(IC)LM-FUZZY TOPOLOGICAL SPACES

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Abstract. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of (weakly) induced LM-fuzzy topological spaces. We discuss the basic properties of (IC)LM-fuzzy topological spaces, and introduce the notions of interior (IC)-fication and exterior (IC)-fication of LM-fuzzy topologies and prove that ICLM-FTop (the category of (IC)LM-fuzzy topological spaces) is an isomorphism-closed full proper subcategory of LM-FTop (the category of LM-fuzzy topological spaces) and ICLM-FTop is a simultaneously bireflective and bicoreflective full subcategory of LM-FTop.

1. Introduction

Since Chang [2] introduced fuzzy theory in topology, many authors discussed various aspects of fuzzy topology. However, in a completely different direction, Hohle [6] created the notion of a topology being viewed as an L-subset of a powerset (in his case, 2^X). Then Kubik [8] and Sostak [17] independently extended Hohle’s notion to L-subsets of L^X.

In 2007, Yue studied induced and weakly induced LM-fuzzy topological spaces, and proved that WILM-FTop (the category of weakly induced LM-fuzzy topological spaces and continuous mappings) is a simultaneously reflective and coreflective full subcategory of LM-FTop (the category of LM-fuzzy topological spaces and continuous mappings) [21]. In 2011, Yao showed that, for L a frame, the category of enriched L-fuzzy topological spaces can be embedded in that of L-generalized convergence spaces as a reflective subcategory and the latter is a cartesian-closed topological category [20]. Moreover, in 2007, Li, Li and Fu defined and studied the properties of (IC)L-cotopological spaces [10], which is the generalization of Martin’s weakly induced I-topological spaces [13] (particularly, Weiss’s induced spaces [19] and Lowen’s topologically generated spaces [11]), and showed that ICLTopc (the category of (IC)L-cotopological spaces and continuous mappings) is simultaneously bireflective and bicoreflective in LTopc (the category of L-cotopological spaces and continuous mappings), which implies that (IC)L-cotopological space has the similar character with weakly induced spaces. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of weakly induced...
and induced LM-fuzzy topological spaces. We discuss the basic properties of LM-fuzzy topological spaces, and introduce the notions of interior (IC)-ication and exterior (IC)-ication of LM-fuzzy topological spaces and prove that ICLM-FTop (the category of (IC)LM-fuzzy topological spaces and continuous mappings) is an isomorphism-closed full proper subcategory of LM-FTop and ICLM-FTop is a simultaneously bireflective and bicoreflective full subcategory of LM-FTop.

For needed categorical notions, please refer to [1, 7, 12, 15].

2. Preliminaries

We now give some definitions and results to be used in this paper. Throughout this paper, L always stands for a completely distributive lattice with locally multiplicative property and M is a completely distributive lattice with the least element 0 and the greatest element 1 (0 ≠ 1). Apparently, the powerset $L^X$ (i.e. the set of all $L$-subset) with the point-wise order is also a completely distributive lattice with locally multiplicative property, in which the least element and the greatest element of $L^X$ will be written as $0_X$ and $1_X$. An element $r \in L - \{0\}$ is called a co-prime element if, for any finite subset $K \subseteq L$ satisfying $r \leq \bigvee K$ (the supremum of $K$), there exists a $k \in K$ such that $r \leq k$. An element $s \in L - \{1\}$ is called a prime element if it is a co-prime element of $L^{op}$ (the opposite lattice $[4]$ of $L$). The set of all co-prime elements (resp., prime elements) of $L$ will be written as $\text{Copr}(L)$ (resp., $\text{Pr}(L)$). We say $a$ is way below (wedge below) $b$, in symbols, $a \ll b$ ($a \wedge b$), if for every directed (arbitrary) subset $D \subset L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. From [5], we know that $\text{Copr}(L)$ is a join-generating set of $L$ if $L$ is a completely distributive lattice. Hence every element in $L$ is also the supremum of all the coprimes wedge below it. By the definition of completely distributive lattice it is easy to see that a complete lattice $L$ is completely distributive iff the operator $\bigvee : \text{Low}(L) \rightarrow L$ taking every lower set to its supremum has a left adjoint $\beta$, and in the case, $\beta(a) = \{b : b \ll a\}$ for all $a \in L$. Hence the wedge below relation has the interpolation property in a completely distributive lattice, that is, $a \ll b$ implies there is some $c \in L$ such that $a \ll c \ll b$.

The way-below relation on a completely distributive lattice $L$ is called locally multiplicative [22] if for every coprime $a \in \text{Copr}(L)$, $a \ll b$ and $a \ll c$ implies $a \ll b \wedge c$ for all $b, c \in L$. If $a$ is a coprime, then $a \ll b$ if and only if $a \ll b$ (see [5, 22]). Hence $L$ is locally multiplicative if for every coprime $a$, $a \ll b$ and $a \ll c$ implies $a \ll b \wedge c$ for all $b, c \in L$. Clearly, $[0,1]$ is locally multiplicative.

