

FURTHER STUDY ON L -FUZZY Q-CONVERGENCE STRUCTURES

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ABSTRACT. In this paper, we discuss the equivalent conditions of pretopological and topological L -fuzzy Q-convergence structures and define T_0 , T_1 , T_2 -separation axioms in L -fuzzy Q-convergence space. Furthermore, L -ordered Q-convergence structure is introduced and its relation with L -fuzzy Q-convergence structure is studied in a categorical sense.

1. Introduction

Since Fischer [8] first introduced convergence structure based on filters in classical sense, the theory developed rapidly. With the development of fuzzy set theory, the classical filter has been generalized to different kinds of fuzzy filters [2, 9, 10, 12, 25]. Based on fuzzy filters, many researchers extended convergence structures to fuzzy convergence structures and studied the relations between fuzzy topological spaces and fuzzy convergence spaces. Using prefilters, Lowen [21, 22] considered fuzzy convergence structures as a generalization of classical convergence structures. Lee et al. [17, 18, 19] also introduced a kind of fuzzy convergence structure and studied its separation. Yao [29] proposed the concept of L -fuzzifying convergence structure which was built on L -filters of ordinary subsets and discussed the relations between L -fuzzifying topological spaces and L -fuzzifying convergence spaces. In [13], Jäger gave the concept of stratified L -fuzzy convergence structure based on stratified L -filters and showed the resulting category has many good categorical properties. Güloğlu and Coker [11] introduced the notion of I -fuzzy convergence structure by means of I -filters which converged to fuzzy points, and proved that this kind of convergence structures and I -fuzzy topologies are one-to-one corresponding. In [24], Pang and Fang introduced L -fuzzy Q-convergence structures and proved that the category of L -fuzzy topological spaces and that of topological L -fuzzy Q-convergence spaces were isomorphic. There are also some relating works in [14, 26, 27, 28].

Many researchers also studied properties of fuzzy convergence spaces. In [15], Jäger gave equivalent conditions of his pretopological and topological stratified L -fuzzy convergence structures. Lee [18, 19], Minkler [23] and Jäger [16] all discussed separation axioms in their fuzzy convergence spaces. Fang [6, 7] introduced the

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concept of stratified L -ordered convergence structure and studied its properties. The aim of this paper is to consider these problems in L -fuzzy Q-convergence spaces.

The structure of this paper is as follows. In Section 2, we state some preliminary concepts and their properties. In Section 3, some characterizations of pretopological and topological L -fuzzy Q-convergence structures are given. In Section 4, we define some separation axioms in L -fuzzy Q-convergence spaces and study their properties. In Section 5, a new kind of convergence structure, named L -ordered Q-convergence structure, is introduced. Moreover, the relations between L -ordered Q-convergence structures and L -fuzzy Q-convergence structures are studied in a categorical sense.

2. Preliminaries

Let $(L, \vee, \wedge, ')$ be a completely distributive De Morgan algebra. $M(L)$ denotes the set of all non-zero coprimes in L . The smallest element and the largest element in L are denoted by 0 and 1, respectively. For $a, b \in L$, we say “ a is wedge below b ” in symbol $a \triangleleft b$ if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. We denote $\beta(a) = \{b \mid b \triangleleft a\}$. Thus $a = \bigvee \beta(a)$ holds for each $a \in L$.

For a nonempty set X , L^X denotes the set of all L -fuzzy subsets on X . The smallest element and the largest element in L^X are denoted by 0_X and 1_X , respectively. L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way, by defining \vee , \wedge , \leq and $'$ point-wise. The set of non-zero coprimes in L^X is denoted by $pt(L^X)$. It is easy to see that $pt(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in M(L)$). We say that x_λ quasi-coincides with A , denoted by $x_\lambda \hat{q}A$, if $\lambda \not\leq A'(x)$ or equivalently $x_\lambda \not\leq A'$. The relation “does not quasi-coincide with” is denoted by $-\hat{q}$. We define a residual implication operation $\rightarrow: L \times L \rightarrow L$ as the right adjoint for the operation of binary meets \wedge by

$$a \rightarrow b = \bigvee \{c \in L \mid a \wedge c \leq b\}.$$

This operator plays a particular role in the sequel. We list some of its properties.

Lemma 2.1. [12] *Suppose that (L, \vee, \wedge) is a completely distributive lattice and \rightarrow is the implication operation corresponding to \wedge . Then for all $a, b, c, d \in L$, $\{a_j\}_{j \in J}$, $\{b_j\}_{j \in J} \subseteq L$, the following conditions hold:*

- (1) $1 \rightarrow a = a$.
- (2) $a \leq b$ if and only if $a \rightarrow b = 1$.
- (3) $a \rightarrow \bigwedge_{j \in J} a_j = \bigwedge_{j \in J} (a \rightarrow a_j)$.
- (4) $\bigvee_{j \in J} a_j \rightarrow b = \bigwedge_{j \in J} (a_j \rightarrow b)$.
- (5) $\bigwedge_{j \in J} a_j \rightarrow \bigwedge_{j \in J} b_j \geq \bigwedge_{j \in J} (a_j \rightarrow b_j)$.

Lemma 2.2. [6] *Let $\mathcal{S}(-, -)$ be the fuzzy inclusion order of L -subsets, i.e., for any $C, D \in L^X$,*

$$\mathcal{S}(C, D) = \bigwedge_{x \in X} C(x) \rightarrow D(x).$$

Then for all $A, B \in L^X$, $\{A_i\}_{i \in I} \subseteq L^X$, the following statements hold:

- (1) $A \leq B \iff \mathcal{S}(A, B) = 1$.

$$(2) \mathcal{S}\left(\bigvee_{i \in I} A_i, A\right) = \bigwedge_{i \in I} \mathcal{S}(A_i, A).$$

$$(3) \mathcal{S}\left(A, \bigwedge_{i \in I} A_i\right) = \bigwedge_{i \in I} \mathcal{S}(A, A_i).$$

Let $f : X \rightarrow Y$ be a mapping. Define $f^\rightarrow : L^X \rightarrow L^Y$ and $f^\leftarrow : L^Y \rightarrow L^X$ by $f^\rightarrow(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, $f^\leftarrow(B) = B \circ f$ for $B \in L^Y$, respectively. Then the following lemma holds.

Lemma 2.3. [6] *Let $f : X \rightarrow Y$ be a mapping. Then for $A, B \in L^X$, $C, D \in L^Y$, it holds that*

$$\mathcal{S}(A, B) \leq \mathcal{S}(f^\rightarrow(A), f^\rightarrow(B)) \text{ and } \mathcal{S}(C, D) \leq \mathcal{S}(f^\leftarrow(C), f^\leftarrow(D)).$$

Definition 2.4. [12] A mapping $\mathcal{F} : L^X \rightarrow L$ is called an L -filter on X iff for all $A, B \in L^X$,

- (F1) $\mathcal{F}(0_X) = 0, \mathcal{F}(1_X) = 1$;
- (F2) $A \leq B \Rightarrow \mathcal{F}(A) \leq \mathcal{F}(B)$;
- (F3) $\mathcal{F}(A \wedge B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$.

It will be called stratified if it satisfies moreover,

$$(Fs) \alpha \wedge \mathcal{F}(A) \leq \mathcal{F}(\alpha \wedge A).$$

The family of all (stratified) L -filters on X is denoted by $(\mathcal{F}_L^s(X)) \mathcal{F}_L(X)$.

Example 2.5. [24] For each $x_\lambda \in pt(L^X)$, we define $\hat{q}(x_\lambda)$ as follows:

$$\forall A \in L^X, \quad \hat{q}(x_\lambda)(A) = \begin{cases} 1, & x_\lambda \hat{q}A, \\ 0, & x_\lambda \neg \hat{q}A. \end{cases}$$

Then $\hat{q}(x_\lambda)$ is an L -filter.

For any mapping $\mathcal{H} : L^X \rightarrow L$, we define a new mapping $\langle \mathcal{H} \rangle : L^X \rightarrow L$ by

$$\forall A \in L^X, \quad \langle \mathcal{H} \rangle(A) = \bigvee_{B \leq A} \mathcal{H}(B).$$

Then we have the following definition.

