Bolzano-Weierstrass theorem, $\{\mu_i^{(m)}\}$ has a convergent subsequence. Let μ_1 denote the limit of the subsequence $\{\delta_1^{(m)}\}$ of $\{\mu_1^{(m)}\}$ and $\{y_{1,\ m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,\ m}\}$ for which the corresponding subsequence $\{\delta_2^{(m)}\}$ of scalar $\{\mu_2^{(m)}\}$ converges, let μ_2 denote the limit. Continuing in this way after n-steps we obtain a subsequence $\{y_{n,\ m}\}$ of $\{y_m\}$ and subsequences $\gamma_i^{(m)}$ of $\{\mu_i^{(m)}\}$ $(\forall i = 1,\ 2, ...,\ n)$ such that

$$y_{n,\ m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i \text{ with } \sum_{i=1}^{n} |\gamma_i^{(m)}| = 1 \text{ and } \gamma_i^{(m)} \to \mu_i \text{ as } m \to \infty.$$

$$\text{Let } y = \sum_{i=1}^{n} \mu_i x_i \text{ with } \sum_{i=1}^{n} |\mu_i| = 1.$$

By Note 1.1, $|y_{n,\ m} - y|_\alpha = |\sum_{i=1}^{n} \gamma_i^{(m)} x_i - \sum_{i=1}^{n} \mu_i x_i|_\alpha = |\sum_{i=1}^{n} (\gamma_i^{(m)} - \mu_i) x_i|_\alpha \leq \sum_{i=1}^{n} |\gamma_i^{(m)} - \mu_i|\ |x_i|_\beta$ for some $\beta \in\ (0,\ \alpha]$.

Thus $y_{n,\ m} \to\ y$. Since all μ_i are not zero and $\{x_1,\ x_2, ...,\ x_n\}$ is a linearly independent set, we have $y \neq \theta$. On the other hand from (2), $\{y_{n,\ m}\} \to\ \theta$. This is a contradiction. This proves the Lemma. □

Theorem 3.7. *Let $(X,\ Q)$ be a generating space of quasi-norm family. If X be finite dimensional then X is complete.*

Proof. Let $\{y_m\}$ be a Cauchy sequence in X.
Let dim $X = n$ and $\{e_1,\ e_2, ...,\ e_n\}$ be a basis of X. Then there exists scalars $\lambda_1^{(m)},\ \lambda_2^{(m)},\ \lambda_3^{(m)}, ...,\ \lambda_n^{(m)}$ such that

$$y_m = \lambda_1^{(m)} e_1 + \lambda_2^{(m)} e_2 + ... + \lambda_n^{(m)} e_n$$

Since $\{y_m\}$ is a Cauchy sequence, so for each $\alpha \in (0, 1]$, there exists $N(\alpha)$ such that

$$|y_m - y_r|_\alpha < \alpha\ \forall\ m,\ r \geq\ N(\alpha).$$

From Lemma 3.6, $\exists\ C > 0$ and $\alpha_0 \in (0,\ 1]$ such that
$\alpha_0 > |y_m - y_r|_{\alpha_0} = |\sum_{i=1}^{n} (\lambda_i^m - \lambda_i^r) e_i|_{\alpha_0} \geq\ C \sum_{i=1}^{n} |\lambda_i^m - \lambda_i^r|\ \forall\ m,\ r \geq\ N(\alpha_0)$
$\Rightarrow \sum_{i=1}^{n} |\lambda_i^m - \lambda_i^r| < \frac{\alpha_0}{C}\ \forall\ m,\ r \geq\ N(\alpha_0) \Rightarrow\ \{\lambda_i^m\}$ are bounded sequences in R or C for each $i = 1,\ 2, ...,\ n$ and by Bolzano-Weierstrass theorem each $\{\lambda_i^{(m)}\}$ has a convergent subsequence converges to some point λ_i. We define $y = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$. Then $y \in\ X$. As in the proof of Lemma 3.6 we conclude that $\{y_m\}$ has a subsequence $\{y_{m,\ n}\}$ which converges to y.
By (QN-3), for each $\alpha \in (0,\ 1]$ there exists $\beta \in (0,\ \alpha]$ such that

$$|y_m - y|_\alpha = |y_m - y_{m,\ n} + y_{m,\ n} - y|_\alpha \leq\ |y_m - y_{m,\ n}|_\beta + |y_m - y|_\beta.$$

$$\Rightarrow \lim_{m\to\infty}\ |y_m - y|_\alpha = 0\ \forall\alpha \in (0,\ 1].$$

$$\Rightarrow\ y_m \to\ y.$$

Hence X is complete. □

Remark 3.8. Any finite dimensional subspace of a generating space of quasi-norm family is closed.

