(IC)LM-FUZZY TOPOLOGICAL SPACES

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ABSTRACT. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of (weakly) induced LM-fuzzy topological spaces. We discuss the basic properties of (IC)LM-fuzzy topological spaces, and introduce the notions of interior (IC)-fication and exterior (IC)-fication of LM-fuzzy topologies and prove that **ICLM-FTop** (the category of (IC)LM-fuzzy topological spaces) is an isomorphism-closed full proper subcategory of **LM-FTop** (the category of LM-fuzzy topological spaces) and **ICLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **LM-FTop**.

1. Introduction

Since Chang [2]introduced fuzzy theory in topology, many authors discussed various aspects of fuzzy topology. However, in a completely different direction, Höhle [6] created the notion of a topology being viewed as an L-subset of a powerset (in his case, 2^X). Then Kubiak [8] and Šostak [17] independently extended Höhle's notion to L-subsets of L^X.

In 2007, Yue studied induced and weakly induced LM-fuzzy topological spaces, and proved that **WILM-FTop** (the category of weakly induced LM-fuzzy topological spaces and continuous mappings) is a simultaneously reflective and coreflective full subcategory of **LM-FTop** (the category of LM-fuzzy topological spaces and continuous mappings) [21]. In 2011, Yao showed that, for L a frame, the category of enriched L-fuzzy topological spaces can be embedded in that of L-generalized convergence spaces as a reflective subcategory and the latter is a cartesian- closed topological category [20]. Moreover, in 2007, Li, Li and Fu defined and studied the properties of (IC)L-cotopological spaces [10], which is the generalization of Martin's weakly induced I-topological spaces [13] (particularly, Weiss's induced spaces [19] and Lowen's topologically generated spaces [11]), and showed that $\mathbf{ICLTop}^c$ (the category of (IC)L-cotopological spaces and continuous mappings) is simultaneously bireflective and bicoreflective in $\mathbf{LTop}^c$ (the category of L-cotopological spaces and continuous mappings), which implies that (IC)L-cotopological space has the similar character with weakly induced spaces. The aim of the present paper is to define and study (IC)LM-fuzzy topological spaces, a generalization of weakly induced

Received: February 2011; Revised: October 2011; Accepted: February 2012

Key words and phrases: LM-fuzzy topology, (IC) LM-fuzzy topological spaces, (IC)-fication of LM-fuzzy topology, Category.

This paper is supported by the Natural Science Foundation of China (11271927),the Foundation of Shaanxi Province (Grant No. 12JK853) and the Foundation of XPU (Grant No. 09XG10).

and induced LM-fuzzy topological spaces. We discuss the basic properties of LM-fuzzy topological spaces, and introduce the notions of interior (IC)-fication and exterior (IC)-fication of LM-fuzzy topological spaces and prove that **ICLM-FTop** (the category of (IC)LM-fuzzy topological spaces and continuous mappings) is an isomorphism-closed full proper subcategory of **LM-FTop** and **ICLM-FTop** is a simultaneously bireflective and bicoreflective full subcategory of **LM-FTop**.

For needed categorical notions, please refer to [1, 7, 12, 15].

2. Preliminaries

We now give some definitions and results to be used in this paper. Throughout this paper, L always stands for a completely distributive lattice with locally multiplicative property and M is a completely distributive lattice with the least element 0 and the greatest element 1 ($0 \neq 1$). Apparently, the powerset L^X (i.e. the set of all L-subset) with the point-wise order is also a completely distributive lattice with locally multiplicative property, in which the least element and the greatest element of L^X will be written as 0_X and 1_X. An element $r \in L - \{0\}$ is called a co-prime element if, for any finite subset $K \subset L$ satisfying $r \leq \bigvee K$ (the supremum of K), there exists a $k \in K$ such that $r \leq k$. An element $s \in L - \{1\}$ is called a prime element if it is a co-prime element of L^{op} (the opposite lattice [4] of L). The set of all co-prime elements (resp., prime elements) of L will be written as $Copr(L)$ (resp., $Pr(L)$). We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$), if for every directed (arbitrary) subset $D \subset L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. From [5], we know that $Copr(L)$ is a join-generating set of L if L is a completely distributive lattice. Hence every element in L is also the supremum of all the coprimes wedge below it. By the definition of completely distributive lattice it is easy to see that a complete lattice L is completely distributive iff the operator $\bigvee : Low(L) \to L$ taking every lower set to its supremum has a left adjoint β, and in the case, $\beta(a) = \{b \mid b \lhd a\}$ for all $a \in L$. Hence the wedge below relation has the interpolation property in a completely distributive lattice, that is, $a \lhd b$ implies there is some $c \in L$ such that $a \lhd c \lhd b$.