For $A \in L^X$ and $p \in L$, let $i_p(A) = \{x : A(x) \ll p\}$, called the strong $p$-cut of $A$. Let $1_U$ denote the characteristic function of $U \subseteq 2^X$, and $[a]$ denote the $L$-subset taking constant value $a$, where $2^X$ is the power set of $X$ and $a \in L$. Each mapping $f : X \rightarrow Y$ induces a mapping $f_\rightarrow^- : L^Y \rightarrow L^X$ (called $L$-forward powerset operator, cf. [14]), defined by

$$f_\rightarrow^-(A)(y) = \bigvee\{A(x) \mid f(x) = y\} \quad (\forall A \in L^Y, \forall y \in Y).$$

The right adjoint to $f_\rightarrow^-$ (called $L$-backward powerset operator, cf. [14]) is denoted as $f_\leftarrow^+$ and given by
\( f_L^+(B) = \bigvee \{ A \in L^X \mid f_L^+(A) \leq B \} = B \circ f \) \((\forall B \in L^Y)\).

It is known that \( f_L^- \) preserves arbitrary unions and that \( f_L^+ \) preserves arbitrary unions, arbitrary intersections, and complements when they exist (canonical examples of such morphisms are given in [14]). Let \( L \) be a locally multiplicative completely distributive lattice. It is easy to verify the following

**Lemma 2.1.** [9, 21] (1) For each \( p \in L \), \( i_p : L^X \to 2^X = \{ E \mid E \subseteq X \} \) preserves arbitrary suprema and finite meets.

(2) For each \( p \in L \), \( A \in L^X \), \( B \in L^Y \) and any mapping \( f : X \to Y \), we have \( i_p(f_L^+(A)) = f^{-1}(i_p(A)) \), \( i_p(f_L^+(B)) = f^{-1}(i_p(B)) \).

(3) \( \tau = \bigvee_{p \in L} [p] \land 1_{i_p(A)} = \bigvee_{p \in {\text{Copr}}(L)} [p] \land 1_{i_p(A)}. \)

An \( LM \)-fuzzy topology (or a Kubiak-Šostak fuzzy topology [8, 17]) on a set \( X \) is defined to be a mapping \( \tau : L^X \to M \) satisfying:

- (FT1) \( \tau(1_X) = \tau(0_X) = 1 \);
- (FT2) \( \forall A, B \in L^X \), \( \tau(A \land B) \geq \tau(A) \land \tau(B) \);
- (FT3) \( \tau(\bigvee_{t \in T} A_t) \geq \bigwedge_{t \in T} \tau(A_t) \), for every family \( \{ A_t \mid t \in T \} \subseteq L^X \).

The value \( \tau(A) \) can be interpreted as the degree of openness of \( A \). A continuous mapping between \( LM \)-fuzzy topological spaces is a mapping \( f : (X, \tau) \to (Y, \delta) \) such that \( \delta(B) \leq \tau(f_L^+(B)) \) for all \( B \in L^Y \), where \( f_L^+(B) \) is defined by \( f_L^+(B)(x) = B(f(x))(\{8,16-17\}) \). When \( L = \{0, 1\} \), this definition reduced to that of \( M \)-fuzzyifying topology. Let \( LM\text{-FTop} \) denote the category of \( LM \)-fuzzy topological spaces and continuous mappings.

In the following of this section, we give some definitions and results about \( LM \)-fuzzy topology.

**Definition 2.2.** [3] Let \( \tau \) be an \( LM \)-fuzzy topology on \( X \).

(1) \( B : L^X \to M \) is called a base of \( \tau \) if \( B \) satisfies the following condition:

\[
\forall A \in L^X, \ \tau(A) = \bigvee_{B_\lambda = A, \lambda \in \Lambda} \bigwedge_{B_\lambda = A, \lambda \in \Lambda} B(B_\lambda),
\]

where the expression \( \bigvee_{B_\lambda = A, \lambda \in \Lambda} \bigwedge_{B_\lambda = A, \lambda \in \Lambda} B(B_\lambda) \) will be denoted by \( B^{(\lambda)}(A) \).

(2) \( \phi : L^X \to M \) is called a subbase of \( \tau \) if \( \phi^{(\tau)} : L^X \to M \) is a base of \( \tau \), where \( \phi^{(\tau)}(A) = \bigvee_{B_\lambda = A, \lambda \in J} \bigwedge_{B_\lambda = A, \lambda \in J} \phi(B_\lambda) \) for all \( A \in L^X \) with \( (\tau) \) standing for “finite intersection”.

**Lemma 2.3.** [3] Let \( \tau \) be an \( LM \)-fuzzy topology on \( X \). Then \( \phi : L^X \to M \) is the subbase of \( \tau \) if and only if \( \phi^{(\tau)}(1_X) = 1 \).