Let $(X\ ,\ Q)$ be a generating space of quasi-norm family. We assume that

(QN-5) $\bigvee_{\alpha\in(0,1]} < \infty\ \forall x \in X.$

Theorem 3.9. *Let $(X,\ Q)$ be a generating space of quasi-norm family satisfying the condition (QN-5). Define $||x||_Q = \bigvee_{\alpha\in(0,\ 1]} |x|_\alpha\ \forall\ x \in\ X$.*

Then $(X,\ ||.||_Q)$ is a normed linear space.

Proof. (N1) Let $x = \theta$, then $|x|_\alpha = 0\ \forall\alpha \in (0,\ 1]\ \Rightarrow\ ||x||_Q = 0$

Conversely let $||x||_Q = 0$, then $\bigvee_{\alpha\in(0,\ 1]} |x|_\alpha = 0,$

$$\Rightarrow\ |x|_\alpha = 0\ \forall\alpha \in (0,\ 1]\ \Rightarrow\ x = \theta.$$

(N2) Clearly $||\lambda\ x||_Q = |\lambda|||x||_Q$ holds.

(N3) Let $\alpha \in (0,\ 1]$ then $\exists\ \beta \in (0,\ \alpha]$ such that

$$|x + y|_\alpha \leq |x|_\beta + |y|_\beta\ \forall\ x,\ y \in X;$$

$$\Rightarrow\ |x + y|_\alpha \leq \bigvee_{\beta\leq\alpha} |x|_\beta + \bigvee_{\beta\leq\alpha} |y|_\beta;$$

$$\Rightarrow\ \bigvee_{\beta\leq\alpha} |x + y|_\beta \leq \bigvee_{\beta\leq\alpha} |x|_\beta + \bigvee_{\beta\leq\alpha} |y|_\beta;$$

$$\Rightarrow\ \bigvee_{\beta\in(0,\ 1]} |x + y|_\beta \leq \bigvee_{\beta\in(0,\ 1]} |x|_\beta + \bigvee_{\beta\in(0,\ 1]} |y|_\beta \text{ by (QN-4)};$$

$$\Rightarrow\ ||x + y||_Q \leq ||x||_Q + ||y||_Q.$$

Thus $(X,\ ||.||_Q)$ is a normed linear space.

$||.||_Q$ is called the norm induced by the family of quasi-norm Q. □

Example 3.10. Let $(X,\ ||.||)$ be a normed linear space. Let for $x \in\ X$

$$|x|_\alpha = \begin{cases} ||x|| & \text{for } \alpha \in (0,\ 0.5] \\ \frac{(1-\alpha)}{(1+\alpha)}||x|| & \text{for } \alpha \in (0.5,\ 1] \end{cases}$$

Then

$Q =\ \{|.|_\alpha : \alpha \in (0,\ 1]\}$ is a quasi norm family and $(X,\ Q)$ is a G.S.Q.N.F and $||x||_Q\ =\ ||x||\ \forall\ x \in\ X$.

Theorem 3.11. *Let $(X,\ Q)$ be a generating space of quasi-norm family satisfying (QN-5). If $A \subset\ X$ is closed in $(X,\ Q)$ then it is closed in $(X,\ ||.||_Q)$.*

Proof. Let $\{x_n\}$ be any sequence in A such that $\lim\limits_{n\to\infty} ||x_n - x||_Q = 0$ for some $x \in\ X$.
Then corresponding to any $\epsilon > 0$ $\$thereexistsanaturalnumber\mathrm{N}(\epsilon)suchthat$

$$||x_n - x||_Q\ < \epsilon \text{ whenever } n \geq\ N(\epsilon);$$

$$\Rightarrow \bigvee_{\alpha\in(0,\ 1]} |x_n - x|_\alpha < \epsilon \text{ whenever } n \geq\ N(\epsilon);$$

$$\Rightarrow\ |x_n - x|_\alpha < \epsilon\ \forall\alpha \in (0,\ 1] \text{ whenever } n \geq\ N(\epsilon);$$

$$\Rightarrow \lim_{n\to\infty} |x_n - x|_\alpha < \epsilon\ \forall\alpha \in (0,\ 1].$$

