ON φ-CONTRACTIONS IN FUZZY METRIC SPACES WITH APPLICATION TO THE INTUITIONISTIC SETTING

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Abstract. We obtain two fixed point theorems for a kind of φ-contractions in complete fuzzy metric spaces, which are applied to easily deduce intuitionistic versions that improve and simplify the recent results of X. Huang, C. Zhu and X. Wen.

1. **Introduction and Preliminaries**

In the last years several authors have studied the existence of fixed points for some kinds of φ-contractive mappings in fuzzy metric spaces, or, equivalently, in Menger spaces (see. e.g. [1, 4, 7, 16, 21, 28]). Recently, Huang, Zhu and Wen [15] have obtained, among other results, an extension of Golet's fixed point theorem [7, Theorem 3] to intuitionistic fuzzy metric spaces.

Here we shall obtain two fixed point theorems for a kind of φ-contractive mappings on complete fuzzy metric spaces from which we easily deduce fixed point results for complete intuitionistic fuzzy metric spaces. Such results improve in several directions the main fixed point theorem of [15], and also provide a simplified proof of it.

Throughout this paper the letters $\mathbb{N}$ and $\mathbb{R}$ will denote the set of positive integer numbers and the set of real numbers, respectively.

Next we recall some pertinent concepts and facts.

According to [29] by a continuous t-norm we mean a binary operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $a * 1 = a$ for every $a \in [0,1]$; (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$; (iv) $*$ is continuous.

Similarly, by a continuous t-conorm [29] we mean a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (i) $\diamond$ is associative and commutative; (ii) $a \diamond 0 = a$ for every $a \in [0,1]$; (iii) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$; (iv) $\diamond$ is continuous.

It is well known, and easy to see, that for each continuous t-norm $*$ and each continuous t-conorm $\diamond$, one has $* \leq \wedge \leq \vee \leq \diamond$, where, as usual, $\wedge$ denotes

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the continuous t-norm of the minimum and $\vee$ denotes the continuous t-conorm of maximum.

It is also well known that if $*$ is a continuous t-norm (respectively, a continuous t-conorm), then $*'$ is a continuous t-conorm (respectively, a continuous t-norm), where $a *' b = 1 - [(1-a) * (1-b)]$ for all $a, b \in [0,1]$.

Now let $*$ be a continuous t-norm, $\diamond$ a continuous t-conorm, X a (non-empty) set and M, N fuzzy sets in $X \times X \times [0, \infty)$. Consider the following conditions for all $x, y, z \in X$:

(M1) $M(x, y, 0) = 0$;
(M2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(M3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(M4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s \geq 0$;
(M5) $M(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous;
(N1) $N(x, y, 0) = 1$;
(N2) $N(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;
(N3) $N(x, y, t) = N(y, x, t)$ for all $t > 0$;
(N4) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $t, s \geq 0$;
(N5) $N(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous;
(MN) $M(x, y, t) + N(x, y, t) \leq 1$.

Then, we have the following notions:

If M satisfies conditions (M1), (M2), (M3), (M4) and (M5), then the pair $(M, *)$ is called a fuzzy metric on X (in the sense of Kramosil and Michalek [17]). In this case, the triple $(X, M, *)$ is called a fuzzy metric space. (Recall [8] that if $(M, *)$ is a fuzzy metric on X, then, for each $x, y \in X$, the function $M(x, y, \_) : [0, \infty) \to [0, 1]$ is non-decreasing.)

If N satisfies (N1), (N2), (N3), (N4) and (N5), we say that $(N, \diamond)$ is co-fuzzy metric on X. In this case, the triple $(X, N, \diamond)$ is called a co-fuzzy metric space.

If $(M, *)$ is a fuzzy metric on X and $(N, \diamond)$ is a co-fuzzy metric on X satisfying condition (MN), then the 4-tuple $(M, N, *, \diamond)$ is called an intuitionistic fuzzy metric on X. In this case, the 5-tuple $(X, M, N, *, \diamond)$ is called an intuitionistic fuzzy metric space [2].