The way-below relation on a completely distributive lattice L is called locally multiplicative [22] if for every coprime $a \in Copr(L)$, $a \ll b$ and $a \ll c$ implies $a \ll b \wedge c$ for all $b,\ c \in L$. If a is a coprime, then $a \ll b$ if and only if $a \lhd b$ (see [5, 22]). Hence L is locally multiplicative if for every coprime a, $a \lhd b$ and $a \lhd c$ implies $a \lhd b \wedge c$ for all $b,\ c \in L$. Clearly, [0,1] is locally multiplicative.

For $A \in L^X$ and $p \in L$, let $\hat{\iota}_p(A) = \{x \mid A(x) \lhd p\}$, called the strong p-cut of A. Let 1_U denote the characteristic function of $U \in 2^X$, and $[a]$ denote the L-subset taking constant value a, where 2^X is the power set of X and $a \in L$. Each mapping $f : X \longrightarrow Y$ induces a mapping $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-forward powerset operator, cf. [14]), defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee \{A(x) \mid f(x) = y\} \quad (\forall A \in L^X, \forall y \in Y).$$

The right adjoint to $f_L^{\rightarrow}$ (called L-backward powerset operator, cf. [14]) is denoted as $f_L^{\leftarrow}$ and given by

$$f_L^{\leftarrow}(B) = \bigvee\{A \in L^X \mid f_L^{\rightarrow}(A) \leq B\} = B \circ f \quad (\forall B \in L^Y).$$

It is known that $f_L^{\rightarrow}$ preserves arbitrary unions and that $f_L^{\leftarrow}$ preserves arbitrary unions, arbitrary intersections, and complements when they exists (canonical examples of such morphisms are given in [14]). Let L be a locally multiplicative completely distributive lattice. It is easy to verify the following

Lemma 2.1. [9, 21] *(1) For each $p \in L$, $\hat{\iota}_p : L^X \to 2^X = \{E \mid E \subset X\}$ preserves arbitrary suprema and finite meets.*

(2) For each $p \in L$, $A \in L^X$, $B \in L^Y$ and any mapping $f : X \to Y$, we have $\hat{\iota}_p(f_L^{\rightarrow}(A)) = f^{\rightarrow}(\hat{\iota}_p(A))$, $\hat{\iota}_p(f_L^{\leftarrow}(B)) = f^{\leftarrow}(\hat{\iota}_p(B))$.

(3) $A = \bigvee_{p\in L}([p] \wedge 1_{\hat{\iota}_p(A)}) = \bigvee_{p\in Copr(L)}([p] \wedge 1_{\hat{\iota}_p(A)})$.

An LM-fuzzy topology (or a Kubiak-Šostak fuzzy topology [8, 17]) on a set X is defined to be a mapping $\tau : L^X \longrightarrow M$ satisfying:

(**FT1**) $\tau(1_X) = \tau(0_X) = 1$;

(**FT2**) $\forall A,\ B \in L^X,\ \tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$;

(**FT3**) $\tau(\bigvee_{t\in T} A_t) \geq \bigwedge_{t\in T} \tau(A_t)$, for every family $\{A_t \mid t \in T\} \subset L^X$.

The value $\tau(A)$ can be interpreted as the degree of openness of A. A continuous mapping between LM-fuzzy topological spaces is a mapping $f : (X, \tau) \longrightarrow (Y, \delta)$ such that $\delta(B) \leq \tau(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, where $f_L^{\leftarrow}(B)$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$([8,16-17]). When $L = \{0, 1\}$, this definition reduced to that of M-fuzziying topology. Let **LM-FTop** denote the category of LM-fuzzy topological spaces and continuous mappings.

In the following of this section, we give some definitions and results about LM-fuzzy topology.

Definition 2.2. [3] Let τ be an LM-fuzzy topology on X.

(1) $\mathcal{B} : L^X \longrightarrow M$ is called a base of τ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in L^X,\ \tau(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A}\ \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda),$$

where the expression $\bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.

(2) $\phi : L^X \longrightarrow M$ is called a subbase of τ if $\phi^{(\sqcap)} : L^X \longrightarrow M$ is a base of τ, where $\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\lambda\in J} B_\lambda = A} \bigwedge_{\lambda\in J} \phi(B_\lambda)$ for all $A \in L^X$ with $(\sqcap)$ standing for "finite intersection".

Lemma 2.3. [3] *Let τ be an LM-fuzzy topology on X. Then $\phi : L^X \longrightarrow M$ is the subbase of τ if and only if $\phi^{(\sqcup)}(1_X) = 1$.*

Definition 2.4. [21] Let (X, τ) be an LM-fuzzy topological space. If $\tau(A) = \bigwedge_{r\in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$ holds for all $A \in L^X$, then (X, τ) is called an induced LM-fuzzy topological space. Let **ILM-FTop** denote the category of induced LM-fuzzy topological spaces and continuous mappings.