Definition 2.6. [3, 4, 5] A mapping $\mathcal{B} : L^X \rightarrow L$ is called an L -filter base on X iff for all $A, B \in L^X$,

- (B1) $\mathcal{B}(0_X) = 0, \bigvee_{A \in L^X} \mathcal{B}(A) = 1$;
- (B2) $\langle \mathcal{B} \rangle(A \wedge B) \geq \mathcal{B}(A) \wedge \mathcal{B}(B)$.

Note that if \mathcal{B} is an L -filter base on X , then $\langle \mathcal{B} \rangle$ is an L -filter and \mathcal{B} is called an L -filter base of $\langle \mathcal{B} \rangle$.

Next we list some definitions and lemmas for L -filters. Since all these results are similar to those for stratified L -filters in [12, 13] and they can be checked easily, we will omit the proof.

On the set $\mathcal{F}_L(X)$ of all L -filters on X , we define an order by $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F}(A) \leq \mathcal{G}(A)$ for all $A \in L^X$. Then we have the following lemmas.

Lemma 2.7. [12] In $(\mathcal{F}_L(X), \leq)$, every nonempty family $\{\mathcal{F}_i\}_{i \in I}$ of L -filters has an infimum $\bigwedge_{i \in I} \mathcal{F}_i$, which can be calculated as

$$\left(\bigwedge_{i \in I} \mathcal{F}_i \right) (A) = \bigwedge_{i \in I} (\mathcal{F}_i(A)), \quad \forall A \in L^X.$$

Lemma 2.8. [12] For a nonempty family $\{\mathcal{F}_i\}_{i \in I}$ of L -filters, the followings are equivalent:

- (1) There exists an L -filter \mathcal{F} such that $\mathcal{F} \geq \mathcal{F}_i$ for all $i \in I$.
- (2) $\mathcal{F}_{i_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{i_n}(A_n) = 0$ if $A_1 \wedge \cdots \wedge A_n = 0_X$ ($n \in \mathbf{N}$, $A_1, \dots, A_n \in L^X$, $\{i_1, \dots, i_n\} \subseteq I$).

In the case of existence, we find

$$\left(\bigvee_{i \in I} \mathcal{F}_i \right) (A) = \bigvee_{n \in \mathbf{N}} \bigvee \{ \mathcal{F}_{i_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{i_n}(A_n) \mid A_1 \wedge \cdots \wedge A_n \leq A \}$$

as the supremum of $\{\mathcal{F}_i\}_{i \in I}$ in $(\mathcal{F}_L(X), \leq)$. We denote it by $\bigvee_{i \in I} \mathcal{F}_i$. For two L -filters \mathcal{F} and \mathcal{G} , we write $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}_L(X)$ if $\mathcal{F} \vee \mathcal{G}$ exists.

Definition 2.9. [12] Let $\mathcal{F} \in \mathcal{F}_L(X)$ and $f : X \rightarrow Y$ be a mapping. Then $f \Rightarrow (\mathcal{F}) : L^Y \rightarrow L$, $A \rightarrow \mathcal{F}(f \leftarrow (A))$ is also an L -filter and is called the image of \mathcal{F} under f .

Lemma 2.10. [13] Let $\mathcal{F} \in \mathcal{F}_L(Y)$ and $f : X \rightarrow Y$ be a mapping. Then the following are equivalent:

- (1) $f \leftarrow (\mathcal{F}) : L^X \rightarrow L$ defined by $f \leftarrow (\mathcal{F})(A) = \bigvee_{f \leftarrow (B) \leq A} \mathcal{F}(B)$ is an L -filter on X .
- (2) $\forall B \in L^Y$, $f \leftarrow (B) = 0_X$ implies $\mathcal{F}(B) = 0$.

In case $f \leftarrow (\mathcal{F}) \in \mathcal{F}_L(X)$, we call $f \leftarrow (\mathcal{F})$ the inverse image of \mathcal{F} under f . Obviously, $f \leftarrow (\mathcal{F})$ exists if f is surjective.

Definition 2.11. [1] (1) A category \mathbf{C} is called a topological category over \mathbf{Set} provided that for any set X , any class J , and family $((X_j, \xi_j))_{j \in J}$ of \mathbf{C} -objects and any family $(f_j : X \rightarrow X_j)_{j \in J}$ of mappings, there exists a unique \mathbf{C} -structure ξ on X which is initial with respect to the source $(f_j : X \rightarrow (X_j, \xi_j))_{j \in J}$. This means that for a \mathbf{C} -object (Y, η) , a mapping $g : (Y, \eta) \rightarrow (X, \xi)$ is a \mathbf{C} -morphism iff for all $j \in J$, $f_j \circ g : (Y, \eta) \rightarrow (X_j, \xi_j)$ is a \mathbf{C} -morphism.

(2) Let \mathbf{B} be a category and E be a class of \mathbf{B} -bimorphisms. A full subcategory \mathbf{A} of \mathbf{B} is called bireflective in \mathbf{B} provided that each \mathbf{B} -object has an \mathbf{A} -reflection arrow in E as a bimorphism. This means that, for any \mathbf{B} -object B , there exists an \mathbf{A} -reflection bimorphism $r : B \rightarrow A$ from B to an \mathbf{A} -object A with the following universal property: for any morphism $f : B \rightarrow A'$ from B into some \mathbf{A} -object A' , there exists a unique \mathbf{A} -morphism $f' : A \rightarrow A'$ such that $f' \circ r = f$.

For more notions related to category theory, we refer to [1].

3. Equivalent Conditions of (Pre)Topological L -fuzzy Q -convergence Structures

In this section, we discuss the equivalent forms of pretopological and topological L -fuzzy Q -convergence structures. We first give the following definition.

Definition 3.1. [24] An L -fuzzy Q -convergence structure (L -fqc, in short) on X is defined to be a mapping $c : \mathcal{F}_L(X) \rightarrow L^X$ such that $\forall x_\lambda \in pt(L^X), \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$,

(LQFC1) $x_\lambda \leq c(\hat{q}(x_\lambda))$;

(LQFC2) $\mathcal{F} \leq \mathcal{G} \Rightarrow c(\mathcal{F}) \leq c(\mathcal{G})$.

The pair (X, c) is called an L -fuzzy Q -convergence space (L -fqc space, in short), and it will be called pretopological if it satisfies

(LQFC3) $x_\lambda \leq c(\mathcal{F}_{x_\lambda}^c)$, where $\mathcal{F}_{x_\lambda}^c = \bigwedge_{x_\lambda \leq c(\mathcal{F})} \mathcal{F}$.

The pair (X, c) is called a topological L -fuzzy Q -convergence space if it satisfies moreover,

(LQFC4) For all $x_\lambda \in pt(L^X)$, the L -filter $\mathcal{F}_{x_\lambda}^c$ has a base $\mathcal{B}_{x_\lambda}^c$ such that $\mathcal{B}_{x_\lambda}^c(A) \leq \mathcal{B}_{y_\mu}^c(A)$ for all $A \in L^X$ with $y_\mu \hat{q} A$.

A continuous mapping between L -fqc spaces (X, c) and (Y, d) is a mapping $f : X \rightarrow Y$ such that for all $\mathcal{F} \in \mathcal{F}_L(X)$, $c(\mathcal{F}) \leq f^{\leftarrow}(d(f_i^{\Rightarrow}(\mathcal{F})))$. The category of pretopological L -fqc spaces and continuous mappings is denoted by L -**QPrFCS**, and L -**QFTCS** denotes the full subcategory of L -**QPrFCS** consisting of topological L -fqc spaces.

For convenience, we denote the category of L -fqc spaces with their continuous mappings by L -**QFCS** and call $\mathcal{F}_{x_\lambda}^c$ the neighborhood filter of x_λ .