**Definition 2.4.** [21] Let \( (X, \tau) \) be an \( LM \)-fuzzy topological space. If \( \tau(A) = \bigwedge_{v \in {\text{Copr}}(L)} \tau(v_{i_v(A)}) \) holds for all \( A \in L^X \), then \( (X, \tau) \) is called an induced \( LM \)-fuzzy topological space. Let \( ILM\text{-FTop} \) denote the category of induced \( LM \)-fuzzy topological spaces and continuous mappings.
Definition 2.5. [21] Let \((X, \tau)\) be an LM-fuzzy topological space. If \(\tau(A) \leq \bigwedge_{t \in \text{Copr}(L)} \tau(1_{t(A)})\) holds for all \(A \in L^X\), then \((X, \tau)\) is called a weakly induced LM-fuzzy topological space. Let \(\text{WILM-FTop}\) denote the category of weakly induced LM-fuzzy topological spaces and continuous mappings.

Definition 2.6. [16, 18] Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of LM-fuzzy topological spaces and \(p_t : \prod_{t \in T} X_t \rightarrow X_t\) be the projection. Then the LM-fuzzy topology on \(\prod_{t \in T} X_t\) whose subbase is defined by

\[
\forall A \in L\prod_{t \in T} X_t, \quad \phi(A) = \bigvee_{t \in T} \tau_t(B)
\]

is called the product LM-fuzzy topology of \(\{\tau_t\}_{t \in T}\), denoted by \(\prod_{t \in T} \tau_t\), and \((\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)\) is called the product space of \(\{(X_t, \tau_t)\}_{t \in T}\).

Definition 2.7. [16] Let \(\{(X_t, \tau_t)\}_{t \in T}\) be a family of LM-fuzzy topological spaces, different \(X_t\)'s be disjoint and \(X = \bigcup_{t \in T} X_t\), and let \(\tau : L^X \rightarrow M\) be defined as follows:

\[
\forall A \in L^X, \quad \tau(A) = \bigwedge_{t \in T} \tau_t(A | X_t)
\]

Then it is easy to verify that \(\tau\) is an LM-fuzzy topology on \(X\), and \(\tau\) is called the sum LM-fuzzy topology of \(\{\tau_t\}_{t \in T}\), denoted by \(\bigoplus_{t \in T} \tau_t\). \(\bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t\) is called the sum space of \(\{(X_t, \tau_t)\}_{t \in T}\).

Definition 2.8. [21] Let \((X, \tau)\) be an LM-fuzzy topological space and \(f : X \rightarrow Y\) be a surjective mapping. It is easy to verify that \(\tau/f^\ast_L : L^Y \rightarrow M\) is an LM-fuzzy topology on \(Y\), where \(\tau/f^\ast_L\) is defined by

\[
\forall A \in L^Y, \quad \tau/f^\ast_L(A) = \tau(f^\ast_L(A)).
\]

\(\tau/f^\ast_L\) is called the LM-fuzzy quotient topology of \(\tau\) with respect to \(f\), and \((Y, \tau/f^\ast_L)\) is called the LM-fuzzy quotient space of \((X, \tau)\) with respect to \(f\).

Definition 2.9. [16, 18] Let \((X, \tau)\) be an LM-fuzzy topological space and \(Y \subset X\). We call \((Y, \tau|Y)\) a subspace of \((X, \tau)\), where \(\tau|Y : L^Y \rightarrow M\) is defined by

\[
\forall B \in L^Y, \quad \tau|Y(B) = \bigvee \{\tau(A) \mid A \in L^X, A|Y = B\}.
\]

Lemma 2.10. [21] Let \(f : (X, \tau) \rightarrow (Y, \delta)\) be a mapping and \(\phi\) be a subbase of \(\delta\). If \(\tau(f^\ast_L(B)) \geq \phi(B)\) for all \(B \in L^Y\), then \(f\) is continuous.

3. (IC)LM-fuzzy Topological Spaces

Definition 3.1. An LM-fuzzy topological space \((X, \tau)\) is said to be an (IC)LM-fuzzy topological space if for all \(A \in L^X\), \(\tau(A) \leq \tau(1_{\text{int}(A)})\).

It is easy to verify that, by Definition 2.4, Definition 2.5 and Definition 3.1, the induced and weakly induced LM-fuzzy topological spaces are the (IC)LM-fuzzy topological space. However, the following example shows that an (IC)LM-fuzzy topological space needn’t be a weakly induced LM-fuzzy topological space.
Example 3.2. Let $L = M$ be the locally multiplicative completely distributive lattices. We consider the map $\tau : L^X \rightarrow M$ defined by

$$\tau(A) = (\bigvee_{x \in X} A(x)) \rightarrow ((\bigwedge_{x \in X} A(x)).$$

Then $\tau$ is an (IC)LM-fuzzy topology on $X$.

Proof. Firstly, $\tau$ is an LM-fuzzy topology on $X$ (see [7], Example 4.2.1(a)). Secondly, we prove that $\tau$ is an (IC)LM-fuzzy topology on $X$. It suffices to show that for all $A \in L^X$, $\tau(A) \leq \tau(1_{i_0(A)})$. Let us suppose that $A \neq 0_X$. (It is easy to prove the conclusion if $A = 0_X$.) If there exists an $x \in X$ such that $A(x) = 0$, then $\tau(A) = 0 = \tau(1_{i_0(A)})$; if $A(x) \neq 0$ for all $x \in X$, then $\tau(1_{i_0(A)}) = 1$. Both of them imply that $\tau(A) \leq \tau(1_{i_0(A)})$ (\forall A \in L^X). Then $\tau$ is an (IC)LM-fuzzy topology on $X$.

