MEASURES OF FUZZY SEMICOMPACTNESS IN $L$-FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, the notion of fuzzy semicompactness degrees is introduced in $L$-fuzzy topological spaces by means of the implication operation of $L$. Characterizations of fuzzy semicompactness degrees in $L$-fuzzy topological spaces are obtained, and some properties of fuzzy semicompactness degrees are researched.

1. Introduction

In 1968, Chang [3] introduced fuzzy theory into topology. In Chang’s fuzzy topology, open sets are fuzzy, but the topology comprising those open sets is a crisp subset of $I$-power set $I^X$. The notion of Chang’s fuzzy topology was extended by Goguen to $L$-topology. Later many other authors gave more results and notions concerning fuzzy topology (see [1], [2], [5]-[9], [11]-[34]).

Especially, the notion of fuzzy compactness degrees and $L$-fuzzy semicompactness are introduced in $L$-fuzzy topological spaces in [14] and [29], respectively. Based on the idea of [14], a natural problem is: Can the degrees of fuzzy semicompactness be defined in an $L$-fuzzy topological space?

The aim of this paper is to present a new notion of fuzzy semicompactness degrees in $L$-fuzzy topological spaces by means of the implication operation of $L$. We will also characterize the fuzzy semicompactness degrees in $L$-fuzzy topological spaces, and research some properties of fuzzy semicompactness degrees.

2. Preliminaries

Throughout this paper, $(L, \lor, \land, \top)$ is a complete DeMorgan frame (i.e., a complete lattice with order-reversing involution satisfying joint-infinite distributive law) [10, 21]. By \bot and \top we denote the smallest element and the largest element in $L$, respectively. The set of non-unit prime elements in $L$ is denoted by $P(L)$. The set of non-zero coprime elements in $L$ is denoted by $M(L)$. We say that $a$ is wedge below $b$ in $L$, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\lor D \geq b$ implies $d \geq a$ for some $d \in D$ [4]. $\beta(b)$ is the greatest minimal family of $b$, $\beta^*(b) = \beta(b) \cap M(L)$. $\alpha(b)$ is the greatest maximal family of $b$, $\alpha^*(b) = \alpha(b) \cap P(L)$.

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In a complete DeMorgan frame $L$, there exists a binary operation $\rightarrow$. Explicitly the implication is given by $a \rightarrow b = \bigvee \{ c \in L : a \land c \leq b \}$.

It is easy to check the following properties of $\rightarrow$.

1. $(a \rightarrow b) \geq c \iff a \land c \leq b$.
2. $a \rightarrow b = \top \iff a \leq b$.
3. $a \rightarrow \bigwedge_i b_i = \bigwedge_i (a \rightarrow b_i)$.
4. $(\bigvee_i a_i) \rightarrow b = \bigwedge_i (a_i \rightarrow b)$.
5. $a \leq b \Rightarrow a \rightarrow c \geq b \rightarrow c, \ c \rightarrow a \leq c \rightarrow b$.
6. $(a \land b) \rightarrow c = a \rightarrow (b \rightarrow c)$.

We interpret $[a \leq b]$ as the degree to which $a \leq b$, then $[a \leq b] = a \rightarrow b$.

Let $X$ be a nonempty set, $L^X$ the set of all $L$-subsets on $X$, $\bot$ the smallest element of $L^X$ and $\top$ the largest element of $L^X$. Then, $L$-fuzzy topology on a set $X$ is a mapping $T : L^X \rightarrow L$ which satisfies the following conditions:

1. $T(\bot) = T(\top) = \top$.
2. $T(U \lor V) \geq T(U) \land T(V) \ (\forall U, V \in L^X)$.
3. $T \left( \bigvee_{j \in J} U_j \right) \geq \bigwedge_{j \in J} T(U_j) \ (\forall \{U_j\}_{j \in J} \subseteq L^X)$.