Since ϵ is arbitrary,

$$\Rightarrow \lim_{n\to\infty} |x_n - x|_\alpha = 0\ \forall\alpha \in (0,\ 1].$$

Since $A \subset\ X$ is closed in $(X,\ Q)$, so $x \in\ A$. Thus A is closed in $(X,\ ||.||_Q)$. □

Lemma 3.12. *(Riesz) Let Y and Z be subspaces of a generating space of quasi-norm family $(X,\ Q)$ satisfying (QN-5) and suppose that Y is a closed and proper subset of Z. Then for every real number $\theta \in (0,\ 1)$ there exist $z \in\ Z$ such that $||z||_Q = 1$ and $||z - y||_Q > \theta\ \forall\ y \in\ Y$.*

Proof. Proof follows from Theorem 3.9, Theorem 3.11 and Lemma 2.8. □

Theorem 3.13. *Let $(X,\ Q)$ be a generating space of quasi-norm family satisfying (QN-5). If $M = \{x \in\ X :\ ||x||_Q = 1\}$ is compact, then X is finite dimensional.*

Proof. It is obvious. □

Definition 3.14. In a linear space X, two quasi-norm families Q_1 and Q_2 are said to be equivalent if there exist $\alpha_0,\ \beta_0 \in (0,\ 1]$ and for each $\alpha \in (0,\ 1]$ there exist $a_\alpha,\ b_\alpha\ >\ 0$ such that
$|x|^1_\alpha \leq\ a_\alpha|x|^2_{\alpha_0}$ and $|x|^2_\alpha \leq\ b_\alpha|x|^1_{\beta_0}\ \forall\ x \in\ X$, where $Q_1 = \{|.|^1_\alpha : \alpha \in (0,\ 1]\}$ and $Q_2 = \{|.|^2_\alpha : \alpha \in (0,\ 1]\}$.

Note 3.15. The relation of equivalent quasi-norm family defines an equivalence on a linear space. Further from Note 3.5, it is evident that equivalent quasi-norm families Q_1 and Q_2 generate equivalent topologies τ_{Q_1} and τ_{Q_2}.

Theorem 3.16. *In a finite dimensional linear space X, any two quasi-norm families are equivalent.*

Proof. Let Q_1 and Q_2 be two quasi-norm families on a finite dimensional linear space X.
Let $Q_1 = \{|.|^1_\alpha : \alpha \in (0,\ 1]\}$ and $Q_2 = \{|.|^2_\alpha : \alpha \in (0,\ 1]\}$.
Let $dim\ X = n$ and $\{e_1,\ e_2, ...,\ e_n\}$ be a basis of X. Then every $x \in\ X$ has a unique representation
$x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$. By Lemma 3.6, $\exists\alpha_0 \in (0,\ 1]$ and $C_2\ >\ 0$ such that

$$|x|^2_{\alpha_0} \geq \ C_2 \ \sum_{i=1}^{n} |\lambda_i| \tag{3}$$

Again for each $\alpha \in (0,\ 1]$, $\exists \beta \in (0,\ \alpha]$ such that

$$|x|^1_\alpha = |\sum_{i=1}^{n} \lambda_i e_i|^1_\alpha \leq \ \sum_{i=1}^{n} |\lambda_i e_i|^1_{\beta(\alpha)} = \sum_{i=1}^{n} |\lambda_i||e_i|^1_{\beta(\alpha)}.$$

Let $a'_\alpha = \ max_{1\leq\ i\leq\ n}|e_i|^1_{\beta(\alpha)}$, then we have

$$|x|^1_\alpha \leq \ a'_\alpha \sum_{i=1}^{n} |\lambda_i| \tag{4}$$

From (3) and (4) we get $|x|^1_\alpha \leq \ \frac{a'_\alpha}{C_2}|x|^2_{\alpha_0}\ \forall \alpha \in (0,\ 1]$.