Generally, the $\tau$ defined above needn’t be a weakly induced LM-fuzzy topology. For example, let $X = \{x_1, x_2\}$ and $L = M = \{0, a, b, 1\}$ be the Diamond lattices. Then $\tau$ defined above is an (IC)LM-fuzzy topology on $X$, but it is not a weakly induced LM-fuzzy topology. In fact, let

$$A(x) = \begin{cases} a, x = x_1 \\ 1, x = x_2 \end{cases}$$

Then $\tau(A) = 1 \rightarrow a = a$ and $\tau(1_{i_0(A)}) = 1 \rightarrow 0 = 0$. Hence, $\tau(A) \neq \bigwedge_{r \in Copr(L)} \tau(1_{i_r(A)})$, which implies that $\tau$ is not a weakly induced LM-fuzzy topology.

Theorem 3.3. Let $(X, \tau)$ be an (IC)LM-fuzzy topological space and $Y \subset X$. Then the subspace $(Y, \tau|_Y)$ is also an (IC)LM-fuzzy topological space.

Proof. For all $A \in L^Y$, we have $\tau|_Y(A) = \bigvee\{\tau(B) \mid B \in L^X, B\upharpoonright Y = A\} \leq \bigvee\{\tau(1_{i_0(B)}) \mid B \in L^X, 1_{i_0(B)}\upharpoonright Y = 1_{i_0(A)}\} \leq \tau(1_{i_0(A)}(A))$, which implies that $(Y, \tau|_Y)$ is an (IC)LM-fuzzy topological space.

Theorem 3.4. Let $(X, \tau)$ be an (IC)LM-fuzzy topological space and $f : X \rightarrow Y$ be a surjective mapping. Then the LM-fuzzy quotient space $(Y, \tau/\overline{f(L)})$ of $(X, \tau)$ with respect to $f$ is an (IC)LM-fuzzy topological space.

Proof. For all $A \in L^Y$, we have $\tau/\overline{f(L)}(A) = \tau(\overline{f(L)}(A)) \leq \tau(1_{i_0(\overline{f(L)}(A)))} = \tau(\overline{f(L)}(1_{i_0(A)}))) = \tau(\overline{f(L)}(1_{i_0(A)})))$, which implies that $(Y, \tau/\overline{f(L)})$ is an (IC)LM-fuzzy topological space.

Theorem 3.5. Let $\{(X_t, \tau_t)\}_{t \in T}$ be a family of (IC)LM-fuzzy topological spaces. Then the product space $(\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t \in T}$ is an (IC)LM-fuzzy topological space.

Proof. Let $\phi$ be the subbase of $\tau$, which is given in Definition 2.6. Then, for all $A \in L^X$, we have

$$\phi(A) = \bigvee_{t \in T \{pr_t\}(B) = A} \tau_t(B)$$
Let \( \text{Theorem 3.6.} \)

\[
\text{Hence, for all } A \in L^X, \quad \tau(A) \leq \tau(L_{i_0}(A)), \text{ which implies that } (\prod_{t \in T} X_t, \prod_{t \in T} \tau_t) \text{ is an (IC)LM-fuzzy topological space.}
\]

\[
\text{Theorem 4.1.} \quad \text{Let } \{\{X_t, \tau_t\}\}_{t \in T} \text{ be a family of (IC)LM-fuzzy topological spaces, different } X_t \text{'s be disjoint. Then the sum space } \left( \bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t \right) \text{ of } \{\{X_t, \tau_t\}\}_{t \in T} \text{ is an (IC)LM-fuzzy topological space iff for all } t \in T, \quad (X_t, \tau_t) \text{ is an (IC)LM-fuzzy topological space.}
\]

\[
\text{Proof. Necessity} \quad \text{For every } t \in T, \quad A_t \in L^{X_t}, \text{ we have}
\]

\[
\tau_t(A_t) = \bigwedge_{t \in T} \tau_t(A_t^*)X_t = \tau(A^*) \leq \tau(L_{i_0}(A^*)) = \bigwedge_{t \in T} \tau_t(L_{i_0}(A^*))X_t = \tau_t(L_{i_0}(A)),
\]

where

\[
A^*(x) = \begin{cases} A(x), & x \in X_t \\ 0, & x \not\in X_t \end{cases}
\]

Thus for every \( t \in T, \quad (X_t, \tau_t) \) is an (IC)LM-fuzzy topological space.

\[
\text{Sufficiency} \quad \text{For every } A \in L^X, \text{ we have}
\]

\[
\tau(A) = \bigwedge_{t \in T} \tau_t(A|X_t) \leq \bigwedge_{t \in T} \tau_t(L_{i_0}(A|X_t)) = \bigwedge_{t \in T} \tau_t(L_{i_0}(A)|X_t) = \tau_t(L_{i_0}(A)).
\]

Hence \( \bigoplus_{t \in T} X_t, \bigoplus_{t \in T} \tau_t \) is an (IC)LM-fuzzy topological space.