The pair $(X, T)$ is called an $L$-fuzzy topological space. $T(U)$ is called the degree of openness of $U$, $T^*(U) = T(U')$ is called the degree of closedness of $U$, where $U'$ is the $L$-complement of $U$. For any family $U \subseteq L^X$, $T(U) = \bigwedge_{A \in U} T(A)$ is called the degree of openness of $U$.

For a subfamily $\Phi \subseteq L^X$, $\{A, B\}$ denotes the set of all finite subfamilies of $\Phi$. For any $a \in L$, $a$ denotes a constant value mapping from $X$ to $L$, its value is $a$.

**Definition 2.1.** [31] An $L$-fuzzy inclusion on $X$ is a mapping $\tilde{\subset} : L^X \times L^X \rightarrow L$ defined by the equality $\tilde{\subset}(A, B) = \bigwedge_{x \in X} (A'(x) \lor B(x))$.

In this paper, we will write $[A \tilde{\subset} B]$, instead of $\tilde{\subset}(A, B)$.

**Definition 2.2.** [25] Let $a \in L \setminus \{\top\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in $L^X$ is said to be

1. an $a$-shading of $G$ if for any $x \in X$, it follows that $G'(x) \lor \bigwedge_{A \in \mathcal{U}} A(x) \not\leq a$.
2. a strong $a$-shading of $G$ if $\bigwedge_{x \in X} (G'(x) \lor \bigwedge_{A \in \mathcal{U}} A(x)) \not\leq a$.

**Definition 2.3.** [25] Let $a \in L \setminus \{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{P}$ in $L^X$ is said to be

1. an $a$-remote family of $G$ if for any $x \in X$, it follows that $G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a$.
2. a strong $a$-remote family of $G$ if $\bigvee_{x \in X} (G(x) \land \bigwedge_{B \in \mathcal{P}} B(x)) \not\geq a$.
3. a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta(G'(x) \lor \bigvee_{A \in \mathcal{P}} A(x))$.
4. a strong $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta(\bigwedge_{x \in X} (G'(x) \lor \bigvee_{A \in \mathcal{P}} A(x)))$.
5. a $Q_a$-cover of $G$ if $a \leq \bigwedge_{x \in X} (G'(x) \lor \bigvee_{A \in \mathcal{P}} A(x))$.
Definition 2.4. [28] Let $\mathcal{T}$ be an $L$-fuzzy topology on $X$. For any $A \in L^X$, define a mapping $\mathcal{T}_s : L^X \rightarrow L$ by

$$\mathcal{T}_s(A) = \bigvee_{B \leq A} (\mathcal{T}(B) \wedge \bigwedge_{x \in A} \bigwedge_{y \geq B} (\mathcal{T}(D))' \wedge x \wedge y).$$

Then $\mathcal{T}_s$ is called the $L$-fuzzy semiopen operator induced by $\mathcal{T}$, where $\mathcal{T}_s(A)$ can be regarded as the degree to which $A$ is semiopen and $\mathcal{T}_s(B) = \mathcal{T}_s(B')$ can be regarded as the degree to which $B$ is semiclosed. For any family $U \subseteq L^X$, $\mathcal{T}_s(U) = \bigwedge_{A \in U} \mathcal{T}_s(A)$ is called the degree of semiopenness of $U$.

Theorem 2.5. [28] Let $\mathcal{T}$ be an $L$-fuzzy topology on $X$ and let $\mathcal{T}_s$ be the $L$-fuzzy semiopen operator induced by $\mathcal{T}$. Then $\mathcal{T}(A) \leq \mathcal{T}_s(A)$ for any $A \in L^X$.

Definition 2.6. [28, 29] A mapping $f : X \rightarrow Y$ between two $L$-fuzzy topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ is called

1. (1) semicontinuous if $\mathcal{T}_1(U) \subseteq (\mathcal{T}_1)_s(f^{-1}_L(U))$ holds for any $U \subseteq Y$.
2. irresolute if $(\mathcal{T}_1)_s(U) \subseteq (\mathcal{T}_1)_s(f^{-1}_L(U))$ holds for any $U \subseteq Y$.
3. strongly irresolute if $(\mathcal{T}_1)_s(U) \subseteq (\mathcal{T}_1)(f^{-1}_L(U))$ holds for any $U \subseteq Y$.