Of course, if $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space, then $(X, M, *)$ is a fuzzy metric space. This simple observation will be relevant later on.

We point out that Park gave in [22] (see also [10, 27]) a more strict notion of intuitionistic fuzzy metric space which is based on the notion of a fuzzy metric due to George and Veeramani [5]. Nevertheless, Park's results and concepts which will be consider here are also valid for intuitionistic fuzzy metric spaces as defined above.

It is well known that, as in the probabilistic case, every fuzzy metric $(M, *)$ on X induces a topology τ_M on X which has as a base the family of open balls $\{B_M(x, \varepsilon, t) : \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $t > 0$ and $\varepsilon \in (0, 1)$. Furthermore τ_M is a metrizable topology (see e.g.

[9, Theorem 1]), i.e., there is a metric d on X such that the topology τ_d induced by d coincides with τ_d.

Observe that a sequence (x_n) in a fuzzy metric space $(X, M, *)$ converges to a point $x \in X$ with respect to τ_M if and only if $\lim_{n\to\infty} M(x, x_n, t) = 1$ whenever $t > 0$.

A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence (compare [6]) if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. $(X, M, *)$ is called complete if every Cauchy sequence converges with respect to τ_M.

Park showed in [22] (see also [25, 26]) that, as for fuzzy metric spaces, every intuitionistic fuzzy metric $(M, N, *, \diamond)$ on a non-empty set X induces a topology $\tau_{(M.N)}$ on X which has as a base the family of open balls $\{B_{(M,N)}(x, \varepsilon, t) : \varepsilon \in (0, 1), t > 0\}$, where $B_{(M,N)}(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1-\varepsilon \text{ and } N(x, y, t) < \varepsilon\}$ for all $x \in X$, $t > 0$ and $\varepsilon \in (0, 1)$.

Similarly to [22], we say that a sequence $(x_n)_n$ in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is a Cauchy sequence if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$. $(X, M, N, *, \diamond)$ is called complete if every Cauchy sequence converges with respect to $\tau_{(M,N)}$.

As direct consequences of condition (MN) above we have the following easy but crucial facts.

Proposition 1.1. (see e.g. [10, 25, 26]) *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then $B_{(M,N)}(x, \varepsilon, t) = B_M(x, \varepsilon, t)$ for all $x \in X$, $t > 0$ and $\varepsilon \in (0, 1)$. Hence the topologies $\tau_{(M.N)}$ and τ_M coincide on X.*

Proposition 1.2. (see e.g. [26]) *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then:*

(A) A sequence in X is a Cauchy sequence in $(X, M, N, *, \diamond)$ if and only if it is a Cauchy sequence in $(X, M, *)$.

(B) $(X, M, N, *, \diamond)$ is complete if and only if $(X, M, *)$ is complete.

We conclude this section with a typical example of an intuitionistic fuzzy metric space.

Example 1.3. [2, 15, 22] Let (X, d) be a metric space. Define $M_d, N_d : X \times X \times [0, \infty) \to [0, 1]$ by $M_d(x, y, 0) = 0$, $N_d(x, y, 0) = 1$, and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$.

Then $(X, M_d, N_d, *, \diamond)$ is an inuitionistic fuzzy metric space for each continuous t-norm $*$ and each continuous t-conorm $\diamond$, with $\tau_d = \tau_{(M,N)}$.

Moreover, from Proposition 1.2 and the well-known fact that $(X, M_d, *)$ is complete if and only (X, d) is complete, it follows that $(X, M_d, N_d, *, \diamond)$ is complete if and only if (X, d) is complete.

2. **Fixed Point Theorems for Complete Fuzzy Metric Spaces**

In [18] Matkowski proved his celebrated (partial) generalization of the Boyd-Wong fixed point theorem [3], namely: If f is a self mapping on a complete metric space (X,d) and there exists a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and $d(fx, fy) \leq \varphi(d(x,y))$ for all $x, y \in X$, then f has a fixed point.