Theorem 3.2. [24] *The category L -**QPrFCS** of pretopological L -fuzzy Q -convergence spaces is topological over **Set**.*

Theorem 3.3. *The category L -**QFCS** of L -fuzzy Q -convergence spaces is topological over **Set**.*

Proof. All necessary steps parallel to the proof of Theorem 3.2 (Proposition 4.7 in [24]), which shows that the category L -**QPrFCS** is topological over **Set**. Hence the proof is omitted. Here we only note that the mapping $c^X : \mathcal{F}_L(X) \rightarrow L^X$ defined by

$$\forall \mathcal{F} \in \mathcal{F}_L(X), \quad c^X(\mathcal{F}) = \bigwedge_{i \in I} f_i^{\leftarrow}(c_i(f_i^{\Rightarrow}(\mathcal{F})))$$

is the initial structure w.r.t. a source $\{f_i : X \rightarrow (X_i, c_i)\}_{i \in I}$ in L -**QFCS**. \square

Definition 3.4. (Product spaces) Let $\prod_{i \in I} c_i$ denote the initial structure w.r.t. the source $\{p_i : X := \prod_{i \in I} X_i \rightarrow (X_i, c_i)\}_{i \in I}$ in L -**QFCS**, i.e.,

$$\forall \mathcal{F} \in \mathcal{F}_L(X), \quad \prod_{i \in I} c_i(\mathcal{F}) = \bigwedge_{i \in I} p_i^{\leftarrow}(c_i(p_i^{\Rightarrow}(\mathcal{F}))).$$

Then $\left(\prod_{i \in I} X_i, \prod_{i \in I} c_i\right)$ is called the product space of $\{(X_i, c_i)\}_{i \in I}$.

Next we will give the characterizations of (pre)topological L -fuzzy Q-convergence structures.

Theorem 3.5. *Let (X, c) be an L -fuzzy Q-convergence space. The following conditions are equivalent:*

- (LQFC3) $x_\lambda \leq c(\mathcal{F}_{x_\lambda}^c)$.
- (LQPC1) $x_\lambda \leq c(\mathcal{F}) \Leftrightarrow \mathcal{F}_{x_\lambda}^c \leq \mathcal{F}$.
- (LQPC2) $c\left(\bigwedge_{i \in I} \mathcal{F}_i\right) = \bigwedge_{i \in I} c(\mathcal{F}_i)$.

Proof. (LQFC3) \Rightarrow (LQPC1) If $x_\lambda \leq c(\mathcal{F})$, then $\mathcal{F}_{x_\lambda}^c = \bigwedge_{x_\lambda \leq c(\mathcal{G})} \mathcal{G} \leq \mathcal{F}$. Conversely, let $\mathcal{F}_{x_\lambda}^c \leq \mathcal{F}$. Then by (LQFC2) and (LQFC3), we have

$$x_\lambda \leq c(\mathcal{F}_{x_\lambda}^c) \leq c(\mathcal{F}).$$

(LQPC1) \Rightarrow (LQPC2) For each $x_\lambda \in pt(L^X)$, it holds that

$$\begin{aligned} x_\lambda \leq \bigwedge_{i \in I} c(\mathcal{F}_i) &\Leftrightarrow \forall i \in I, x_\lambda \leq c(\mathcal{F}_i) \\ &\Leftrightarrow \forall i \in I, \mathcal{F}_{x_\lambda}^c \leq \mathcal{F}_i \quad (\text{by (LQPC1)}) \\ &\Leftrightarrow \mathcal{F}_{x_\lambda}^c \leq \bigwedge_{i \in I} \mathcal{F}_i \\ &\Leftrightarrow x_\lambda \leq c\left(\bigwedge_{i \in I} \mathcal{F}_i\right). \quad (\text{by (LQPC1)}) \end{aligned}$$

(LQPC2) \Rightarrow (LQFC3) By (LQPC2), it follows that

$$x_\lambda \leq \bigwedge_{x_\lambda \leq c(\mathcal{F})} c(\mathcal{F}) = c\left(\bigwedge_{x_\lambda \leq c(\mathcal{F})} \mathcal{F}\right) = c(\mathcal{F}_{x_\lambda}^c). \quad \square$$

Remark 3.6. Generally, there are three different ways to characterize the pretopological condition of convergence structures. Lee [17] requires that the neighborhood filter of a fuzzy point x_λ converge to x_λ . The second form requires the convergence structure be closed for arbitrary meets. Xu [27] defined his pretopological limit structure in this way. Another form is that a filter \mathcal{F} converges to x if and only if \mathcal{F} contains the neighborhood filter of x . Jäger [13] and Yao [29] all adapted this form in lattice-valued situation. Li and Jin [20] showed that pretopological stratified L -convergence structures (stratified L -ordered convergence structures [6]) could be characterized in both the first and the third form. From the above theorem, we see that pretopological L -fuzzy Q-convergence structures can be characterized in all the three ways.

The topological convergence structure is usually defined by the neighborhood filter, and sometimes it is characterized by the diagonal condition which is equivalent to the former way [15]. The axiom (LQFC4) in L -fuzzy Q-convergence structure is complicated and doesn't satisfy each of the two ways. The following theorem will give a simple characterization by means of the neighborhood filter $\mathcal{F}_{x_\lambda}^c$.

Theorem 3.7. Let (X, c) be a pretopological L -fuzzy Q -convergence space. Then the followings are equivalent:

(LQFC4) For all $x_\lambda \in pt(L^X)$, the L -filter $\mathcal{F}_{x_\lambda}^c$ has a base $\mathcal{B}_{x_\lambda}^c$ such that $\mathcal{B}_{x_\lambda}^c(A) \leq \mathcal{B}_{y_\mu}^c(A)$ for all $A \in L^X$ with $y_\mu \hat{q}A$.

$$(LQTCS) \quad \mathcal{F}_{x_\lambda}^c(A) = \bigvee_{x_\lambda \hat{q}B \leq A} \bigwedge_{y_\mu \hat{q}B} \mathcal{F}_{y_\mu}^c(B).$$

Proof. (LQFC4) \Rightarrow (LQTCS) It suffices to prove $\mathcal{F}_{x_\lambda}^c(A) \leq \bigvee_{x_\lambda \hat{q}B \leq A} \bigwedge_{y_\mu \hat{q}B} \mathcal{F}_{y_\mu}^c(B)$. By

(LQFC4), there exists an L -filter base $\mathcal{B}_{x_\lambda}^c$ of $\mathcal{F}_{x_\lambda}^c$ such that

$$\forall y_\mu \in pt(L^X), A \in L^X, y_\mu \hat{q}A \Rightarrow \mathcal{B}_{x_\lambda}^c(A) \leq \mathcal{B}_{y_\mu}^c(A).$$

Take any $\alpha \in M(L)$, $\alpha \triangleleft \mathcal{F}_{x_\lambda}^c(A) = \langle \mathcal{B}_{x_\lambda}^c \rangle(A) = \bigvee_{B \leq A} \mathcal{B}_{x_\lambda}^c(B)$. Then there exists B_α such that $B_\alpha \leq A$ and $\alpha \leq \mathcal{B}_{x_\lambda}^c(B_\alpha) \leq \mathcal{F}_{x_\lambda}^c(B_\alpha)$. For each $y_\mu \hat{q}B_\alpha$, we have $\mathcal{B}_{x_\lambda}^c(B_\alpha) \leq \mathcal{B}_{y_\mu}^c(B_\alpha) \leq \mathcal{F}_{y_\mu}^c(B_\alpha)$. Therefore $\mathcal{B}_{x_\lambda}^c(B_\alpha) \leq \bigwedge_{y_\mu \hat{q}B_\alpha} \mathcal{F}_{y_\mu}^c(B_\alpha)$. This means

that

$$\begin{aligned} \alpha \leq \mathcal{B}_{x_\lambda}^c(B_\alpha) &\leq \mathcal{F}_{x_\lambda}^c(B_\alpha) \wedge \bigwedge_{y_\mu \hat{q}B_\alpha} \mathcal{F}_{y_\mu}^c(B_\alpha) \\ &\leq \bigvee_{B \leq A} \left(\mathcal{F}_{x_\lambda}^c(B) \wedge \bigwedge_{y_\mu \hat{q}B} \mathcal{F}_{y_\mu}^c(B) \right) \\ &= \bigvee_{x_\lambda \hat{q}B \leq A} \bigwedge_{y_\mu \hat{q}B} \mathcal{F}_{y_\mu}^c(B). \quad (\mathcal{F}_{x_\lambda}^c(B) \leq \hat{q}(x_\lambda)(B) = 0 \text{ when } x_\lambda \neg \hat{q}B) \end{aligned}$$

From the arbitrariness of α , we get that $\mathcal{F}_{x_\lambda}^c(A) \leq \bigvee_{x_\lambda \hat{q}B \leq A} \bigwedge_{y_\mu \hat{q}B} \mathcal{F}_{y_\mu}^c(B)$.