$$\text{i.e. } |x|^1_\alpha \leq \ a_\alpha |x|^2_{\alpha_0}\ \forall \alpha \in (0,\ 1].$$

Interchanging the role of $|.|^1_\alpha$ and $|.|^2_\alpha$ we can prove that there exists $\beta_0 \in (0,\ 1]$ and for each $\alpha \in (0,\ 1]$ there is $b_\alpha\ >\ 0$ such that $|x|^2_\alpha \leq\ b_\alpha |x|^1_{\beta_0}\ \forall\ x \in\ X$. □

Lemma 3.17. *Any compact subset M of a generating space of strong quasi-norm family $(X,\ Q)$ is closed and bounded.*

Proof. Let $\{x_n\}$ be a sequence in M such that $lim\ x_n = x$. Since M is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point in M. But $lim\ x_n = x$ implies $lim\ x_{n_k} = x$ and hence $x \in\ M$. So M is closed.
If possible, let M be unbounded. Then there exists $\alpha_0\ \in\ (0,\ 1]$ and for each $n \in\ N$, $x_n \in\ M$ such that $|x_n|_{\alpha_0} \geq\ n$. So $\{x_n\}$ is a sequence in M. Since M is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $lim\ x_{n_k} = x \in\ M$.
Thus $|x_{n_k}|_{\alpha_0} \geq\ n_k\ \Rightarrow\ |x|_{\alpha_0} \geq\ \infty$ [By Lemma 2.7],
which is a contradiction. Hence M is bounded. □

Theorem 3.18. *In a finite dimensional generating space of strong quasi-norm family $(X,\ Q)$, any subset M of X is compact iff M is closed and bounded.*

Proof. First, we suppose that M is compact. Then by Lemma 3.17, it follows that M is closed and bounded.
Next we suppose that M is closed and bounded. Let $dimX = n$ and $\{e_1,\ e_2, ..., \ e_n\}$ be a basis for X. We consider any sequence $\{x_m\}$ in M. Then for each $m \in\ N$, x_m has a unique representation

$$x_m = \xi_1^{(m)} e_1 + \xi_2^{(m)} e_2 + ... + \xi_n^{(m)} e_n \text{ (say).}$$

Since M is bounded, for each $\alpha \in (0,\ 1]$ there exists $M(\alpha) > 0$ such that

$$|x_m|_\alpha \leq\ M(\alpha)\ \forall\ m \tag{5}$$

By Lemma 3.6, $\exists\ \alpha_0 \in (0,\ 1]$ and $C\ >\ 0$ such that

$$|x_m|_{\alpha_0} \geq\ C \sum_{i=1}^{n} |\xi_i^{(m)}|.$$

From (5) we have $M(\alpha_0) \geq\ |x_m|_{\alpha_0} \geq\ C \sum_{i=1}^{n} |\xi_i^{(m)}|$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i^{(m)}| \leq \frac{M(\alpha_0)}{C} \tag{6}$$

From (6), we see that $\{\xi_i^{(m)}\}$ is a bounded sequence of scalars for each $i = 1, 2, ..., n$ and by Bolzano-Weierstrass theorem each $\{\xi_i^{(m)}\}$ has a subsequence $\{\xi_i^{(m_i)}\}$ which converges to some point ξ_i. As in the proof of Lemma 3.6 we conclude that $\{x_m\}$ has a subsequence $\{z_m\}$ which converges to $z = \sum_{i=1}^{n} \xi_i e_i$. Since M is closed, $z \in M$. This shows that the arbitrary sequence $\{x_m\}$ in M has a subsequence which converges in M. Hence M is compact. □

4. Generalized Fuzzy Normed Linear Space and a Decomposition Theorem

In this Section we generalize the fuzzy norm introduced by Bag & Samanta [1] and deduce a G.S.Q.N.F from it.

Definition 4.1. Let X be a linear space over the field F (real or complex) and * be a continuous t-norm. A fuzzy subset N on $X \times R$ (R-set of all real numbers) is called a fuzzy norm on X if and only if for $x,\ y \in X$ and $c\ \in F$

(N1) $\forall t \in R$ with $t \leq 0,\ N(x,\ t) = 0$
(N2) $(\forall t \in R\ ,\ t > 0,\ N(x,\ t) = 1)$ iff $x = \theta$.
(N3) $\forall t \in R\ ,\ t > 0,\ N(cx,\ t) = N(x,\ \frac{t}{|c|})$ if $c\ \neq 0$.
(N4) $\forall s\ ,\ t \in R,\ x,\ y\ \in X,\ N(x + y\ , t + s\ ) \geq\ N(x,\ t) * N(y,\ s)$.
(N5) $\lim_{t\to\infty} N(x,\ t) = 1$.
The triplet $(X,\ N,\ *)$ will be referred to as a fuzzy normed linear space.