Recently, Jachymski [16] obtained a probabilistic version of Matkowski's theorem by using continuous t-norms of Hadžić type (or h-type) [12, 13] and an appropriate notion of φ-contraction. In our main result we shall prove a fuzzy metric version of Matkowski's theorem with a different approach to the ones given in [16]. In fact, our notion of φ-contraction is motivated by the recent work of Huang, Zhu and Wen [15], where the authors extended the notion of C-contraction, as given by Hicks [14], to intuitionistic fuzzy metric spaces and obtained some fixed point theorems in this framework.

Definition 2.1. By a fuzzy φ-contraction on a fuzzy metric space $(X, M, *)$ we mean a mapping $f : X \to X$ such that there is a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and

$$M(x, y, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t),$$

for all $x, y \in X$ and $t > 0$.

Recall that if $\varphi : [0,\infty) \to [0,\infty)$ is non-decreasing and satisfies $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, then $\varphi(t) < t$ for all $t > 0$.

Theorem 2.2. *Every fuzzy φ-contraction on a complete fuzzy metric space has a unique fixed point.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a φ-contraction on $(X, M, *)$. Then, there is a function φ verifying the conditions of Definition 2.1.

Take $t_0 > 1$. Then, for each $x, y \in X$ we have $M(x, y, t_0) > 1 - t_0$, so

$$M(fx, fy, \varphi(t_0)) > 1 - \varphi(t_0).$$

Repeating the process, we obtain that

$$M(f^n x, f^n y, \varphi^n(t_0)) > 1 - \varphi^n(t_0),$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Now choose $x_0 \in X$. We show that $(f^n x_0)_n$ is a Cauchy sequence in $(X, M, *)$.

Indeed, given $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi^n(t_0) < \min\{\varepsilon, t\}$ for all $n \geq n_\varepsilon$. Let $m > n \geq n_0$. Then $m = n + k$ for some $k \in \mathbb{N}$, and thus

$$\begin{aligned} M(f^n x_0, f^m x_0, t) &= M(f^n x_0, f^n f^k x_0, t) \\ &\geq M(f^n x_0, f^n f^k x_0, \varphi^n(t_0)) \\ &> 1 - \varphi^n(t_0) \\ &> 1 - \varepsilon. \end{aligned}$$

Consequently $(f^n x_0)_n$ is a Cauchy sequence in $(X, M, *)$. So there is $z \in X$ such that the sequence $(f^n x_0)_n$ converges to z for τ_M, i.e., $\lim_{n\to\infty} M(z, f^n x_0, t) = 1$ for all $t > 0$.

Next we show that z is the unique fixed point of f.

Indeed, given $t > 0$ and $\varepsilon \in (0, t)$ there exists $n_\varepsilon \in \mathbb{N}$ such that $M(z, f^n x_0, \varepsilon) > 1 - \varepsilon$ for all $n \geq n_\varepsilon$. Hence

$$M(fz, f^{n+1}x_0, \varphi(\varepsilon)) > 1 - \varphi(\varepsilon),$$

for all $n \geq n_\varepsilon$. Since $\varphi(\varepsilon) < \varepsilon < t$, we deduce that

$$\begin{aligned} M(fz, f^{n+1}x_0, t) &\geq M(fz, f^{n+1}x_0, \varepsilon) \geq M(fz, f^{n+1}x_0, \varphi(\varepsilon)) \\ &> 1 - \varphi(\varepsilon) > 1 - \varepsilon, \end{aligned}$$

for all $n \geq n_\varepsilon$. Hence $\lim_{n\to\infty} M(fz, f^n x_0, t) = 1$ for all $t > 0$, and thus, $z = fz$.