(LQTCS) \Rightarrow (LQFC4) Let $\mathcal{B}_{x_\lambda}^c(A) = \mathcal{F}_{x_\lambda}^c(A) \wedge \bigwedge_{y_\mu \hat{q}A} \mathcal{F}_{y_\mu}^c(A)$. We will check that $\mathcal{B}_{x_\lambda}^c$ is just the L -filter base which satisfies (LQFC4).

Firstly, we prove $\mathcal{B}_{x_\lambda}^c$ is an L -filter base.

$$\begin{aligned} \text{(B1)} \quad \mathcal{B}_{x_\lambda}^c(0_X) &\leq \mathcal{F}_{x_\lambda}^c(0_X) = 0, \\ \bigvee_{A \in L^X} \mathcal{B}_{x_\lambda}^c(A) &= \bigvee_{A \in L^X} \left(\mathcal{F}_{x_\lambda}^c(A) \wedge \bigwedge_{y_\mu \hat{q}A} \mathcal{F}_{y_\mu}^c(A) \right) \\ &\geq \mathcal{F}_{x_\lambda}^c(1_X) \wedge \bigwedge_{y_\mu \hat{q}(1_X)} \mathcal{F}_{y_\mu}^c(1_X) = 1. \end{aligned}$$

(B2) Take any $A, B \in L^X$. Then

$$\begin{aligned} \langle \mathcal{B}_{x_\lambda}^c \rangle(A \wedge B) &= \bigvee_{C \leq A \wedge B} \mathcal{B}_{x_\lambda}^c(C) \\ &\geq \mathcal{B}_{x_\lambda}^c(A \wedge B) \\ &= \mathcal{F}_{x_\lambda}^c(A \wedge B) \wedge \bigwedge_{y_\mu \hat{q}(A \wedge B)} \mathcal{F}_{y_\mu}^c(A \wedge B) \\ &= \mathcal{F}_{x_\lambda}^c(A) \wedge \mathcal{F}_{x_\lambda}^c(B) \wedge \bigwedge_{y_\mu \hat{q}A, y_\mu \hat{q}B} \left(\mathcal{F}_{y_\mu}^c(A) \wedge \mathcal{F}_{y_\mu}^c(B) \right) \\ &\quad \left(y_\mu \hat{q}(A \wedge B) \Leftrightarrow y_\mu \hat{q}A, y_\mu \hat{q}B \right) \\ &\geq \mathcal{B}_{x_\lambda}^c(A) \wedge \mathcal{B}_{x_\lambda}^c(B). \end{aligned}$$

Secondly, we check that $\mathcal{B}_{x_\lambda}^c$ is an L -filter base of $\mathcal{F}_{x_\lambda}^c$.

$$\begin{aligned}
\langle \mathcal{B}_{x_\lambda}^c \rangle(A) &= \bigvee_{B \leq A} \mathcal{B}_{x_\lambda}^c(B) \\
&= \bigvee_{B \leq A} \left(\mathcal{F}_{x_\lambda}^c(B) \wedge \bigwedge_{y_\mu \hat{q} B} \mathcal{F}_{y_\mu}^c(B) \right) \\
&= \bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \mathcal{F}_{y_\mu}^c(B) \quad (\mathcal{F}_{x_\lambda}^c(B) = 0 \text{ when } x_\lambda \neg \hat{q} B) \\
&= \mathcal{F}_{x_\lambda}^c(A).
\end{aligned}$$

Finally, for any $z_\nu \in pt(L^X)$, $A \in L^X$ with $z_\nu \hat{q} A$, we have

$$\begin{aligned}
\mathcal{B}_{x_\lambda}^c(A) &= \mathcal{F}_{x_\lambda}^c(A) \wedge \bigwedge_{y_\mu \hat{q} A} \mathcal{F}_{y_\mu}^c(A) \\
&\leq \bigwedge_{y_\mu \hat{q} A} \mathcal{F}_{y_\mu}^c(A) \\
&= \mathcal{F}_{z_\nu}^c(A) \wedge \bigwedge_{y_\mu \hat{q} A} \mathcal{F}_{y_\mu}^c(A) = \mathcal{B}_{z_\nu}^c(A).
\end{aligned}$$

Therefore (LQFC4) holds. \square

Remark 3.8. In (LQTCS), we use the neighborhood filter $\mathcal{F}_{x_\lambda}^c$ to define topological L -fuzzy Q-convergence structure and it's more succinct than (LQFC4). Although the diagonal condition is another way to define topological convergence structure [15], we can't find the diagonal condition of topological L -fuzzy Q-convergence structure. So we will leave this problem for further research.

4. Separation Axioms

In this section, we introduce separation axioms to L -fuzzy Q-convergence space and investigate some properties of the initial space with respect to separation axioms.

Definition 4.1. Let (X, c) be an L -fuzzy Q-convergence space. If it satisfies

- (T_0) $\forall x_\lambda, y_\mu \in pt(L^X)$, $x_\lambda \leq c(\hat{q}(y_\mu))$ and $y_\mu \leq c(\hat{q}(x_\lambda)) \Rightarrow x = y$,
then it is called a T_0 -space. If it satisfies
- (T_1) $\forall x_\lambda, y_\mu \in pt(L^X)$, $x_\lambda \leq c(\hat{q}(y_\mu))$ or $y_\mu \leq c(\hat{q}(x_\lambda)) \Rightarrow x = y$,
then it is called a T_1 -space. If it satisfies
- (T_2) $\forall x_\lambda, y_\mu \in pt(L^X)$, $\forall \mathcal{F} \in \mathcal{F}_L(X)$, $x_\lambda \leq c(\mathcal{F})$ and $y_\mu \leq c(\mathcal{F}) \Rightarrow x = y$,
then it is called a T_2 -space.

Proposition 4.2. Let (X, c) be a pretopological L -fuzzy Q-convergence space. Then the following conclusions hold:

- (1) (T_0) $\Leftrightarrow \forall x_\lambda, y_\mu \in pt(L^X)$, $\mathcal{F}_{x_\lambda}^c \leq \hat{q}(y_\mu)$ and $\mathcal{F}_{y_\mu}^c \leq \hat{q}(x_\lambda) \Rightarrow x = y$.
- (2) (T_1) $\Leftrightarrow \forall x_\lambda, y_\mu \in pt(L^X)$, $\mathcal{F}_{x_\lambda}^c \leq \hat{q}(y_\mu)$ or $\mathcal{F}_{y_\mu}^c \leq \hat{q}(x_\lambda) \Rightarrow x = y$.
- (3) (T_2) $\Leftrightarrow \forall x_\lambda, y_\mu \in pt(L^X)$, $\mathcal{F}_{x_\lambda}^c \vee \mathcal{F}_{y_\mu}^c \in \mathcal{F}_L(X) \Rightarrow x = y$.

Proof. By Theorem 3.5, the proof is easy and omitted. \square

In crisp set theory, if a set of topological spaces $\{(X_i, \tau_i)\}_{i \in I}$ satisfies T_0 -axiom (resp. T_1 and T_2), then their product space $\left(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i\right)$, as a special initial space, also satisfies T_0 -axiom (resp. T_1 and T_2). For L -fuzzy Q-convergence space, we also have similar conclusions.