Definition 2.5. [21] Let (X,τ) be an LM-fuzzy topological space. If $\tau(A) \leq \bigwedge_{r\in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$ holds for all $A \in L^X$, then (X,τ) is called a weakly induced LM-fuzzy topological space. Let **WILM-FTop** denote the category of weakly induced LM-fuzzy topological spaces and continuous mappings.

Definition 2.6. [16, 18] Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces and $p_t : \prod_{t\in T} X_t \longrightarrow X_t$ be the projection. Then the LM-fuzzy topology on $\prod_{t\in T} X_t$ whose subbase is defined by

$$\forall A \in L^{\prod_{t\in T} X_t},\ \phi(A) = \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(B)$$

is called the product LM-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\prod_{t\in T} \tau_t$, and $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ is called the product space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.7. [16] Let $\{(X_t,\tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different $X_t's$ be disjoint and $X = \bigcup_{t\in T} X_t$, and let $\tau : L^X \longrightarrow M$ be defined as follows:

$$\forall A \in L^X,\ \tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t)$$

Then it is easy to verify that τ is an LM-fuzzy topology on X, and τ is called the sum LM-fuzzy topology of $\{\tau_t\}_{t\in T}$, denoted by $\bigoplus_{t\in T} \tau_t$. $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} \tau_t)$ is called the sum space of $\{(X_t,\tau_t)\}_{t\in T}$.

Definition 2.8. [21] Let (X,τ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. It is easy to verify that $\tau/f_L^{\rightarrow} : L^Y \longrightarrow M$ is an LM-fuzzy topology on Y, where $\tau/f_L^{\rightarrow}$ is defined by

$$\forall A \in L^Y,\ \tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)).$$

$\tau/f_L^{\rightarrow}$ is called the LM-fuzzy quotient topology of τ with respect to f, and $(Y, \tau/f_L^{\rightarrow})$ is called the LM-fuzzy quotient space of (X,τ) with respect to f.

Definition 2.9. [16, 18] Let (X,τ) be an LM-fuzzy topological space and $Y \subset X$. We call $(Y, \tau|Y)$ a subspace of (X,τ), where $\tau|Y : L^Y \longrightarrow M$ is defined by

$$\forall B \in L^Y,\ \tau|Y(B) = \bigvee\{\tau(A) \mid A \in L^X,\ A|Y = B\}.$$

Lemma 2.10. [21] *Let $f : (X,\tau) \longrightarrow (Y,\delta)$ be a mapping and ϕ be a subbase of δ. If $\tau(f_L^{\leftarrow}(B)) \geq \phi(B)$ for all $B \in L^Y$, then f is continuous.*

3. (IC)LM-fuzzy Topological Spaces

Definition 3.1. An LM-fuzzy topological space (X,τ) is said to be an (IC)LM-fuzzy topological space iff for all $A \in L^X,\ \tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$.

It is easy to verify that, by Definition 2.4, Definition 2.5 and Definition 3.1, the induced and weakly induced LM-fuzzy topological spaces are the (IC)LM-fuzzy topological space. However, the following example shows that an (IC)LM-fuzzy topological space needn't be a weakly induced LM-fuzzy topological space.

Example 3.2. Let $L = M$ be the locally multiplicative completely distributive lattices. We consider the map $\tau : L^X \longrightarrow M$ defined by

$$\tau(A) = (\bigvee_{x \in X} A(x)) \to ((\bigwedge_{x \in X} A(x)).$$

Then τ is an (IC)LM-fuzzy topology on X.

Proof. Firstly, τ is an LM-fuzzy topology on X (See [7],Example 4.2.1(a)). Secondly, we prove that τ is an (IC)LM-fuzzy topology on X. It suffices to show that for all $A \in L^X,\ \tau(A) \le \tau(1_{\hat{\iota}_0(A)})$. Let us suppose that $A \ne 0_X$ (It is easy to prove the conclusion if $A = 0_X$). If there exists an $x \in X$ such that $A(x) = 0$, then $\tau(A) = 0 = \tau(1_{\hat{\iota}_0(A)})$; if $A(x) \ne 0$ for all $x \in X$, then $\tau(1_{\hat{\iota}_0(A)}) = 1$. Both of them imply that $\tau(A) \le \tau(1_{\hat{\iota}_0(A)})$ $(\forall A \in L^X)$. Then τ is an (IC)LM-fuzzy topology on X. □

Generally, the τ defined above needn't be a weakly induced LM-fuzzy topology. For example, let $X = \{x_1, x_2\}$ and $L = M = \{0, a, b, 1\}$ be the Diamond lattices. Then τ defined above is an (IC)LM-fuzzy topology on X, but it is not a weakly induced LM-fuzzy topology. In fact, let

$$A(x) = \begin{cases} a, x = x_1 \\ 1, x = x_2 \end{cases}$$

Then $\tau(A) = 1 \to a = a$ and $\tau(1_{\hat{\iota}_b(A)}) = 1 \to 0 = 0$. Hence, $\tau(A) \nleq \bigwedge_{r \in Copr(L)} \tau(1_{\hat{\iota}_r(A)})$, which implies that τ is not a weakly induced LM-fuzzy topology.