Definition 2.7. [29] Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space. $G \in L^X$ is said to be $L$-fuzzy semicompact if for every family $U \subseteq L^X$, it follows that

$$\bigwedge_{A \in U} \mathcal{T}_s(A) \wedge \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in U} A(x)) \leq \bigvee_{V \in 2^U} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in V} A(x)).$$

Definition 2.8. [14] Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. The fuzzy compactness degree $cd_\mathcal{T}$ of $G$ is defined as

$$cd_\mathcal{T}(G) = \bigwedge_{U \subseteq L^X} (\mathcal{T}(U) \rightarrow (\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in U} A(x)) \rightarrow \bigvee_{V \in 2^U} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in V} A(x)))).$$

Theorem 2.9. [21, 25] Let $f : X \rightarrow Y$ be a set mapping and $f^*_L : L^X \rightarrow L^Y$ is induced by $f$. Then for any $\mathcal{P} \subseteq L^X$, we have that

$$\bigwedge_{y \in Y} (f^*_L(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y)) = \bigwedge_{x \in X} (G'(x) \vee \bigvee_{B \in \mathcal{P}} f^*_L(B)(x)).$$

3. Measures of Fuzzy Semicompactness

In [14], Li and Shi generalized the notion of fuzzy compactness to $L$-fuzzy topological spaces, and gave the definition of fuzzy compactness degrees in $L$-fuzzy topological spaces. Based on [14], we will generalize the notion of fuzzy semicompactness to $L$-fuzzy topological spaces. In order to do this, let us recall fuzzy semicompactness in $L$-topology [23].

Let $(X, \mathcal{T})$ be an $L$-topological space and $G \in L^X$. $G$ is fuzzy semicompact if and only if for every family $\mathcal{U}$ of semiopen $L$-sets, it follows that

$$\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \leq \bigvee_{V \in 2^\mathcal{U}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in V} A(x)).$$
This implies that for every family $\mathcal{U}$ of semiopen $L$-sets,

$$
\left[ [G \bar{\cup} \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2^{|\mathcal{U}|}} [G \bar{\cup} \mathcal{V}] \right] = \top.
$$

We know that an $L$-topology $\mathcal{T}$ can be looked as a special $L$-fuzzy topology. Therefore, $A \in L^X$ is a semiopen set if and only if $\mathcal{T}_s(A) = \top$ [28]. Thus $G$ is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$
\mathcal{T}_s(\mathcal{U}) \leq \left[ [G \bar{\cup} \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2^{|\mathcal{U}|}} [G \bar{\cup} \mathcal{V}] \right].
$$

Therefore we can naturally generalize the notion of fuzzy semicompactness degrees to $L$-fuzzy topological spaces as follows:

**Definition 3.1.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. The fuzzy semicompactness degree $scd_{\mathcal{T}}(G)$ is defined as

$$
scd_{\mathcal{T}}(G) = \bigwedge_{U \subseteq L^X} (\mathcal{T}_s(U) \to (\bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V}] \to \bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V}] \to \bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V} \to \bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V} \to \bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V} \to \bigvee_{\mathcal{V} \in 2^{|U|}} [G \bar{\cup} \mathcal{V}]).
$$

**Theorem 3.2.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. Then

$$
scd_{\mathcal{T}}(G) \leq cd_{\mathcal{T}}(G).
$$

**Proof.** Straightforward. \(\square\)

**Theorem 3.3.** Let $(X, \mathcal{T})$ be an $L$-topological space and $G \in L^X$. $G$ is fuzzy semicompactness in $(X, \mathcal{T})$ if and only if $scd_{\chi_{\mathcal{T}}}(G) = \top$.