Theorem 4.3. *If all L -fuzzy Q-convergence spaces (X_i, c_i) ($i \in I$) are T_0 -spaces (resp. T_1, T_2 -spaces) and the family of mappings $(f_i : X \rightarrow X_i)_{i \in I}$ separates points (i.e., for $x \neq y$ there is an $i \in I$ such that $f_i(x) \neq f_i(y)$), then the initial space (X, c^X) is a T_0 -space (resp. T_1, T_2 -space).*

Proof. We only prove T_0 and the proofs of T_1 and T_2 are similar.

In order to prove that (X, c^X) is a T_0 -space, we need only prove for any $x_\lambda, y_\mu \in pt(L^X)$ with $x \neq y$ and $x_\lambda \leq c^X(\hat{q}(y_\mu))$, it holds that $y_\mu \not\leq c^X(\hat{q}(x_\lambda))$.

By the definition of c^X in Theorem 3.3, it follows that

$$x_\lambda \leq c^X(\hat{q}(y_\mu)) = \bigwedge_{i \in I} f_i^+(c_i(f_i^{\rightarrow}(\hat{q}(y_\mu)))).$$

Then $f_i(x)_\lambda \leq c_i(\hat{q}(f_i(y)_\mu))$ for each $i \in I$. Since $(f_i : X \rightarrow X_i)_{i \in I}$ separates points, there exists an $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$. By T_0 -axiom of (X_{i_0}, c_{i_0}) , we have

$$f_{i_0}(y)_\mu \not\leq c_{i_0}(\hat{q}(f_{i_0}(x)_\lambda)) = c_{i_0}(f_{i_0}^{\rightarrow}(\hat{q}(x_\lambda))).$$

This implies

$$y_\mu \not\leq f_{i_0}^+(c_{i_0}(\hat{q}(f_{i_0}(x)_\lambda))) = f_{i_0}^+(c_{i_0}(f_{i_0}^{\rightarrow}(\hat{q}(x_\lambda)))).$$

Therefore

$$y_\mu \not\leq \bigwedge_{i \in I} f_i^+(c_i(f_i^{\rightarrow}(\hat{q}(x_\lambda)))) = c^X(\hat{q}(x_\lambda)).$$

Thus the initial space (X, c^X) is a T_0 -space. \square

Corollary 4.4. *If all L -fuzzy Q-convergence spaces (X_i, c_i) ($i \in I$) are T_0 -spaces (resp. T_1, T_2 -spaces), then their product space $\left(\prod_{i \in I} X_i, \prod_{i \in I} c_i\right)$ is a T_0 -space (resp. T_1, T_2 -space).*

In Definition 4.1, the fuzzy points are arbitrary. If we refine them with the same height, then the following separation axioms are obtained.

Definition 4.5. Let (X, c) be an L -fuzzy Q-convergence space. If it satisfies

$$(qT_0) \quad \forall x_\lambda, y_\lambda \in pt(L^X), \quad x_\lambda \leq c(\hat{q}(y_\lambda)) \text{ and } y_\lambda \leq c(\hat{q}(x_\lambda)) \Rightarrow x = y,$$

then it is called a qT_0 -space. If it satisfies

$$(qT_1) \quad \forall x_\lambda, y_\lambda \in pt(L^X), \quad x_\lambda \leq c(\hat{q}(y_\lambda)) \text{ or } y_\lambda \leq c(\hat{q}(x_\lambda)) \Rightarrow x = y,$$

then it is called a qT_1 -space. If it satisfies

$$(qT_2) \quad \forall x_\lambda, y_\lambda \in pt(L^X), \quad \forall \mathcal{F} \in \mathcal{F}_L(X), \quad x_\lambda \leq c(\mathcal{F}) \text{ and } y_\lambda \leq c(\mathcal{F}) \Rightarrow x = y,$$

then it is called a qT_2 -space.

Similarly, the following two propositions hold obviously and we omit the proof.

Proposition 4.6. *Let (X, c) be a pretopological L -fuzzy Q -convergence space. Then the following conclusions hold:*

- (1) $(qT_0) \Leftrightarrow \forall x_\lambda, y_\lambda \in pt(L^X), \mathcal{F}_{x_\lambda}^c \leq \hat{q}(y_\lambda)$ and $\mathcal{F}_{y_\lambda}^c \leq \hat{q}(x_\lambda) \Rightarrow x = y$.
- (2) $(qT_1) \Leftrightarrow \forall x_\lambda, y_\lambda \in pt(L^X), \mathcal{F}_{x_\lambda}^c \leq \hat{q}(y_\lambda)$ or $\mathcal{F}_{y_\lambda}^c \leq \hat{q}(x_\lambda) \Rightarrow x = y$.
- (3) $(qT_2) \Leftrightarrow \forall x_\lambda, y_\lambda \in pt(L^X), \mathcal{F}_{x_\lambda}^c \vee \mathcal{F}_{y_\lambda}^c \in \mathcal{F}_L(X) \Rightarrow x = y$.

Proposition 4.7. *If all L -fuzzy Q -convergence spaces (X_i, c_i) ($i \in I$) are qT_0 -spaces (resp. qT_1, qT_2 -spaces) and the family of mappings $(f_i : X \rightarrow X_i)_{i \in I}$ separates points, then the initial space (X, c^X) is a qT_0 -space (resp. qT_1, qT_2 -space).*

Next we discuss qT_0 (resp. qT_1, qT_2)-separation of product space of L -fuzzy Q -convergence spaces. We first give the following lemma.

Lemma 4.8. *Suppose that $0 \in L$ is a prime. Let X_i be a nonempty set, $\mathcal{F}_i \in \mathcal{F}_L(X_i)$ and $p_i : \prod_{j \in I} X_j \rightarrow X_i$ be the projection mapping for each $i \in I$. Then*

- (1) $\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i) \in \mathcal{F}_L(\prod_{j \in I} X_j)$.
- (2) $p_j^{\rightarrow}(\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i)) \geq \mathcal{F}_j$ for all $j \in I$.

Proof. (1) It suffices to prove that (i) $p_i^{\leftarrow}(\mathcal{F}_i)$ exists for all $i \in I$ and (ii) $\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i)$ exists.

For (i), condition (2) in Lemma 2.10 is trivially satisfied since p_i is surjective.

For (ii), we need only verify that $\{p_i^{\leftarrow}(\mathcal{F}_i)\}_{i \in I}$ satisfies condition (2) in Lemma 2.8. Let $n = 2$ and $A_1 \wedge A_2 = 0_X$ ($X := \prod_{i \in I} X_i$). Then

$$\begin{aligned}
& p_1^{\leftarrow}(\mathcal{F}_1)(A_1) \wedge p_2^{\leftarrow}(\mathcal{F}_2)(A_2) \\
&= \bigvee_{p_1^{\leftarrow}(B_1) \leq A_1} \mathcal{F}_1(B_1) \wedge \bigvee_{p_2^{\leftarrow}(B_2) \leq A_2} \mathcal{F}_2(B_2) \\
&= \bigvee_{p_1^{\leftarrow}(B_1) \leq A_1} \bigvee_{p_2^{\leftarrow}(B_2) \leq A_2} \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \\
&\leq \bigvee_{p_1^{\leftarrow}(B_1) \wedge p_2^{\leftarrow}(B_2) \leq A_1 \wedge A_2} \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \\
&= \bigvee_{p_1^{\leftarrow}(B_1) \wedge p_2^{\leftarrow}(B_2) = 0_X} \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \\
&= \bigvee \{ \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \mid \bigvee_{x \in X} B_1(p_1(x)) \wedge B_2(p_2(x)) = 0 \} \\
&= \bigvee \{ \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \mid \bigvee_{(x_1, x_2) \in X_1 \times X_2} B_1(x_1) \wedge B_2(x_2) = 0 \} \\
&= \bigvee \{ \mathcal{F}_1(B_1) \wedge \mathcal{F}_2(B_2) \mid \bigvee_{x_1 \in X_1} B_1(x_1) \wedge \bigvee_{x_2 \in X_2} B_2(x_2) = 0 \} \\
&\leq \bigvee_{\alpha \wedge \beta = 0} \mathcal{F}_1(\underline{\alpha}) \wedge \mathcal{F}_2(\underline{\beta}) \\
&= 0. \quad (0 \text{ is a prime})
\end{aligned}$$

This proves $\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i) \in \mathcal{F}_L(\prod_{j \in I} X_j)$.