Theorem 3.3. *Let (X, τ) be an (IC)LM-fuzzy topological space and $Y \subset X$. Then the subspace $(Y, \tau|Y)$ is also an (IC)LM-fuzzy topological space.*

Proof. For all $A \in L^Y$, we have $\tau|Y(A) = \bigvee\{\tau(B) \mid B \in L^X,\ B|Y = A\} \le \bigvee\{\tau(1_{\hat{\iota}_0(B)}) \mid B \in L^X,\ 1_{\hat{\iota}_0(B)}|Y = 1_{\hat{\iota}_0(A)}\} \le \tau|Y(1_{\hat{\iota}_0(A)})$, which implies that $(Y, \tau|Y)$ is an (IC)LM-fuzzy topological space. □

Theorem 3.4. *Let (X, τ) be an (IC)LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. Then the LM-fuzzy quotient space $(Y, \tau/f_L^{\rightarrow})$ of (X, τ) with respect to f is an (IC)LM-fuzzy topological space.*

Proof. For all $A \in L^Y$, we have $\tau/f_L^{\rightarrow}(A) = \tau(f_L^{\leftarrow}(A)) \le \tau(1_{\hat{\iota}_0(f_L^{\leftarrow}(A))})$ $= \tau(f_L^{\leftarrow}(1_{\hat{\iota}_0(A)})) = \tau/f_L^{\rightarrow}(1_{\hat{\iota}_0(A)}))$, which implies that $(Y, \tau/f_L^{\rightarrow})$ is an (IC)LM-fuzzy topological space. □

Theorem 3.5. *Let $\{(X_t, \tau_t)\}_{t \in T}$ be a family of (IC)LM-fuzzy topological spaces. Then the product space $(\prod_{t \in T} X_t, \prod_{t \in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t \in T}$ is an (IC)LM-fuzzy topological space.*

Proof. Let ϕ be the subbase of τ, which is given in Definition 2.6. Then, for all $A \in L^X$, we have

$$\phi(A) = \bigvee_{t \in T} \bigvee_{(p_t)_L^{\leftarrow}(B) = A} \tau_t(B)$$

$$\leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(B)=A} \tau_t(1_{\hat{\iota}_0(B)})$$

$$\leq \bigvee_{t\in T} \bigvee_{(p_t)_L^{\leftarrow}(C)=1_{\hat{\iota}_0(A)}} \tau_t(C)$$

$$= \phi(1_{\hat{\iota}_0(A)}).$$

Hence, for all $A \in L^X,\ \tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$, which implies that $(\prod_{t\in T} X_t, \prod_{t\in T} \tau_t)$ is an (IC)LM-fuzzy topological space. □

Theorem 3.6. *Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of (IC)LM-fuzzy topological spaces, different X_t's be disjiont. Then the sum space $(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} \tau_t)$ of $\{(X_t, \tau_t)\}_{t\in T}$ is an (IC)LM-fuzzy topological space iff for all $t \in T,\ (X_t, \tau_t)$ is an (IC)LM-fuzzy topological space.*

Proof. **Necessity** For every $t \in T,\ A_t \in L^{X_t}$, we have

$$\tau_t(A_t) = \bigwedge_{t\in T} \tau_t(A^*|X_t) = \tau(A^*) \leq \tau(1_{\hat{\iota}_0(A^*)}) = \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A^*)}|X_t) = \tau_t(1_{\hat{\iota}_0(A)}),$$

where

$$A^*(x) = \begin{cases} A(x),\ x \in X_t \\ 0,\ x \notin X_t \end{cases}$$

Thus for every $t \in T,\ (X_t, \tau_t)$ is an (IC)LM-fuzzy topological space.