4. (IC)-fication of LM-fuzzy Topology

\[
\text{Theorem 4.1.} \quad \text{Let } (X, \tau) \text{ be an LM-fuzzy topological space and } I(\tau) : L^X \rightarrow M \text{ be defined by } I(\tau)(A) = \tau(A) \land \tau(L_{i_0}(A)) \quad (\forall A \in L^X). \text{ Then } I(\tau) \text{ is the largest (IC)LM-fuzzy topology on } X \text{ which is contained in } \tau. \text{ We call } I(\tau) \text{ the interior (IC)-fication of } \tau.
\]

\[
\text{Proof.} \quad \text{For every } A \in L^X, \text{ we have}
\]

\[
I(\tau)(L_{i_0}(A)) = \tau(L_{i_0}(A)) \geq \tau(A) \land \tau(L_{i_0}(A)) = I(\tau)(A), \text{ which implies that } I(\tau) \text{ is the (IC)LM-fuzzy topology on } X \text{ which is contained in } \tau. \text{ Furthermore, let } \delta \leq \tau \text{ and } \delta \text{ be an (IC)LM-fuzzy topology. Then for all } A \in L^X, \delta(A) \leq \delta(L_{i_0}(A)) \leq \tau(L_{i_0}(A)), \text{ and thus } \delta(A) \leq \tau(A) \land \tau(L_{i_0}(A)) = I(\tau)(A), \text{ i.e., } \delta \leq I(\tau). \text{ Hence } I(\tau) \text{ is the largest (IC)LM-fuzzy topology on } X \text{ which is contained in } \tau.
\]

\[
\text{Let } (X, \tau) \text{ be an LM-fuzzy topological space and } \phi^\tau : L^X \rightarrow M \text{ be defined by}
\]

\[
\phi^\tau(A) = \begin{cases} \bigvee \{ \tau(B) | L_{i_0}(B) = A \}, & A \text{ is a characteristic function, i.e., the range of } \tau(A), \quad A \in \{0,1\} \\ \tau(A), & \text{otherwise} \end{cases}
\]
It is easy to verify that \( \phi^\tau \) is a subbase of one LM-fuzzy topology. We denote this LM-fuzzy topology by \( E(\tau) \) and call it the exterior (IC)-fication of \( \tau \) (see Theorem 4.2). \( \square \)

**Theorem 4.2.** For an LM-fuzzy topology \( \tau \) on \( X \), \( E(\tau) \) is the smallest (IC)LM-fuzzy topology on \( X \) which contains \( \tau \).

**Proof.** Firstly, we show that \( E(\tau) \) is the (IC)LM-fuzzy topology on \( X \). In fact, for every \( A \in L^X \), we have

\[
E(\tau)(A) = \bigvee_{\lambda \in \Lambda} \bigwedge_{B_\lambda = A} \bigwedge_{\gamma \in \Gamma} \bigvee_{\alpha \in \Delta} \bigwedge_{\beta \in \Lambda} \phi^\tau(C_{\lambda\beta}),
\]
and

\[
E(\tau)(1_{i_0(A)}) = \bigvee_{\lambda \in \Lambda} \bigwedge_{B_\lambda = A} \bigwedge_{\gamma \in \Gamma} \bigvee_{\alpha \in \Delta} \bigwedge_{\beta \in \Lambda} \phi^\tau(C_{\lambda\beta}) 
\geq \bigvee_{\lambda \in \Lambda} \bigwedge_{B_\lambda = A} \bigwedge_{\gamma \in \Gamma} \bigvee_{\alpha \in \Delta} \bigwedge_{\beta \in \Lambda} \phi^\tau(1_{i_0(C_{\lambda\beta})}).
\]

We only need to consider the case that \( C_{\lambda\beta} \) is not a characteristic function. Since

\[
\phi^\tau(1_{i_0(C_{\lambda\beta})}) = \bigvee \{ \tau(B) \mid i_0(B) = i_0(C_{\lambda\beta}) \} \geq \tau(C_{\lambda\beta}) = \phi^\tau(C_{\lambda\beta}),
\]
we have

\[
E(\tau)(A) \leq E(\tau)(1_{i_0(A)}),
\]
which implies \( E(\tau) \) is the (IC)LM-fuzzy topology on \( X \) which contains \( \tau \).

Secondly, we prove that \( E(\tau) \) is the smallest (IC)LM-fuzzy topology on \( X \) which contains \( \tau \). Let \( \tau \leq \eta \) and \( \eta \) be an (IC)LM-fuzzy topology on \( X \). We need to prove that \( E(\tau) \leq \eta \). It suffices to show that \( \phi^\tau(A) \leq \eta(A) \) for all \( A \in L^X \). If \( A \) is a characteristic function, then

\[
\phi^\tau(A) = \bigvee \{ \tau(B) \mid i_0(B) = A \} \leq \bigvee \{ \eta(B) \mid i_0(B) = A \} \leq \bigvee \{ \eta(1_{i_0(B)}) \mid i_0(B) = A \} = \eta(A).
\]

It is clearly that \( \phi^\tau(A) \leq \eta(A) \) when \( A \) is not a characteristic function. Thus \( E(\tau) \leq \eta \). Therefore \( E(\tau) \) is the smallest (IC)LM-fuzzy topology on \( X \) which contains \( \tau \). \( \square \)

**Corollary 4.3.** \((X, \delta)\) is an (IC)LM-fuzzy topological space iff any two of \( E(\delta) \), \( I(\delta) \), \( \delta \) are equal (equivalently, all the three are equal).