(2) Take any $A_j \in L^{X_j}$. Then

$$\begin{aligned} p_j^{\rightarrow} \left(\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i) \right) (A_j) &= \bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i)(p_j^{\leftarrow}(A_j)) \\ &\geq p_j^{\leftarrow}(\mathcal{F}_j)(p_j^{\leftarrow}(A_j)) \\ &= \bigvee_{p_j^{\leftarrow}(B_j) \leq p_j^{\leftarrow}(A_j)} \mathcal{F}_j(B_j) \\ &\geq \mathcal{F}_j(A_j). \end{aligned}$$

From the arbitrariness of A_j , we obtain $p_j^{\rightarrow}(\bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i)) \geq \mathcal{F}_j$ for all $j \in I$. \square

Proposition 4.9. *Suppose that 0 is a prime. Let (X_i, c_i) ($i \in I$) be L -fuzzy Q -convergence spaces. Then they are all qT_0 -spaces (resp. qT_1 , qT_2 -spaces) if and only if their product space $\left(\prod_{i \in I} X_i, \prod_{i \in I} c_i \right)$ is a qT_0 -space (resp. qT_1 , qT_2 -space).*

Proof. We only prove qT_2 . By Proposition 4.7, the necessity is obvious. It is enough to prove the sufficiency.

In order to prove (X_{i_0}, c_{i_0}) ($\forall i_0 \in I$) is a qT_2 -space, we need only prove that for any $(x_{i_0})_\lambda, (y_{i_0})_\lambda \in pt(L^{X_{i_0}})$, $\mathcal{F}_{i_0} \in \mathcal{F}_L(X_{i_0})$ with $x_{i_0} \neq y_{i_0}$ and $(x_{i_0})_\lambda \leq c_{i_0}(\mathcal{F}_{i_0})$, it follows that $(y_{i_0})_\lambda \not\leq c_{i_0}(\mathcal{F}_{i_0})$.

Take $x, y \in \prod_{i \in I} X_i$ such that $p_{i_0}(x) = x_{i_0}$, $p_{i_0}(y) = y_{i_0}$ and $p_i(x) = p_i(y)$ for $i \neq i_0$. By Lemma 4.8 (1), we define $\mathcal{F} \in \mathcal{F}_L\left(\prod_{i \in I} X_i\right)$ as follows:

$$\mathcal{F} := \bigvee_{i \in I} p_i^{\leftarrow}(\mathcal{F}_i),$$

where $\mathcal{F}_i = \hat{q}(p_i(x)_\lambda)$ for $i \neq i_0$ and $\mathcal{F}_{i_0} = \mathcal{F}_{i_0}$ for $i = i_0$. By Definition 3.4 and Lemma 4.8 (2), it is easy to check that

$$x_\lambda \leq \left(\prod_{i \in I} c_i \right) (\mathcal{F}) = \bigwedge_{i \in I} p_i^{\leftarrow}(c_i(p_i^{\rightarrow}(\mathcal{F}))).$$

Since $\left(\prod_{i \in I} X_i, \prod_{i \in I} c_i \right)$ is a qT_2 -space and $x \neq y$, it follows that

$$y_\lambda \not\leq \left(\prod_{i \in I} c_i \right) (\mathcal{F}) = \bigwedge_{i \in I} p_i^{\leftarrow}(c_i(p_i^{\rightarrow}(\mathcal{F}))).$$

If $i \neq i_0$, then by Lemma 4.8 (2), we have

$$p_i(y)_\lambda = p_i(x)_\lambda \leq c_i(\hat{q}(p_i(x)_\lambda)) \leq c_i(p_i^{\rightarrow}(\mathcal{F})).$$

This implies that

$$p_{i_0}(y)_\lambda \not\leq c_{i_0}(p_{i_0}^{\rightarrow}(\mathcal{F})) \geq c_{i_0}(\mathcal{F}_{i_0}).$$

Therefore $(y_{i_0})_\lambda \not\leq c_{i_0}(\mathcal{F}_{i_0})$, as desired. \square

5. L -ordered \mathbf{Q} -convergence Structures

In [6], Fang defined fuzzy inclusion order in $\mathcal{F}_L(X)$ in the following way:

$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \quad \mathcal{S}_F(\mathcal{F}, \mathcal{G}) = \bigwedge_{A \in L^X} \mathcal{F}(A) \rightarrow \mathcal{G}(A).$$

Inspired by this, we modify the axioms (LQFC2) and (LQFC3) in the definition of L -fuzzy \mathbf{Q} -convergence structures and construct a new kind of convergence structures. Furthermore, the relations between them are discussed.

Definition 5.1. An L -ordered \mathbf{Q} -convergence structure on X is defined to be a mapping $c : \mathcal{F}_L(X) \rightarrow L^X$ such that $\forall x_\lambda \in pt(L^X), \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$,

$$(LQFC1) \quad x_\lambda \leq c(\hat{q}(x_\lambda));$$

$$(LOCS2) \quad \mathcal{S}_F(\mathcal{F}, \mathcal{G}) \leq \mathcal{S}(c(\mathcal{F}), c(\mathcal{G})).$$

The pair (X, c) is called an L -ordered \mathbf{Q} -convergence space. It will be called pretopological if it satisfies

$$(LOCS3) \quad \mathcal{S}_F(\mathcal{F}_{x_\lambda}^c, \mathcal{F}) \leq \mathcal{S}(x_\lambda, c(\mathcal{F})).$$

The pair (X, c) is called a topological L -ordered \mathbf{Q} -convergence space if it satisfies moreover,

$$(LQTCS) \quad \mathcal{F}_{x_\lambda}^c(A) = \bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \mathcal{F}_{y_\mu}^c(B).$$

A continuous mapping between L -ordered \mathbf{Q} -convergence spaces (X, c) and (Y, d) is a mapping $f : X \rightarrow Y$ such that for all $\mathcal{F} \in \mathcal{F}_L(X)$, $c(\mathcal{F}) \leq f^+(d(f^\Rightarrow(\mathcal{F})))$.

Let $L\text{-}\mathbf{OQCS}$ denote the category of L -ordered \mathbf{Q} -convergence spaces with continuous mappings, and $L\text{-}\mathbf{OQPrCS}$ the full subcategory of $L\text{-}\mathbf{OQCS}$ consisting of pretopological L -ordered \mathbf{Q} -convergence spaces.

Theorem 5.2. An (pretopological, topological) L -ordered \mathbf{Q} -convergence space must be an (pretopological, topological) L -fuzzy \mathbf{Q} -convergence space.

Proof. By Lemma 2.2 (1) and Theorem 3.5, the conclusion is obvious. \square

The inversion of the above theorem doesn't hold, the following example demonstrate this.

Example 5.3. Let $X = \{x, y\}$ and $L = \{0, \frac{1}{2}, 1\}$ be a chain. We define a mapping $c : \mathcal{F}_L(X) \rightarrow L^X$ by

$$\forall \mathcal{F} \in \mathcal{F}_L(X), z \in X, \quad c(\mathcal{F})(z) = \begin{cases} 1, & \hat{q}(z_{\frac{1}{2}}) \leq \mathcal{F}, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to check (X, c) is an (pretopological) L -fuzzy \mathbf{Q} -convergence space in the sense of Definition 3.1. But on the other hand, the space (X, c) is not an (pretopological) L -ordered \mathbf{Q} -convergence space. To see this, we introduce a mapping $\mathcal{F}^* : L^X \rightarrow L$ by

$$\forall A \in L^X, \quad \mathcal{F}^*(A) = \begin{cases} 1, & \text{if } A = 1_X, \\ \frac{1}{2}, & \text{if } A(x) = 1, A(y) \neq 1, \\ \frac{1}{2}, & \text{if } A(x) = \frac{1}{2}, \\ 0, & \text{if } A(x) = 0. \end{cases}$$

It's routine to verify $\mathcal{F}^* \in \mathcal{F}_L(X)$.