Sufficiency For every $A \in L^X$, we have

$$\tau(A) = \bigwedge_{t\in T} \tau_t(A|X_t) \leq \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A|X_t)}) = \bigwedge_{t\in T} \tau_t(1_{\hat{\iota}_0(A)}|X_t) = \tau(1_{\hat{\iota}_0(A)}).$$

Hence $(\bigoplus_{t\in T} X_t,\ \bigoplus_{t\in T} \tau_t)$ is an (IC)LM-fuzzy topological space. □

4. (IC)-fication of LM-fuzzy Topology

Theorem 4.1. *Let (X, τ) be an LM-fuzzy topological space and $I(\tau) : L^X \longrightarrow M$ be defined by $I(\tau)(A) = \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)})$ $(\forall A \in L^X)$. Then $I(\tau)$ is the largest (IC)LM-fuzzy topology on X which is contained in τ. We call $I(\tau)$ the interior (IC)-fication of τ.*

Proof. For every $A \in L^X$, we have $I(\tau)(1_{\hat{\iota}_0(A)}) = \tau(1_{\hat{\iota}_0(A)}) \geq \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)}) = I(\tau)(A)$, which implies that $I(\tau)$ is the (IC)LM-fuzzy topology on X which is contained in τ. Furthermore, let $\delta \leq \tau$ and δ be an (IC)LM-fuzzy topology. Then for all $A \in L^X,\ \delta(A) \leq \delta(1_{\hat{\iota}_0(A)}) \leq \tau(1_{\hat{\iota}_0(A)})$, and thus $\delta(A) \leq \tau(A) \wedge \tau(1_{\hat{\iota}_0(A)}) = I(\tau)(A)$, i.e., $\delta \leq I(\tau)$. Hence $I(\tau)$ is the largest (IC)LM-fuzzy topology on X which is contained in τ.

Let (X, τ) be an LM-fuzzy topological space and $\phi^\tau : L^X \longrightarrow M$ be defined by

$$\phi^\tau(A) = \begin{cases} \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = A\}, & A \text{ is a characteristic function, i.e., the range of } A \text{ is } \{0,1\} \\ \tau(A), & \text{otherwise} \end{cases}$$

It is easy to verify that ϕ^τ is a subbase of one LM-fuzzy topology. We denote this LM-fuzzy topology by $E(\tau)$ and call it the exterior (IC)-fication of τ (see Theorem 4.2). $\square$

Theorem 4.2. *For an LM-fuzzy topology τ on X, $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ.*

Proof. Firstly, we show that $E(\tau)$ is the (IC)LM-fuzzy topology on X. In fact, for every $A \in L^X$, we have

$$E(\tau)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(C_{\lambda\beta}),$$

and

$$E(\tau)(1_{\hat{\iota}_0(A)}) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = 1_{\hat{\iota}_0(A)}} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(C_{\lambda\beta})$$

$$\geq \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^\tau(1_{\hat{\iota}_0(C_{\lambda\beta})}).$$

We only need to consider the case that $C_{\lambda\beta}$ is not a characteristic function. Since

$$\phi^\tau(1_{\hat{\iota}_0(C_{\lambda\beta})}) = \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = \hat{\iota}_0(C_{\lambda\beta})\} \geq \tau(C_{\lambda\beta}) = \phi^\tau(C_{\lambda\beta}),$$

we have

$$E(\tau)(A) \leq E(\tau)(1_{\hat{\iota}_0(A)}),$$

which implies $E(\tau)$is the (IC)LM-fuzzy topology on X which contains τ.

Secondly, we prove that $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ. Let $\tau \leq \eta$ and η be an (IC)LM-fuzzy topology on X. We need to prove that $E(\tau) \leq \eta$. It suffices to show that $\phi^\tau(A) \leq \eta(A)$ for all $A \in L^X$. If A is a characteristic function, then

$$\phi^\tau(A) = \bigvee\{\tau(B) \mid \hat{\iota}_0(B) = A\} \leq \bigvee\{\eta(B) \mid \hat{\iota}_0(B) = A\}$$

$$\leq \bigvee\{\eta(1_{\hat{\iota}_0(B)}) \mid \hat{\iota}_0(B) = A\} = \eta(A).$$

It is clearly that $\phi^\tau(A) \leq \eta(A)$ when A is not a characteristic function. Thus $E(\tau) \leq \eta$. Therefore $E(\tau)$ is the smallest (IC)LM-fuzzy topology on X which contains τ. $\square$

Corollary 4.3. *(X, δ) is an (IC)LM-fuzzy topological space iff any two of $E(\delta)$, $I(\delta)$, δ are equal (equivalently, all the three are equal).*

Theorem 4.4. *(1) Let (X, δ) be an LM-fuzzy topological space and (Y, τ) be an (IC)LM-fuzzy topological space. Then $f : (X, \delta) \longrightarrow (Y, \tau)$ is a continuous mapping iff $f : (X, I(\delta)) \longrightarrow (Y, I(\tau)) = (Y, \tau)$ is.*

(2) Let (X, δ) be an LM-fuzzy topological space and (Y, τ) be an (IC)LM-fuzzy topological space. Then $f : (Y, \tau) \longrightarrow (X, \delta)$ is a continuous mapping iff $f : (Y, E(\tau)) = (Y, \tau) \longrightarrow (X, E(\delta))$ is.