Finally, suppose that $u \in X$ satisfies $u = fu$. Then

$$M(u, z, \varphi^n(t_0)) = M(f^n u, f^n z, \varphi^n(t_0)) \geq 1 - \varphi^n(t_0),$$

for all $n \in \mathbb{N}$. Given an arbitrary $t > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $\varphi^n(t_0) < t$ for all $n \geq n_\varepsilon$. Hence

$$M(u, z, t) \geq 1 - \varphi^n(t_0),$$

for all $n \geq n_\varepsilon$. Since $\lim_{n\to\infty} \varphi^n(t_0) = 0$, we conclude that $M(u, z, t) = 1$. Since t is arbitrary, it follows that $u = z$. The proof is complete. □

Motivated by the notions of a fuzzy contractive sequence and of a fuzzy contractive mapping given in [11], Radu introduced in [24] the notion of a strict B-contraction for which he obtained a nice fixed point theorem for any complete fuzzy metric space $(X, M, *)$ satisfying $* \geq *_L$, where by $*_L$ we denote the well-known Lukasiewicz t-norm. Later on, Mihet [19] generalized the notion of a strict B-contraction, introducing the concept of $(\varphi - k)$-B contraction. Then, he proved [19, Theorem 3.11] that if $(X, M*)$ is a complete fuzzy metric space and $f : X \to X$ is a $(\varphi - k)$-B contraction satisfying $M(x, fx, t) > 0$ for some $x \in X$ and $t > 0$, then f has a (non necessarily unique) fixed point in X.

Let us recall [19, Definition 3.6] that a self-mapping f on a fuzzy metric space $(X, M, *)$ is said to be a $(\varphi - k)$-B contraction if there is a function $\varphi \in \Phi$ and a constant $k \in (0, 1)$ satisfying the following condition

$$M(x, y, t) > 1 - \lambda \Rightarrow M(fx, fy, kt) > 1 - \varphi(\lambda),$$

for all $x, y \in X$, $t > 0$ and $\lambda \in (0, 1)$, where by Φ is denoted the class of all functions $\varphi : (0, 1) \to (0, 1)$ such that φ is an increasing bijection satisfying $\varphi(\lambda) < \lambda$ for all $\lambda \in (0, 1)$.

Note that if $\varphi \in \Phi$, then $\lim_{n\to\infty} \varphi^n(\lambda) = 0$ for all $\lambda \in (0, 1)$.

The following two easy examples illustrate the differences between Theorem 2.2 and Mihet's theorem cited above.

Example 2.3. Let $X = \{0, 1\}$, and $(M, \wedge)$ be the complete fuzzy metric on X given by $M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) = 0$ otherwise. Define $f : X \to X$ by $fx = x$ for all $x \in X$. Clearly f is a $(\varphi - k)$-B contraction for every

$\varphi \in \Phi$ and $k \in (0,1)$. So, the hypotheses of Mihet's theorem are verified. However f is not a fuzzy φ-contraction because f has no a unique fixed point. In fact, if f was a fuzzy φ-contraction for some φ satisfying the conditions of Definition 2.1, then, from $M(0,1,t) > 1-t$ we would have $M(f0, f1, \varphi(t)) > 1 - \varphi(t)$. Hence $\varphi(t) > 1$ for all $t > 1$, and thus $\lim_{n\to\infty} \varphi^n(t) \geq 1$ whenever $t > 1$, which contradicts the definition of fuzzy φ-contraction. So, indeed, we cannot apply Theorem 2.2 to this example.

Example 2.4. Let $X = [0,1]$, and $(M, \wedge)$ be the complete fuzzy metric on X given by $M(x,y,t) = |x-y|$ if $t > 0$ and $|x-y| < t$, and $M(x,y,t) = 0$ otherwise. Define $f : X \to X$ by $fx = x/2$ for all $x \in X$, and $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(0) = 0$, $\varphi(t) = 1/(n+1)$ whenever $t \in (1/(n+1), 1/n]$, $n \in \mathbb{N}$, and $\varphi(t) = t/2$ whenever $t > 1$. We prove that f is a fuzzy φ-contraction on $(X, M, *)$.

It is clear that φ is non-decreasing with $t/2 \leq \varphi(t) < t$ for all $t > 0$. Moreover, it is easy to check that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

Let $M(x,y,t) > 1-t$. If $M(x,y,t) = 0$, it follows that $t > 1$ and $|x-y| \geq t$, which is not possible because $x, y \in [0,1]$. Hence, we shall suppose that $M(x,y,t) = 1$. Then $|x-y| < t$, so $|x-y| < 2\varphi(t)$, i.e., $M(fx, fy, \varphi(t)) = 1$. We have shown that f is a fuzzy φ-contraction on $(X, M, \wedge)$, and thus the hypotheses of Theorem 2.2 are verified. However, the restriction of φ to $(0,1)$ does not belong to the class Φ because it is not one-to-one on $(0,1)$.