**Theorem 4.4.** (1) Let \((X, \delta)\) be an LM-fuzzy topological space and \((Y, \tau)\) be an (IC)LM-fuzzy topological space. Then \( f : (X, I(\delta)) \rightarrow (Y, I(\tau)) \) is a continuous mapping iff \( f : (X, E(\delta)) \rightarrow (Y, E(\tau)) \) is.

(2) Let \((X, \delta)\) be an LM-fuzzy topological space and \((Y, \tau)\) be an (IC)LM-fuzzy topological space. Then \( f : (Y, \tau) \rightarrow (X, \delta) \) is a continuous mapping iff \( f : (Y, E(\tau)) \rightarrow (X, E(\delta)) \) is.
Proof. (1) It suffices to show necessity. Suppose that \( f : (X, \delta) \rightarrow (Y, \tau) \) is continuous. So \( \tau(B) \leq \delta(f^{-1}_L(B)) \) for every \( B \in Y \). Since \( (Y, \tau) \) is an (IC)LM-fuzzy topological space, we have \( \tau(B) \leq \tau(1_{i_0(B)}) \), and hence \( \tau(B) \leq \delta(f^{-1}_L(B)) \land \tau(1_{i_0(B)}) \leq \delta(f^{-1}_L(B)) \land \delta(f^{-1}(1_{i_0(B)})) = \delta(f^{-1}_L(B)) \land \delta(1_{i_0(f^{-1}(B))}) = I(\delta)(f^{-1}_L(B)) \), which implies that \( f : (X, I(\delta)) \rightarrow (Y, I(\tau)) = (Y, \tau) \) is a continuous mapping.

(2) It suffices to show necessity. It is sufficient to show \( \phi^A(A) \leq \delta(f^{-1}_L(A)) \) for all \( A \in L^X (\forall U \subset X) \) by the definition of \( E(\delta) \) and Lemma 2.10. Since \( f : (Y, \tau) \rightarrow (X, \delta) \) is a continuous mapping, we have \( \delta(A) \leq \tau(f^{-1}_L(A)) \). It is also because \( (Y, \tau) \) is an (IC)LM-fuzzy topological space, thus \( \tau(A) \leq \tau(1_{i_0(A)}) \), and hence \( \phi^A(A) = \bigvee \{ \delta(B) \mid i_0(B) = U \} \leq \bigvee \{ \tau(f^{-1}_L(B)) \mid i_0(B) = U \} \leq \bigvee \{ \tau(1_{i_0(f^{-1}(B))) \mid i_0(B) = U \} = \tau(f^{-1}_L(A)) \), which implies that \( f : (Y, E(\tau)) = (Y, \tau) \rightarrow (X, E(\delta)) \) is a continuous mapping. \( \square \)

Let \( i : \text{(IC)LM-FTop} \rightarrow \text{LM-FTop} \) be the inclusion functor. By Theorem 4.1, Theorem 4.2 and Theorem 4.4, we have

**Theorem 4.5.** \( I, E : \text{LM-FTop} \rightarrow \text{(IC)LM-FTop} \) are functors, \( I \text{-} i \) and \( i \text{-} E \).

**Corollary 4.6.** ICLM-FTop is an isomorphism-closed full proper subcategory of LM-FTop which is simultaneously bireflective and bico-reflective in LM-FTop, and given an LM-fuzzy topological space \( (X, \delta) \), its reflection and coreflection are given by \( id_X : (X, \delta) \rightarrow (X, I(\delta)) \) and \( id_X : (X, E(\delta)) \rightarrow (X, \delta) \) respectively, where \( id_X : X \rightarrow X \) is the identity mapping.

As every right adjoint preserves limits and every left adjoint preserves colimits, we have the following Corollaries:

**Corollary 4.7.** (1) Let \( \{(X_t, \tau_t)\}_{t \in T} \) be a family of LM-fuzzy topological spaces. Then \( \left( \prod_{t \in T} X_t, E(\prod_{t \in T} \tau_t) \right) = \left( \prod_{t \in T} X_t, \prod_{t \in T} E(\tau_t) \right) \).

(2) Let \( \{(X_t, \tau_t)\}_{t \in T} \) be a family of LM-fuzzy topological spaces, different \( X_t \)'s be disjoint. Then \( \left( \bigoplus_{t \in T} X_t, I(\bigoplus_{t \in T} \tau_t) \right) = \left( \bigoplus_{t \in T} X_t, \bigoplus_{t \in T} I(\tau_t) \right) \).