Proof. (1) It suffices to show necessity. Suppose that $f : (X, \delta) \longrightarrow (Y, \tau)$ is continuous. So $\tau(B) \leq \delta(f_L^{\leftarrow}(B))$ for every $B \in L^Y$. Since (Y, τ) is an (IC)LM-fuzzy topological space, we have $\tau(B) \leq \tau(1_{\hat{\iota}_0(B)})$, and hence $\tau(B) \leq \delta(f_L^{\leftarrow}(B)) \wedge \tau(1_{\hat{\iota}_0(B)}) \leq \delta(f_L^{\leftarrow}(B)) \wedge \delta(f_L^{\leftarrow}(1_{\hat{\iota}_0(B)})) = \delta(f_L^{\leftarrow}(B)) \wedge \delta(1_{\hat{\iota}_0(f_L^{\leftarrow}(B))}) = I(\delta)(f_L^{\leftarrow}(B))$, which implies that $f : (X, I(\delta)) \longrightarrow (Y, I(\tau)) = (Y, \tau)$ is a continuous mapping.

(2) It suffices to show necessity. It is sufficient to show $\phi^{\delta}(A) \leq \tau(f_L^{\leftarrow}(A))$ for all $A = 1_U \in L^X$ $(\forall U \subset X)$ by the definition of $E(\delta)$ and Lemma 2.10. Since $f : (Y, \tau) \longrightarrow (X, \delta)$ is a continuous mapping, we have $\delta(A) \leq \tau(f_L^{\leftarrow}(A))$. It is also because (Y, τ) is an (IC)LM-fuzzy topological space, thus $\tau(A) \leq \tau(1_{\hat{\iota}_0(A)})$, and hence $\phi^{\delta}(A) = \bigvee\{\delta(B) \mid \hat{\iota}_0(B) = U\} \leq \bigvee\{\tau(f_L^{\leftarrow}(B)) \mid \hat{\iota}_0(B) = U\} \leq \bigvee\{\tau(1_{\hat{\iota}_0(f_L^{\leftarrow}(B))}) \mid \hat{\iota}_0(B) = U\} = \tau(f_L^{\leftarrow}(A))$, which implies that $f : (Y, E(\tau)) = (Y, \tau) \longrightarrow (X, E(\delta))$ is a continuous mapping. □

Let $\mathbf{i}$: **(IC)LM-FTop**$\longrightarrow$ **LM-FTop** be the inclusion functor. By Theorem 4.1, Theorem 4.2 and Theorem 4.4, we have

Theorem 4.5. **I**, **E** *:* **LM-FTop**$\longrightarrow$**(IC)LM-FTop** *are functors,* **I**$\dashv$**i** *and* **i**$\dashv$ **E**.

Corollary 4.6. **ICLM-FTop** *is an isomorphism-closed full proper subcategory of* **LM-FTop** *which is simultaneously bireflective and bicoreflective in* **LM-FTop***, and given an LM-fuzzy topological space (X, δ), its reflection and coreflection are given by $id_X : (X, \delta) \longrightarrow (X, I(\delta))$ and $id_X : (X, E(\delta)) \longrightarrow (X, \delta)$ respectively, where $id_X : X \longrightarrow X$ is the identity mapping.*

As every right adjoint preserves limits and every left adjoint preserves colimits, we have the following Corollaries:

Corollary 4.7. *(1) Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces. Then $\left(\prod_{t\in T} X_t, E(\prod_{t\in T} \delta_t)\right) = \left(\prod_{t\in T} X_t, \prod_{t\in T} E(\delta_t)\right)$.*

(2) Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different X_t's be disjiont. Then $\left(\bigoplus_{t\in T} X_t, I(\bigoplus_{t\in T} \delta_t)\right) = \left(\bigoplus_{t\in T} X_t, \bigoplus_{t\in T} I(\delta_t)\right)$.

Corollary 4.8. *(1) Let (X, δ) be an LM-fuzzy topological space and $Y \subset X$. Then $E(\delta|Y) = E(\delta)|Y$.*

(2) Let (X, δ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. $(Y, \delta/f_L^{\rightarrow})$ is the LM-fuzzy quotient space of (X, δ) with respect to f. Then $I(\delta/f_L^{\rightarrow}) = I(\delta)/f_L^{\rightarrow}$.