Note that both in Example 2.3 and Example 2.4, there exist $x, y \in X$ and $t > 0$ for which $M(x,y,t) = 0$. Answering a natural question posed by one of the referees we give other two examples that also allow us to distinguish between Theorem 2.2 and Mihet's theorem, and for which the considered fuzzy metric space $(X, M, *)$ verifies that $M(x,y,t) > 0$ for all $x, y \in X$ and $t > 0$.

Example 2.5. Let $X = [0,\infty)$ and let d be the Euclidean metric on X. Then $(X, M_d, \wedge)$ is a complete fuzzy metric space and $M_d(x,y,t) > 0$ for all $x, y \in X$ and $t > 0$ (see Example 1.3). Define $f : X \to X$ by $fx = x/4$ for all $x \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t/(2-t)$ whenever $t \in [0,1)$ and $\varphi(t) = 1$ whenever $t \geq 1$. Since $\lim_{n\to\infty} \varphi^n(t) = 1$ for all $t \geq 1$ we deduce that f is not a fuzzy φ-contraction, so we cannot apply Theorem 2.2 to this case. However, it is easy to check that f is a $(\psi - k)$-B contraction for ψ the restriction of φ to $(0,1)$ and $k = 1/2$. Moreover $M_d(x, fx, t) > 0$ for $x \in X\backslash\{0\}$ and $t > 0$, so the hypotheses of Mihet's theorem are verified.

Example 2.6. Let $(X, M_d, \wedge)$ defined as in Example 2.5, let $fx = 0$ for all $x \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ defined as in Example 2.4. Obviously, f is a φ-contraction, and hence we can apply Theorem 2.2. However, the restriction of φ to $(0,1)$ does not belong to the class Φ (see Example 2.4).

Next we generalize our notion of fuzzy φ-contraction as follows.

Definition 2.7. (compare [7, 15]) By a fuzzy (g, φ)-contraction on a fuzzy metric space $(X, M, *)$ we mean a mapping $f : X \to X$ such that there exist a bijective mapping $g : X \to X$ and a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying

$\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and

$$M(gx, gy, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t),$$

for all $x, y \in X$ and $t > 0$.

Now we extend Theorem 2.2 to the case of fuzzy (g, φ)-contractions. Actually, this will be done by a direct application of that theorem. It is interesting to point out that Mihet [20] already used a similar approach to deduce the fixed point theorem proved by Golet in [7, Theorem 3] from the ones proved by Radu in [23] for C-contractions.

Theorem 2.8. *Let $(X, M, *)$ be a complete fuzzy metric space. If $f : X \to X$ is a fuzzy (g, φ)-contraction on $(X, M, *)$, then there is a unique $z \in X$ such that $fz = gz$.*

Proof. Since f is a fuzzy (g, φ)-contraction, there exist a bijective mapping $g : X \to X$ and function φ verifying the conditions of Definition 2.7. Define $h := g^{-1}$. Then $g(h) = 1_X$.

Clearly $f(h)$ is a fuzzy φ-contraction on $(X, M, *)$: Indeed, let $M(x, y, t) > 1-t$. Since $x = g(h)x$ and $= g(h)y$, it follows that $M(f(h)x, f(h)y, \varphi(t)) > 1 - \varphi(t)$.

Therefore, by Theorem 2.2, there exists a unique $u \in X$ such that $u = f(h)u$. Let $z = hu$. Then $fz = f(h)u = u = g(h)u = gz$. Finally, if $fv = gv$, it follows that $fhgv = gv$, so gv is a fixed point of fh, and thus $u = gv$. Since $u = gz$ and g is injective we conclude that $v = z$. This completes the proof. □

3. Application to Intuitionistic Fuzzy Metric Spaces

The fixed point theorems obtained in Section 2 admit an easy and natural extension to the intuitionistic framework. In fact, them directly follow from Proposition 1.2, and Theorems 2.2 and 2.8, respectively.