**Corollary 4.8.** (1) Let \( (X, \delta) \) be an LM-fuzzy topological space and \( Y \subset X \). Then \( E(\delta)(Y) = E(\delta)|Y \).

(2) Let \( (X, \delta) \) be an LM-fuzzy topological space and \( f : X \rightarrow Y \) be a surjective mapping. \( (Y, \delta/f^{-1} \) \( ) \) is the LM-fuzzy quotient space of \( (X, \delta) \) with respect to \( f \). Then \( I(\delta/f^{-1}) = I(\delta)/f^{-1} \).

**Theorem 4.9.** (1) Let \( (X, \tau) \) be an LM-fuzzy topological space and \( U \subset X \). Then \( I(\tau)|U \leq I(\tau)|U \).

(2) Let \( (X, \delta) \) be an LM-fuzzy topological space and \( f : X \rightarrow Y \) be a surjective mapping. \( (Y, \delta/f^{-1}) \) is the LM-fuzzy quotient space of \( (X, \delta) \) with respect to \( f \). Then \( E(\delta/f^{-1}) \leq E(\delta)/f^{-1} \).

**Proof.** (1) It is easy to verify that \( I(\tau)|U \leq I(\tau)|U \) by Theorem 3.3 and the definition of \( I(\delta) \).
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(2) It is easy to verify that \( E(\delta/fL^+_B) \leq E(\delta)/fL^+_A \) by Theorem 3.4 and the definition of \( E(\delta) \).

\[ \square \]

**Remark 4.10.** The following two counterexamples show that the above inequality in Theorem 4.9 cannot be replaced by equalities.

1. Let \( X = L = M = [0, 1], U = [0, 0.5], \tau : LX \to M \) is defined by

\[ \tau(0_X) = \tau(1_X) = \tau([0.5] \land 1_U) = 1 \]

and for others \( A \in LX, \tau(A) = 0. \)

It is easy to verify that \( I(\tau) : LX \to M \) is defined by

\[ I(\tau)(0_X) = I(\tau)(1_X) = 1 \]

and for others \( A \in LX, I(\tau)(A) = 0, \)

and hence \( I(\tau)|U : LU \to M \) is defined by

\[ I(\tau)|U(0_U) = I(\tau)|U(1_U) = 1 \]

and for others \( B \in LU, I(\tau)|U(B) = 0. \)

On the other hand, we have \( \tau|U : LU \to M \) defined by

\[ \tau|U(0_U) = \tau|U(1_U) = \tau([0.5] \land 1_U) = 1 \]

and hence \( I(\tau)|U = \tau(U). \) Therefore, \( I(\tau)|U \leq I(\tau|U) \) and \( I(\tau|U) \neq I(\tau|U). \)

2. Let \( X = [-1, 1], Y = L = M = [0, 1], \delta : LX \to M \) is defined by

\[ \delta(0_X) = \delta(1_X) = \delta([0.5] \land 1_{[0.5, 1]}) = \delta(1_{[-1, -0.5]}) = 1 \]

and for others \( A \in LX, \tau(A) = 0. \)

It is easy to verify that \( E(\delta) : LX \to M \) is defined by

\[ E(\delta)(0_X) = E(\delta)(1_X) = E(\delta)([0.5] \land 1_{[0.5, 1]}) = E(\delta)(1_{[-1, -0.5]}) \]

\[ = E(\delta)(1_{[-1, -0.5]} \cup [0.5, 1]) = 1 \]

and for others \( A \in LX, E(\delta)(A) = 0 \)

and \( E(\delta)/fL^+_B : LY \to M \) is defined by

\[ E(\delta)/fL^+_B(0_Y) = E(\delta)/fL^+_B(1_Y) = E(\delta)/fL^+_B(1_{[0.25, 1]}) = 1 \]

and for others \( B \in LY, E(\delta)/fL^+_B(B) = 0. \)

On the other hand, we have \( \delta/fL^+_B : LY \to M \) is defined by

\[ \delta/fL^+_B(0_Y) = \delta/fL^+_B(1_Y) = 1 \]

and hence \( E(\delta)/fL^+_B = \delta/fL^+_B. \) Therefore, \( E(\delta)/fL^+_B \leq E(\delta)/fL^+_B \) and \( E(\delta)/fL^+_B \neq E(\delta)/fL^+_B. \)

Moreover, we also have

**Theorem 4.11.** Let \( \{(X_t, \delta_t)\}_{t \in T} \) be a family of LM-fuzzy topological spaces, different \( X_t \)'s be disjoint. Then \( \bigoplus_{t \in T} E(\delta_t) = E(\bigoplus_{t \in T} \delta_t). \)