**Proof.** Let $(X, \mathcal{T})$ be an $L$-topological space. The mapping $\chi_{\mathcal{T}} : L^X \to L$ defined by

$$
\chi_{\mathcal{T}}(A) = \left\{ \begin{array}{ll}
\top, & A \in \mathcal{T}, \\
\bot, & A \notin \mathcal{T}.
\end{array} \right.
$$

is a special $L$-fuzzy topology. Then $A \in L^X$ is a semiopen set in $L$-topology $\mathcal{T}$ if and only if $(\chi_{\mathcal{T}})_s(A) = \top$. Thus by the definition of fuzzy semicompactness and the properties of $\to$, we know that $G$ is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$
(\chi_{\mathcal{T}})_s(\mathcal{U}) \leq \left[ [G \bar{\cup} \mathcal{U}] \leq \bigvee_{\mathcal{V} \in 2^{|\mathcal{U}|}} [G \bar{\cup} \mathcal{V}] \right].
$$

This implies that $G$ is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$
(\chi_{\mathcal{T}})_s(\mathcal{U}) \to (\bigvee_{\mathcal{V} \in 2^{|\mathcal{U}|}} [G \bar{\cup} \mathcal{V}] = \top.
$$

By the definition of $scd_{\chi_{\mathcal{T}}}$, the conclusion is hold. \(\square\)

**Theorem 3.4.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. $G$ is $L$-fuzzy semicompactness in $(X, \mathcal{T})$ if and only if $scd_{\mathcal{T}}(G) = \top$. 
Proof. By the definition of \( L \)-fuzzy semicompactness, we know that \( G \) is \( L \)-fuzzy semicompactness in \((X, T)\) if and only if for every family \( U \subseteq L^X \), it follows that
\[
(T_s(U) \land [G \tilde{c} \lor u]) \leq \bigvee_{V \in 2^U} [G \tilde{c} \lor V].
\]
By the properties of \( \rightarrow \), we obtain that \( G \) is \( L \)-fuzzy semicompactness in \((X, T)\) if and only if for every family \( U \subseteq L^X \), it follows that
\[
T_s(U) \rightarrow ([G \tilde{c} \lor u] \rightarrow \bigvee_{V \in 2^U} [G \tilde{c} \lor V]) = \top.
\]
By the definition of \( scd_T \), the conclusion is hold. \( \square \)

Lemma 3.5. Let \((X, T)\) be an \( L \)-fuzzy topological space and \( G \in L^X \). Then \( \text{scd}_T(G) \geq a \) if and only if for any \( U \subseteq L^X \),
\[
\bigwedge_{U \subseteq L^X} (T_s(U) \land [G \tilde{c} \lor u] \land a) \leq \bigvee_{V \in 2^U} [G \tilde{c} \lor V].
\]

Proof. For any \( a \in L \), \( \text{scd}_T(G) \geq a \), i.e.,
\[
\bigwedge_{U \subseteq L^X} (T_s(U) \rightarrow ([G \tilde{c} \lor u] \rightarrow \bigvee_{V \in 2^U} [G \tilde{c} \lor V])) \geq a
\]
if and only if for any \( U \subseteq L^X \),
\[
T_s(U) \rightarrow ([G \tilde{c} \lor u] \rightarrow \bigvee_{V \in 2^U} [G \tilde{c} \lor V]) \geq a
\]
if and only if (by the property (6) of \( \rightarrow \)) for any \( U \subseteq L^X \),
\[
(T_s(U) \land [G \tilde{c} \lor u]) \rightarrow \bigvee_{V \in 2^U} [G \tilde{c} \lor V] \geq a
\]
if and only if (by the property (1) of \( \rightarrow \)) for any \( U \subseteq L^X \),
\[
T_s(U) \land [G \tilde{c} \lor u] \land a \leq \bigvee_{V \in 2^U} [G \tilde{c} \lor V].
\]