Theorem 3.1. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space. If $f : X \to X$ is a fuzzy φ-contraction on the fuzzy metric space $(X, M, *)$, then f has a unique fixed point.*

Theorem 3.2. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space. If $f : X \to X$ is a fuzzy (g, φ)-contraction on the fuzzy metric space $(X, M, *)$, then there is a unique $z \in X$ such that $fz = gz$.*

In [8] Grabiec introduced the following notion of completeness for fuzzy metric spaces, in order to obtain fuzzy extensions of the classical Banach contraction principle and the Edelstein fixed point theorem, respectively.

Definition 3.3. [8] (a) A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is called G-Cauchy if for each $p \in \mathbb{N}$ and each $t > 0$, $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$.

(b) A fuzzy metric space is called G-complete if every G-Cauchy sequence converges.

Alaca, Turkoglu and Yildiz [2] generalized Definition 3.3 to the intuitionistic setting as follows.

Definition 3.4. [2] (a) A sequence $(x_n)_n$ in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be a G-Cauchy sequence if for each $p \in \mathbb{N}$ and each $t > 0$, $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ and $\lim_{n\to\infty} N(x_n, x_{n+p}, t) = 0$.

(b) An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is called G-complete if for every G-Cauchy sequence $(x_n)_n$ there is $x \in X$ such that $\lim_{n\to\infty} M(x, x_n, t) = 1$ and $\lim_{n\to\infty} N(x, x_n, t) = 0$ for all $t > 0$.

Similarly to Proposition 1.2, we have the following.

Proposition 3.5. [25, Propositions 1 and 2] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then:*

(A) A sequence in X is a G-Cauchy sequence in $(X, M, N, *, \diamond)$ if and only if it is a G-Cauchy sequence in $(X, M, *)$.

(B) $(X, M, N, *, \diamond)$ is G-complete if and only if $(X, M, *)$ is G-complete.

Obviously, every Cauchy sequence in a fuzzy metric space is a G-Cauchy sequence, and hence, every G-complete fuzzy metric space is complete. So, by Propositions 1.2 and 3.5, every G-complete intuitionistic fuzzy metric space is complete. The converse is not true, in general. In fact, there exist compact, and hence complete, fuzzy metric spaces that are not G-complete [30]. Furthermore, if we denote by e the Euclidean metric on $\mathbb{R}$, then $(\mathbb{R}, M_e, *)$ (see Example 1.3) provides a distinguished example of a complete fuzzy metric space that is not G-complete (see [31]). Hence, by Proposition 3.5, $(\mathbb{R}, M_e, N_e, *, \diamond)$ is a complete intuitionistic fuzzy metric space that is not G-complete.

Huang, Zhu and Wen discussed in [15] the existence of fixed points for (g, φ)-contractions in the realm of G-complete intuitionistic fuzzy metric spaces. To this end, they introduced the following notion.

Definition 3.6. [15, Definition 11] An intuitionistic fuzzy (g, φ)-contraction on an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is a mapping $f : X \to X$ such that there exist a bijective mapping $g : X \to X$ and a non-decreasing right continuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and

$$M(gx, gy, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t),$$

and

$$N(gx, gy, t) < t \Rightarrow N(fx, fy, \varphi(t)) < \varphi(t),$$

for all $x, y \in X$ and $t > 0$.

Then, they proved, among other, the following main result.

Theorem 3.7. [15, Theorem 1] *Let $(X, M, N, *, \diamond)$ be a G-complete intuitionistic fuzzy metric space. If $f : X \to X$ is an intuitionistic fuzzy (g, φ)-contraction on $(X, M, N, *, \diamond)$, then there is a unique $z \in X$ such that $fz = gz$.*

Note that Theorem 3.7 is a direct consequence of Theorem 3.2. In fact, G-completeness can be relaxed to completeness, and both right continuity of φ and the contractive condition of N in Definition 3.6 can be omitted.

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