**Proof.** \( E(\delta) \leq \bigoplus_{t \in T} E(\delta_t) \) by Theorem 3.6 and the definition of \( E(\delta). \) Conversely, let \( \lambda \in Copr(L) \) and \( \lambda \cup \bigoplus_{t \in T} E(\delta_t)(A) (\forall A \in LX), \) i.e.,

\[ \lambda \cup \bigoplus_{t \in T} E(\delta_t)(A) = \bigwedge_{t \in T} E(\delta_t)(A|X_t) \]
\[= \bigwedge_{t \in T} \bigvee_{\lambda \in \Lambda^t} \bigwedge_{C_{\lambda} = A \cup \lambda} \bigvee_{\beta \in \Lambda^t} \bigwedge_{D^{\lambda}_t = D^\lambda_A} \phi_{\delta_t}(E^\lambda_{\beta \lambda}).\]

Then for all \(t \in T\), there exists \(\{D^\lambda_t\}_{\lambda \in \Lambda^t} \subset L^{X_t}\) such that

(i) \(\bigvee_{\lambda \in \Lambda^t} D^\lambda_t = A|X_t;\)

(ii) For each \(\lambda \in \Lambda^t\), there exists \(\{E^\lambda_{\beta \lambda}\}_{\beta \in \Lambda^t} \subset L^{X_t}\) such that \(\bigcap_{\beta \in \Lambda^t} E^\lambda_{\beta \lambda} = D^\lambda_t;\)

(iii) For each \(\beta \in \Lambda^t\), we have \(\lambda \leq \phi_{\delta_t}(E^\lambda_{\beta \lambda}).\)

Let \((D^\lambda_t)^* \in L^X, (E^\lambda_{\beta \lambda})^* \in L^X\) be defined as follows:

\[(D^\lambda_t)^*(x) = \begin{cases} D^\lambda_t(x), & x \in X_t \\ 0, & x \notin X_t \end{cases} \]

\[(E^\lambda_{\beta \lambda})^*(x) = \begin{cases} E^\lambda_{\beta \lambda}(x), & x \in X_t \\ 0, & x \notin X_t \end{cases} \]

Then we have

\[\bigvee_{t \in T} \bigvee_{\lambda \in \Lambda^t} (D^\lambda_t)^* = A, (\cap_{\beta \in \Lambda^t} (E^\lambda_{\beta \lambda})^*) = (D^\lambda_t)^* \text{ and } \phi_{\delta_t}(E^\lambda_{\beta \lambda}) = \phi_{\bigvee_{t \in T} \delta_t}((E^\lambda_{\beta \lambda})^*).\]

Hence \(\lambda \leq \phi_{\bigvee_{t \in T} \delta_t}((E^\lambda_{\beta \lambda})^*).\) Note that

\[E(\bigoplus_{t \in T} \delta_t)(A) = \bigvee_{t \in T} \bigwedge_{\lambda \in \Lambda^t} A \cup \lambda \bigvee_{\beta \in \Lambda^t} C_{\beta \lambda} = B_{\lambda} \bigvee_{\beta \in \Lambda^t} \phi_{\bigvee_{t \in T} \delta_t}(C_{\beta \lambda}).\]

We have \(\lambda \leq E(\bigoplus_{t \in T} \delta_t)(A),\) and hence \(\bigoplus_{t \in T} E(\delta_t) \leq E(\delta).\)

**Remark 4.12.** Let \(\{X_t, \delta_t\}_{t \in T}\) be a family of LM-fuzzy topological spaces. Then \((\prod_{t \in T} X_t, I(\prod_{t \in T} \delta_t)) \geq (\prod_{t \in T} X_t, \prod_{t \in T} I(\delta_t))\), and the inequality cannot be replaced by equality.

**Proof.** It is easy to verify that \((\prod_{t \in T} X_t, I(\prod_{t \in T} \delta_t)) \geq (\prod_{t \in T} X_t, \prod_{t \in T} I(\delta_t))\) by Theorem 3.5 and the definition of \(I(\delta)\). The following example shows that the inequality cannot be replaced by equality.

Let \(X = L = M = [0, 1], \tau : L^X \rightarrow M\) is defined by

\[\tau(0_X) = \tau(1_X) = \tau([0.5] \land 1_{[0,0.5]}) = \tau(1_{[0.5,1]}) = 1\]

and for others \(A \in L^X, \tau(A) = 0.\)

Then \(I(\tau) : L^X \rightarrow M\) is defined by

\[I(\tau)(0_X) = I(\tau)(1_X) = I(\tau)(1_{[0.5,1]}) = 1\]

and for others \(A \in L^X, I(\tau)(A) = 0,\) and hence

\[I(\tau) \times I(\tau)(([0.5] \land 1_X \times [0,0.5], [0,0.5] \times X) \lor 1_{[0,0.5] \times [0,0.5,1]} = 0.\]

On the other hand, we have

\[\tau \times \tau(([0.5] \land 1_X \times [0,0.5], [0,0.5] \times X) \lor 1_{[0,0.5] \times [0,0.5,1]} = 1\]

and

\[I(\tau \times \tau)(([0.5] \land 1_X \times [0,0.5], [0,0.5] \times X) \lor 1_{[0,0.5] \times [0,0.5,1]} = 1.\]
Therefore $I(\tau) \times I(\tau) \leq I(\tau \times \tau)$ and $I(\tau) \times I(\tau) \neq I(\tau \times \tau)$. □

References


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