Theorem 4.9. *(1) Let (X, τ) be an LM-fuzzy topological space and $U \subset X$. Then $I(\tau)|U \leq I(\tau|U)$.*

(2) Let (X, δ) be an LM-fuzzy topological space and $f : X \longrightarrow Y$ be a surjective mapping. $(Y, \delta/f_L^{\rightarrow})$ is the LM-fuzzy quotient space of (X, δ) with respect to f. Then $E(\delta/f_L^{\rightarrow}) \leq E(\delta)/f_L^{\rightarrow}$.

Proof. (1) It is easy to verify that $I(\tau)|U \leq I(\tau|U)$ by Theorem 3.3 and the definition of $I(\delta)$.

(2) It is easy to verify that $E(\delta/f_L^{\rightarrow}) \leq E(\delta)/f_L^{\rightarrow}$ by Theorem 3.4 and the definition of $E(\delta)$. □

Remark 4.10. The following two counterexamples show that the above inequality in Theorem 4.9 cannot be replaced by equalities.

(1) Let $X = L = M = [0,1], U = [0, 0.5]$, $\tau : L^X \longrightarrow M$ is defined by

$$\tau(0_X) = \tau(1_X) = \tau([0.5] \wedge 1_U) = 1 \text{ and for others } A \in L^X, \tau(A) = 0.$$

It is easy to verify that $I(\tau) : L^X \longrightarrow M$ is defined by

$$I(\tau)(0_X) = I(\tau)(1_X) = 1 \text{ and for others } A \in L^X,\ I(\tau)(A) = 0,$$

and hence $I(\tau)|U : L^U \longrightarrow M$ is defined by

$$I(\tau)|U(0_U) = I(\tau)|U(1_U) = 1 \text{ and for others } B \in L^U,\ I(\tau)|U(B) = 0.$$

On the other hand, we have $\tau|U : L^U \longrightarrow M$ defined by

$$\tau|U(0_U) = \tau|U(1_U) = \tau([0.5] \wedge 1_U) = 1 \text{ and for others } B \in L^U,\ \tau|U(B) = 0,$$

and hence $I(\tau|U) = \tau|U$. Therefore, $I(\tau)|U \leq I(\tau|U)$ and $I(\tau)|U \neq I(\tau|U)$.

(2) Let $X = [-1,1], Y = L = M = [0,1]$, $\delta : L^X \longrightarrow M$ is defined by

$$\delta(0_X) = \delta(1_X) = \delta([0.5] \wedge 1_{[0.5,1]}) = \delta(1_{[-1,-0.5]}) = 1$$
$$\text{and for others } A \in L^X, \tau(A) = 0.$$

It is easy to verify that $E(\delta) : L^X \longrightarrow M$ is defined by

$$E(\delta)(0_X) = E(\delta)(1_X) = E(\delta)([0.5] \wedge 1_{[0.5,1]}) = E(\delta)(1_{[0.5,1]}) = E(\delta)(1_{[-1,-0.5]})$$
$$= E(\delta)(1_{[-1,-0.5] \cup [0.5,1]}) = 1 \text{ and for others } A \in L^X, E(\delta)(A) = 0$$

and $E(\delta)/f_L^{\rightarrow} : L^Y \longrightarrow M$ is defined by

$$E(\delta)/f_L^{\rightarrow}(0_Y) = E(\delta)/f_L^{\rightarrow}(1_Y) = E(\delta)/f_L^{\rightarrow}(1_{[0.25,1]}) = 1$$
$$\text{and for others } B \in L^Y, E(\delta)/f_L^{\rightarrow}(B) = 0.$$

On the other hand, we have $\delta/f_L^{\rightarrow} : L^Y \longrightarrow M$ is defined by

$$\delta/f_L^{\rightarrow}(0_Y) = \delta/f_L^{\rightarrow}(1_Y) = 1 \text{ and for others } B \in L^Y, \delta/f_L^{\rightarrow}(B) = 0$$

and hence $E(\delta/f_L^{\rightarrow}) = \delta/f_L^{\rightarrow}$. Therefore, $E(\delta/f_L^{\rightarrow}) \leq E(\delta)/f_L^{\rightarrow}$ and $E(\delta/f_L^{\rightarrow}) \neq E(\delta)/f_L^{\rightarrow}$.