Note 4.2. Since $min \geq\ *$ ($*$ is a continuous t-norm), it is easy to see that if $(X,\ N)$ is a fuzzy normed linear space in the sense of Definition 2.9, then $(X,\ N,\ *)$ is also a fuzzy normed linear space in the sense of Definition 4.1 (i.e., a generalized fuzzy normed linear space).

Remark 4.3. From (N2) and (N4) it is clear that $N(x,t)$ is nondecreasing with respect to t.

Note 4.4. From (N4) it is clear that
$N(x_1 + x_2 + ... + x_n\ ,\ t_1 + t_2 + ... + t_n)\ \geq\ N(x_1,t_1) * N(x_2,\ t_2) * N(x_3,\ t_3) * .... * N(x_n,\ t_n)$.

Example 4.5. Let X be a linear space over a field F and let $N : X \times R \to [0,\ 1]$ be defined by

$$N(x,\ t) = \begin{cases} \frac{t-||x||}{t+||x||} & \text{for } t > ||x|| \\ 0 & \text{for } t \leq\ ||x||. \end{cases}$$

Then $(X,\ N,\ *)$ is a fuzzy normed linear space for any continuous t-norm $*$.

Example 4.6. Let $(X,\ ||.||)$ be a normed linear space over the field F(real or complex) and let $N : X \times R \to [0,\ 1]$ be defined by

$$N(x,\ t) = \begin{cases} 0 & \text{for } t\ \leq ||x|| \\ 1 & \text{for } t >\ ||x|| \end{cases}$$

Then N is a fuzzy norm on X and $(X,\ N,\ *)$ is a fuzzy normed linear space for any continuous t-norm $*$.

Example 4.7. Let $(X = R^2)$ be a linear space over the field R (the set of real numbers),
$x = (x_1,\ x_2) \in\ X$ and $N : X \times R \to [0,\ 1]$ be defined by

$$N(x,\ t) = \begin{cases} \frac{t^2}{(t+|x_1|)(t+|x_2|)} & \text{for } t\ > 0 \\ 0 & \text{for } t \leq\ 0 \end{cases}$$

Then $(X,\ N,\ *)$ is a fuzzy normed linear space in the sense of Definition 4.1 for the continuous t-norm $*$ defined by $a * b = ab\ \forall\ a,\ b, \in\ [0\ 1]$, but $(X,\ N)$ is not a fuzzy normed linear space in the sense of Definition 2.9.

Proof. First we show that $(X,\ N)$ is not a fuzzy normed linear space in the sense of Definition 2.9. Let $x = (1,\ 0),\ y = (0,\ 1),\ t = s = 1$ then

$$N(x,\ t) = \tfrac{1}{2} = N(y,\ s) \text{ and } N(x + y,\ t + s) = \tfrac{4}{9}.$$

Thus $N(x + y,\ t + s) < min\{N(x,\ t),\ N(y,\ s)\}$.
So N does not satisfy the condition $(N4)$ in the sense of Definition 2.9. Thus $(X,\ N)$ is not a fuzzy normed linear space in this sense.

Next we show that $(X,\ N,\ *)$ is a fuzzy normed linear space in the sense of Definition 4.1 for the continuous t-norm $*$ defined by $a * b = ab\ \forall\ a,\ b \in\ [0\ 1]$.

(N1) $\forall t \in R$ with $t \leq 0,\ N(x,\ t) = 0$ (from definition).

(N2) $\forall t \in R,\ t >\ 0,\ N(x,\ t) = 1$
$\Rightarrow\ \frac{t^2}{(t+|x_1|)(t+|x_2|)} = 1$
$\Rightarrow\ t^2 = t^2 + t(|x_1| + |x_2|) + |x_1x_2|$
$\Rightarrow\ t(|x_1| + |x_2|) + |x_1x_2| = 0\ \forall\ t > 0$
$\Rightarrow\ |x_1| = 0,\ |x_2| = 0$ i.e. $x = \theta$.