In this case, for $A \in L^X$ defined by $A(x) = 1$, $A(y) \neq 1$, we have $\hat{q}(x_{\frac{1}{2}})(A) = 1 > \frac{1}{2} = \mathcal{F}^*(A)$. This implies $\hat{q}(x_{\frac{1}{2}}) \not\leq \mathcal{F}^*$. Further it holds that

$$\begin{aligned} \mathcal{S}_F(\hat{q}(x_{\frac{1}{2}}), \mathcal{F}^*) &= \bigwedge_{B \in L^X} \left(\hat{q}(x_{\frac{1}{2}})(B) \rightarrow \mathcal{F}^*(B) \right) \\ &= (1 \rightarrow 1) \wedge (1 \rightarrow \frac{1}{2}) \wedge (0 \rightarrow \frac{1}{2}) \wedge (0 \rightarrow 0) \\ &= \frac{1}{2}, \end{aligned}$$

which can be checked by considering the nine different L -sets $B \in L^X$. But we also obtain that

$$\begin{aligned} \mathcal{S}(c(\hat{q}(x_{\frac{1}{2}}), c(\mathcal{F}^*))) &= \left(c(\hat{q}(x_{\frac{1}{2}}))(x) \rightarrow c(\mathcal{F}^*)(x) \right) \wedge \left(c(\hat{q}(x_{\frac{1}{2}}))(y) \rightarrow c(\mathcal{F}^*)(y) \right) \\ &\leq c(\hat{q}(x_{\frac{1}{2}}))(x) \rightarrow c(\mathcal{F}^*)(x) \\ &= (1 \rightarrow 0) \\ &= 0. \end{aligned}$$

It follows that

$$\mathcal{S}(c(\hat{q}(x_{\frac{1}{2}}), c(\mathcal{F}^*))) = 0 \not\geq \frac{1}{2} = \mathcal{S}_F(\hat{q}(x_{\frac{1}{2}}), \mathcal{F}^*).$$

This means that this L -fuzzy Q-convergence space (X, c) doesn't satisfy the axiom (LOCS2). Therefore we point out that an L -fuzzy Q-convergence space may not be an L -ordered Q-convergence space.

Let X be a nonempty set, and $C_{loc}(X)$ denote the fibre

$$\{c : c \text{ is an } L\text{-ordered Q-convergence structure on } X\}$$

of X . We can define a partial order on $C_{loc}(X)$ as follows:

$$c_1 \leq c_2 \Leftrightarrow \text{id} : (X, c_2) \rightarrow (X, c_1) \text{ is continuous.}$$

That is to say

$$c_1 \leq c_2 \Leftrightarrow \forall x \in X, \mathcal{F} \in \mathcal{F}_L(X), c_2(\mathcal{F})(x) \leq c_1(\mathcal{F})(x).$$

Then the following theorem holds.

Theorem 5.4. $(C_{loc}(X), \leq)$ is a complete lattice.

Proof. Firstly, we define $c_{sm} : \mathcal{F}_L(X) \rightarrow L^X$ by

$$\forall \mathcal{F} \in \mathcal{F}_L(X), c_{sm}(\mathcal{F}) = 1_X.$$

Obviously c_{sm} is the minimal element.

Secondly, let $\{c_j\}_{j \in J} \subseteq C_{loc}(X)$. Define $\sup_{j \in J} c_j : \mathcal{F}_L(X) \rightarrow L^X$ as follows:

$$\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \left(\sup_{j \in J} c_j \right)(\mathcal{F})(x) = \bigwedge_{j \in J} c_j(\mathcal{F})(x).$$

Then we claim that $\sup_{j \in J} c_j$ is an L -ordered Q -convergence structure on X .

$$(LQFC1) \quad (\sup_{j \in J} c_j)(\hat{q}(x_\lambda))(x) = \bigwedge_{j \in J} c_j(\hat{q}(x_\lambda))(x) \geq \bigwedge_{j \in J} \lambda = \lambda.$$

(LOCS2) For each $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$, it follows that

$$\begin{aligned} & \mathcal{S}\left(\left(\sup_{j \in J} c_j\right)(\mathcal{F}), \left(\sup_{j \in J} c_j\right)(\mathcal{G})\right) \\ &= \bigwedge_{x \in X} \left(\bigwedge_{j \in J} c_j(\mathcal{F})(x) \rightarrow \bigwedge_{j \in J} c_j(\mathcal{G})(x) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{j \in J} (c_j(\mathcal{F})(x) \rightarrow c_j(\mathcal{G})(x)) \quad (\text{by Lemma 2.1 (5)}) \\ &= \bigwedge_{j \in J} \mathcal{S}(c_j(\mathcal{F}), c_j(\mathcal{G})) \\ &\geq \mathcal{S}_F(\mathcal{F}, \mathcal{G}). \end{aligned}$$

Finally, it's trivial to check that $\sup_{j \in J} c_j$ is the minimal upper bound. \square

Lemma 5.5. [6] *Let $f : X \rightarrow Y$ be a mapping. Then for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$,*

$$\mathcal{S}_F(\mathcal{F}, \mathcal{G}) \leq \mathcal{S}_F(f^\Rightarrow(\mathcal{F}), f^\Rightarrow(\mathcal{G})).$$

Lemma 5.6. *Let (Y, c_Y) be an L -ordered Q -convergence space, f be a mapping from X to Y . Define $c_X : \mathcal{F}_L(X) \rightarrow L^X$ such that*

$$\forall \mathcal{F} \in \mathcal{F}_L(X), \quad c_X(\mathcal{F}) = f^\leftarrow(c_Y(f^\Rightarrow(\mathcal{F}))).$$

Then c_X is an L -ordered Q -convergence structure on X .

Proof. That c_X satisfies (LQFC1) and (LOCS2) is verified as follows:

$$(LQFC1) \quad c_X(\hat{q}(x_\lambda))(x) = f^\leftarrow(c_Y(f^\Rightarrow(\hat{q}(x_\lambda))))(x) = c_Y(\hat{q}(f(x)_\lambda))(f(x)) \geq \lambda.$$

(LOCS2) Take any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$. Then

$$\begin{aligned} \mathcal{S}(c_X(\mathcal{F}), c_X(\mathcal{G})) &= \mathcal{S}(f^\leftarrow(c_Y(f^\Rightarrow(\mathcal{F}))), f^\leftarrow(c_Y(f^\Rightarrow(\mathcal{G})))) \\ &\geq \mathcal{S}(c_Y(f^\Rightarrow(\mathcal{F})), c_Y(f^\Rightarrow(\mathcal{G}))) \quad (\text{by Lemma 2.3}) \\ &\geq \mathcal{S}_F(f^\Rightarrow(\mathcal{F}), f^\Rightarrow(\mathcal{G})) \\ &\geq \mathcal{S}_F(\mathcal{F}, \mathcal{G}). \quad (\text{by Lemma 5.5}) \end{aligned}$$

Completes the proof. \square

Theorem 5.7. *The category $L\text{-OQCS}$ of all L -ordered Q -convergence spaces is a bireflective full subcategory in $L\text{-QFCS}$.*

Proof. Let $(X, \bar{c}) \in |L\text{-QFCS}|$ and $E_{\bar{c}} = \{c \mid (X, c) \in |L\text{-OQCS}|, c \leq \bar{c}\}$. By Theorem 5.4, we define $(X, c^*) \in |L\text{-OQCS}|$ such that

$$\forall \mathcal{F} \in \mathcal{F}_L(X), \quad x \in X, \quad c^*(\mathcal{F})(x) = \bigwedge_{c \in E_{\bar{c}}} c(\mathcal{F})(x).$$

We claim that $id_X : (X, \bar{c}) \rightarrow (X, c^*)$ is the $L\text{-OQCS}$ -reflection.

For this it suffices to prove:

- (1) $id_X : (X, \bar{c}) \rightarrow (X, c^*)$ is continuous.