Moreover, we also have

Theorem 4.11. *Let $\{(X_t,\ \delta_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces, different X_t's be disjioint. Then $\bigoplus_{t\in T} E(\delta_t) = E(\bigoplus_{t\in T} \delta_t)$.*

Proof. $E(\delta) \leq \bigoplus_{t\in T} E(\delta_t)$ by Theorem 3.6 and the definition of $E(\delta)$. Conversely, let $\lambda \in Copr(L)$ and $\lambda \lhd \bigoplus_{t\in T} E(\delta_t)(A)$ $(\forall A \in L^X)$, i.e.,

$$\lambda \lhd \bigoplus_{t\in T} E(\delta_t)(A) = \bigwedge_{t\in T} E(\delta_t)(A|X_t)$$

$$= \bigwedge_{t\in T} \bigvee_{\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t} \bigwedge_{\lambda\in\Lambda^t} \bigvee_{(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda}=D^t_\lambda} \bigwedge_{\beta\in\Lambda^t_\lambda} \phi^{\delta_t}(E^t_{\beta\lambda}).$$

Then for all $t\in T$, there exists $\{D^t_\lambda\}_{\lambda\in\Lambda^t}\subset L^{X_t}$ such that

(i) $\bigvee_{\lambda\in\Lambda^t} D^t_\lambda = A|X_t$;

(ii) For each $\lambda\in\Lambda^t$, there exists $\{E^t_{\lambda\beta}\}_{\beta\in\Lambda^t_\lambda}\subset L^{X_t}$ such that $(\sqcap)_{\beta\in\Lambda^t_\lambda} E^t_{\beta\lambda} = D^t_\lambda$;

(iii) For each $\beta\in\Lambda^t_\lambda$, we have $\lambda\leq\phi^{\delta_t}(E^t_{\beta\lambda})$.

Let $(D^t_\lambda)^*\in L^X,\ (E^t_{\beta\lambda})^*\in L^X$ be defined as follows:

$$(D^t_\lambda)^*(x) = \begin{cases} D^t_\lambda(x),\ x\in X_t \\ 0,\ x\notin X_t \end{cases}$$

$$(E^t_{\beta\lambda})^*(x) = \begin{cases} E^t_{\beta\lambda}(x),\ x\in X_t \\ 0,\ x\notin X_t \end{cases}$$

Then we have

$$\bigvee_{t\in T}\bigvee_{\lambda\in\Lambda^t}(D^t_\lambda)^* = A, (\sqcap)_{\beta\in\Lambda^t_\lambda}(E^t_{\beta\lambda})^* = (D^t_\lambda)^* \text{ and } \phi^{\delta_t}(E^t_{\beta\lambda}) = \phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})^*).$$

Hence $\lambda\leq\phi^{\bigoplus_{t\in T}\delta_t}((E^t_{\beta\lambda})^*)$. Note that

$$E(\bigoplus_{t\in T}\delta_t)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\beta\lambda} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^{\bigoplus_{t\in T}\delta_t}(C_{\beta\lambda}).$$

We have $\lambda\leq E(\bigoplus_{t\in T}\delta_t)(A)$, and hence $\bigoplus_{t\in T} E(\delta_t)\leq E(\delta)$. □

Remark 4.12. Let $\{(X_t,\delta_t)\}_{t\in T}$ be a family of LM-fuzzy topological spaces. Then $\big(\prod_{t\in T} X_t, I(\prod_{t\in T}\delta_t)\big)\geq\big(\prod_{t\in T} X_t, \prod_{t\in T} I(\delta_t)\big)$, and the inequality cannot be replaced by equality.

Proof. It is easy to verify that $\big(\prod_{t\in T} X_t, I(\prod_{t\in T}\delta_t)\big)\geq\big(\prod_{t\in T} X_t, \prod_{t\in T} I(\delta_t)\big)$ by Theorem 3.5 and the definition of $I(\delta)$. The following example shows that the inequality cannot be replaced by equality.

Let $X = L = M = [0,1]$, $\tau: L^X\longrightarrow M$ is defined by

$$\tau(0_X) = \tau(1_X) = \tau([0.5]\wedge 1_{[0,0.5]}) = \tau(1_{(0.5,1]}) = 1$$

and for others $A\in L^X, \tau(A) = 0$.

Then $I(\tau): L^X\longrightarrow M$ is defined by

$$I(\tau)(0_X) = I(\tau)(1_X) = I(\tau)(1_{(0.5,1]}) = 1 \text{ and for others } A\in L^X, I(\tau)(A) = 0,$$

and hence

$$I(\tau)\times I(\tau)(([0.5]\wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X})\vee 1_{(0.5,1]\times(0.5,1]}) = 0.$$

On the other hand, we have

$$\tau\times\tau(([0.5]\wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X})\vee 1_{(0.5,1]\times(0.5,1]}) = 1$$

and

$$I(\tau\times\tau)(([0.5]\wedge 1_{X\times[0,0.5]\cup[0,0.5]\times X})\vee 1_{(0.5,1]\times(0.5,1]}) = 1.$$

Therefore $I(\tau) \times I(\tau) \leq I(\tau \times \tau)$ and $I(\tau) \times I(\tau) \neq I(\tau \times \tau)$. □

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