Conversely, if $x = \theta$, then $N(x,\ t) = 1\ \forall t \in R,\ t >\ 0$ (from definition).

(N3) $\forall\ t \in R,\ t >\ 0,\ N(cx,\ t) = \frac{t^2}{(t+|cx_1|)(t+|cx_2|)}\ = \frac{(\frac{t}{c})^2}{(\frac{t}{|c|}+|x_1|)(\frac{t}{|c|}+|x_2|)}$
$= N(x,\ \frac{t}{|c|})$ if $c \neq\ 0$.

(N4) Let $x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in\ X$ and $t,\ s > 0$. Note that
$N(x + y,\ t + s) = \frac{(t+s)^2}{(t+s+|x_1+y_1|)(t+s+|x_2+y_2|)}\ \geq\ \frac{(t+s)^2}{((t+s)+|x_1|+|y_1|)((t+s)+|x_2|+|y_2|)}$ and
$N(x,\ t) * N(y,\ s) = \frac{t^2s^2}{(t+|x_1|)(t+|x_2|)(s+|y_1|)(s+|y_2|)}$,
and it is not difficult to verify that

$(t+s)^2(t+|x_1|)(t+|x_2|)(s+|y_1|)(s+|y_2|) \geq t^2s^2((t+s)+|x_1|+|y_1|)((t+s)+|x_2|+|y_2|)$.

Hence N(4) holds.

(N5) $\lim_{t\to\infty} N(x,\ t) = \lim_{t\to\infty} \dfrac{t^2}{(t+|x_1|)(t+|x_2|)} = 1.$

Hence proved. □

Note 4.8. Definitions of Convergent sequences and Cauchy sequences are the same as in Definition 2.10[1] and Definition 2.11[1] respectively.

Proposition 4.9. *The limit of a sequence in a fuzzy normed linear space* $(X,\ N,\ *)$, *if exists, is unique.*

Proposition 4.10. *Every convergent sequence is a Cauchy sequence.*

Proposition 4.11. *Let* $(X,\ N,\ *)$ *be a fuzzy normed linear space and* $\{x_n\}$ *be a convergent sequence which converges to* x *in* X*. Then any subsequence* $\{x_{n_k}\}$ *also converges to* x*.*

Theorem 4.12. *Let* $(X,\ N,\ *)$ *be a fuzzy normed linear space. We assume that* $N(x,\ .)$ *is continuous, as well as strictly increasing for all* $t > 0$*. For* $\alpha \in (0,\ 1]$ *we define*

$|x|_\alpha = \bigwedge\{t > 0\ :\ N(x,\ t) \geq 1-\alpha\}$

and

$$Q = \{|.|_\alpha : \alpha \in (0,\ 1]\}.$$

Then $(X,\ Q)$ *is a generating space of quasi-norm family such that* $|.|_\alpha$ *is continuous for* $\alpha \in\ (0,\ 1]$*.*

Example 4.13. In Theorem 4.12, if we take $a * b = ab$ then Q=$\{|.|_\alpha : \alpha \in (0,\ 1]\}$. is a quasi-norm family and $(X,\ Q)$ is a generating space of quasi-norm family with $\beta = 1 - \sqrt{1-\alpha}$.

Note 4.14. Let $(X,\ N,\ *)$ be a fuzzy norm linear space. Then $|x|_\alpha$ is lower semi-continuous for $\alpha \in (0,\ 1]$.

For some $\alpha_0 \in (0,\ 1],\ t_0 \in [0,\ \infty)$, $|x|_{\alpha_0} > t_0$

$\Rightarrow\ N(x,\ t_0) < 1 - \alpha_0$

$\Rightarrow\ \exists\ \epsilon > 0$ such that $N(x,\ t_0) < 1 - \alpha_0 - \epsilon\ < 1 - \alpha_0$

$\Rightarrow\ |x|_\alpha \geq\ t_0\ \forall \alpha \in (\alpha_0 - \epsilon,\ \alpha_0 + \epsilon) \cap (0,\ 1]$.

$\Rightarrow\ |x|_{(.)}$ is lower semi-continuous.

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