(2) For each L -ordered Q -convergence space (Y, c_Y) , and each mapping $f : X \rightarrow Y$, the continuity of $f : (X, \bar{c}) \rightarrow (Y, c_Y)$ implies the continuity of $f : (X, c^*) \rightarrow (Y, c_Y)$.

(1) is true since $c^* \in E_{\bar{c}}$, that is to say,

$$c^* \leq \bar{c} \Leftrightarrow \forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \bar{c}(\mathcal{F})(x) \leq c^*(\mathcal{F})(x).$$

(2) By definition, we need only prove

$$\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, c^*(\mathcal{F})(x) \leq c_Y(f \Rightarrow (\mathcal{F}))(f(x)).$$

By Lemma 5.6, c_X defined by $c_X(\mathcal{F}) = f^{\leftarrow}(c_Y(f \Rightarrow (\mathcal{F})))$ is an L -ordered Q -convergence structure on X . Further by the continuity of $f : (X, \bar{c}) \rightarrow (Y, c_Y)$, we have

$$\bar{c}(\mathcal{F})(x) \leq f^{\leftarrow}(c_Y(f \Rightarrow (\mathcal{F}))) (x) = c_X(\mathcal{F})(x).$$

Then from the arbitrariness of \mathcal{F} and x , we obtain that $c_X \leq \bar{c}$ and $c_X \in E_{\bar{c}}$. This shows $c_X \leq c^*$.

Therefore

$$c^*(\mathcal{F})(x) \leq c_X(\mathcal{F})(x) = c_Y(f \Rightarrow (\mathcal{F}))(f(x)).$$

The continuity of $f : (X, c^*) \rightarrow (Y, c_Y)$ is proved. \square

By Theorems 3.3 and 5.7, we have

Corollary 5.8. *The category L -OQCS of all L -ordered Q -convergence spaces is topological over **Set**.*

We write $C_{ploc}(X)$ for the set of all pretopological L -ordered Q -convergence structures on X and still write \leq for the restriction of the order on $C_{loc}(X)$ to $C_{ploc}(X)$. Let $\{c_j\}_{j \in J}$ be a nonempty family of pretopological L -ordered Q -convergence structures on X . It can be checked that the supremum $\sup_{j \in J}$ of $\{c_j\}_{j \in J}$ in $(C_{ploc}(X), \leq)$ is also defined by

$$\forall \mathcal{F} \in \mathcal{F}_L(X), x \in X, \left(\sup_{j \in J} c_j \right) (\mathcal{F})(x) = \bigwedge_{j \in J} c_j(\mathcal{F})(x).$$

Now we will discuss the relations between pretopological L -fuzzy Q -convergence structures and pretopological L -ordered Q -convergence structures. For this the following lemma is necessary.

Lemma 5.9. *Suppose $\{(X, c_i)\}_{i \in I}$ are all L -ordered Q -convergence spaces and $\sup_{j \in J} c_j$ is defined as above. Then $\mathcal{F}_{x_\lambda}^{\sup_{j \in J} c_j} \geq \bigvee_{j \in J} \mathcal{F}_{x_\lambda}^{c_j}$.*

Proof. Take any $x_\lambda \in pt(L^X)$, $\mathcal{F} \in \mathcal{F}_L(X)$. Then

$$\begin{aligned} \mathcal{F}_{x_\lambda}^{\sup_{j \in J} c_j} &= \bigwedge_{x_\lambda \leq (\sup_{j \in J} c_j)(\mathcal{F})} \mathcal{F} = \bigwedge_{\lambda \leq \bigwedge_{j \in J} c_j(\mathcal{F})(x)} \mathcal{F} \\ &\geq \bigvee_{j \in J} \bigwedge_{\lambda \leq c_j(\mathcal{F})(x)} \mathcal{F} \\ &= \bigvee_{j \in J} \mathcal{F}_{x_\lambda}^{c_j}. \end{aligned}$$

The conclusion is proved. \square

Theorem 5.10. $(C_{ploc}(X), \leq)$ is a complete lattice.

Proof. The proof is the same as that of Theorem 5.4. It is sufficient to show that $\sup_{j \in J} c_j$ satisfies (LOCS3).

Take $x_\lambda \in pt(L^X)$, $\mathcal{F} \in \mathcal{F}_L(X)$. Then

$$\begin{aligned} \mathcal{S}\left(x_\lambda, \left(\sup_{j \in J} c_j\right)(\mathcal{F})\right) &= \bigwedge_{j \in J} \mathcal{S}(x_\lambda, c_j(\mathcal{F})) \quad (\text{by Lemma 2.2 (3)}) \\ &\geq \bigwedge_{j \in J} \mathcal{S}_F(\mathcal{F}_{x_\lambda}^{c_j}, \mathcal{F}) \\ &= \mathcal{S}_F\left(\bigvee_{j \in J} \mathcal{F}_{x_\lambda}^{c_j}, \mathcal{F}\right) \quad (\text{by Lemma 2.2 (2)}) \\ &\geq \mathcal{S}_F\left(\mathcal{F}_{x_\lambda}^{\sup_{j \in J} c_j}, \mathcal{F}\right). \quad (\text{by Lemma 5.9}) \end{aligned}$$

Hence $\sup_{j \in J} c_j$ satisfies (LOCS3). \square

Theorem 5.11. The category $L\text{-OQPrCS}$ of all pretopological L -ordered Q -convergence spaces is a bireflective full subcategory in $L\text{-QFPrCS}$.

Proof. The proof is the same as that of Theorem 5.6. We need only prove the following conclusion.

For a mapping $f : X \rightarrow (Y, c_Y)$, where (Y, c_Y) is a pretopological L -ordered Q -convergence space, the mapping $c_X : \mathcal{F}_L(X) \rightarrow L^X$ defined by

$$\forall \mathcal{F} \in \mathcal{F}_L(X), \quad c_X(\mathcal{F}) = f^{\leftarrow}(c_Y(f^{\Rightarrow}(\mathcal{F})))$$

is a pretopological L -ordered Q -convergence structure. For this it is enough to prove that c_X satisfies (LOCS3). First we have

$$\begin{aligned} f^{\Rightarrow}(\mathcal{F}_{x_\lambda}^{c_X}) &= f^{\Rightarrow}\left(\bigwedge_{x_\lambda \leq c_X(\mathcal{F})} \mathcal{F}\right) = \bigwedge_{x_\lambda \leq c_X(\mathcal{F})} f^{\Rightarrow}(\mathcal{F}) \\ &= \bigwedge_{\lambda \leq c_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))} f^{\Rightarrow}(\mathcal{F}) = \bigwedge_{f(x)_\lambda \leq c_Y(f^{\Rightarrow}(\mathcal{F}))} f^{\Rightarrow}(\mathcal{F}) \\ &\geq \bigwedge_{f(x)_\lambda \leq c_Y(\mathcal{G})} \mathcal{G} \\ &= \mathcal{F}_{f(x)_\lambda}^{c_Y}. \end{aligned}$$

Then it follows that

$$\begin{aligned} \mathcal{S}_F(\mathcal{F}_{x_\lambda}^{c_X}, \mathcal{F}) &\leq \mathcal{S}_F(f^{\Rightarrow}(\mathcal{F}_{x_\lambda}^{c_X}), f^{\Rightarrow}(\mathcal{F})) \quad (\text{by Lemma 5.4}) \\ &\leq \mathcal{S}_F(\mathcal{F}_{f(x)_\lambda}^{c_Y}, f^{\Rightarrow}(\mathcal{F})) \quad (\text{by Lemma 2.2 (2)}) \\ &\leq \mathcal{S}(f(x)_\lambda, c_Y(f^{\Rightarrow}(\mathcal{F}))) \\ &= \lambda \rightarrow c_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \\ &= \lambda \rightarrow c_X(\mathcal{F})(x) \\ &= \mathcal{S}(x_\lambda, c_X(\mathcal{F})). \end{aligned}$$

This proves the conclusion. \square

From Theorems 3.2 and 5.11, the following result holds.

Corollary 5.12. *The category L -OQPrCS of all pretopological L -ordered Q -convergence spaces is topological over **Set**.*

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