Theorem 3.6. Let \((X, T)\) be an \( L \)-fuzzy topological space and \( G \in L^X \). Then \( \text{scd}_T(G) \geq a \) if and only if for any \( P \subseteq L^X \),
\[
\bigvee_{F \in P} T_s^r(F) \lor \bigvee_{x \in X} \left( G(x) \land \bigwedge_{F \in P} F(x) \right) \lor a \geq \bigwedge_{U \subseteq L^X} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{F \in H} F(x) \right).
\]

Proof. It can be easily obtained by Lemma 3.5 and the definition of \( T_s^r \). \( \square \)

Theorem 3.7. Let \((X, T)\) be an \( L \)-fuzzy topological space and \( G \in L^X \). Then
\[
\text{scd}_T(G) = \bigvee \{ a \in L : T_s(U) \land [G \tilde{c} \lor u] \land a \leq \bigvee_{V \in 2^U} [G \tilde{c} \lor V], \forall U \subseteq L^X \}.
\]
Proof. By Lemma 3.5, we know that \( \text{scd}_T(G) \) is an upper bound of
\[
\{ a \in L : T_s(\mathcal{U}) \land |G\setminus \mathcal{U}| \land a \leq \bigvee_{V \in \mathcal{U}^2} |G\setminus V|, \forall \mathcal{U} \subseteq L^X \}.
\]
Since
\[
\text{scd}_T(G) = \bigwedge_{U \subseteq L^X} (T_s(\mathcal{U}) \rightarrow (|G\setminus \mathcal{U}| \rightarrow \bigvee_{V \in \mathcal{U}^2} [G\setminus V])),
\]
then for every family \( \mathcal{U} \subseteq L^X \), we have
\[
\text{scd}_T(G) \leq T_s(\mathcal{U}) \rightarrow (|G\setminus \mathcal{U}| \rightarrow \bigvee_{V \in \mathcal{U}^2} [G\setminus V])
\]
\[
= (T_s(\mathcal{U}) \land |G\setminus \mathcal{U}|) \rightarrow \bigvee_{V \in \mathcal{U}^2} [G\setminus V].
\]
By the property (1) of \( \rightarrow \), we obtain that for every family \( \mathcal{U} \subseteq L^X \),
\[
T_s(\mathcal{U}) \land |G\setminus \mathcal{U}| \land \text{scl}_T(G) \leq \bigvee_{V \in \mathcal{U}^2} [G\setminus V],
\]
thus
\[
\text{scl}_T(G) \in \{ a \in L : T_s(\mathcal{U}) \land |G\setminus \mathcal{U}| \land a \leq \bigvee_{V \in \mathcal{U}^2} [G\setminus V], \forall \mathcal{U} \subseteq L^X \}.
\]
Therefore, the conclusion is hold. \( \square \)

In order to write simply, for any mapping \( T : L^X \rightarrow L \), denote \( T_b = \{ a \in L^X : T(a) \geq b \} \).

**Theorem 3.8.** Let \((X, T)\) be an L-fuzzy topological space and \( G \in L^X, a \in L \{ \perp \} \).
The following conditions are equivalent:

1. \( \text{scl}_T(G) \geq a \).
2. For any \( b \in P(L) \), \( b \not\in a \), each strong b-shading \( \mathcal{U} \) of \( G \) with \( T_s(\mathcal{U}) \not\subseteq b \) has a finite subfamily \( \mathcal{V} \) which is a strong b-shading of \( G \).
3. For any \( b \in P(L) \), \( b \not\in a \), each strong b-shading \( \mathcal{U} \) of \( G \) with \( T_s(\mathcal{U}) \not\subseteq b \), there exists a finite subfamily \( \mathcal{V} \) of \( \mathcal{U} \) and \( r \in a^*(b) \) such that \( \mathcal{V} \) is an \( r \)-shading of \( G \).
4. For any \( b \in P(L) \), \( b \not\in a \), each strong b-shading \( \mathcal{U} \) of \( G \) with \( T_s(\mathcal{U}) \not\subseteq b \), there exists a finite subfamily \( \mathcal{V} \) of \( \mathcal{U} \) and \( r \in a^*(b) \) such that \( \mathcal{V} \) is a strong \( r \)-shading of \( G \).
5. For any \( b \in M(L) \), \( b \not\in a' \), each strong b-remote family \( \mathcal{P} \) of \( G \) with \( T^*_s(\mathcal{P}) \not\subseteq b' \) has a finite subfamily \( \mathcal{H} \) which is a strong b-remote family of \( G \).
6. For any \( b \in M(L) \), \( b \not\in a' \), each strong b-remote family \( \mathcal{P} \) of \( G \) with \( T^*_s(\mathcal{P}) \not\subseteq b' \), there exists a finite subfamily \( \mathcal{H} \) of \( \mathcal{P} \) and \( r \in \beta^*(b) \) such that \( \mathcal{H} \) is an \( r \)-remote family of \( G \).
7. For any \( b \in M(L) \), \( b \not\in a' \), each strong b-remote family \( \mathcal{P} \) of \( G \) with \( T^*_s(\mathcal{P}) \not\subseteq b' \), there exists a finite subfamily \( \mathcal{H} \) of \( \mathcal{P} \) and \( r \in \beta^*(b) \) such that \( \mathcal{H} \) is a strong \( r \)-remote family of \( G \).
8. For any \( b \leq a, r \in \beta(b) \), \( b, r \not\perp \), each \( Q_{b, r} \)-cover \( \mathcal{U} \subseteq (T_s)_b \) of \( G \) has a finite subfamily \( \mathcal{V} \) which is a \( Q_{b, r} \)-cover of \( G \).
9. For any \( b \leq a, r \in \beta(b) \), \( b, r \not\perp \), each \( Q_{b, r} \)-cover \( \mathcal{U} \subseteq (T_s)_b \) of \( G \) has a finite subfamily \( \mathcal{V} \) which is a strong \( \beta_{b, r} \)-cover of \( G \).
(10) For any \( b \leq a, r \in \beta(b), \) \( b, r \neq \bot \), each \( Q_b \)-cover \( U \subseteq (\mathcal{T}_s)_b \) of \( G \) has a finite subfamily \( V \) which is a \( \beta \)-cover of \( G \).

(11) For any \( b \leq a, r \in \beta(b), \) \( b, r \neq \bot \), each strong \( \beta_b \)-cover \( U \subseteq (\mathcal{T}_s)_b \) of \( G \) has a finite subfamily \( V \) which is a \( \beta \)-cover of \( G \).

(12) For any \( b \leq a, r \in \beta(b), \) \( b, r \neq \bot \), each strong \( \beta_b \)-cover \( U \subseteq (\mathcal{T}_s)_b \) of \( G \) has a finite subfamily \( V \) which is a \( \beta \)-cover of \( G \).

(13) For any \( b \leq a, r \in \beta(b), \) \( b, r \neq \bot \), each strong \( \beta_b \)-cover \( U \subseteq (\mathcal{T}_s)_b \) of \( G \) has a finite subfamily \( V \) which is a \( \beta \)-cover of \( G \).

In Theorem 3.8 (8)-(13), if we replace \( b, r \neq \bot \) and \( r \in \beta(b) \) with \( b \in M(L) \) and \( r \in \beta^*(b) \), then the conclusions are still right.

**Theorem 3.9.** Let \((X, \tau)\) be an \( L \)-fuzzy topological space and \( G \in L^X, a \in L \setminus \{ \bot \} \).

If for any \( c, d \in L \), \( \beta(c \land d) = \beta(c) \land \beta(d) \). Then the following conditions are equivalent:

1. \( scd_{\tau}(G) \geq a \).
2. For any \( b \in \beta(a), \) \( b \neq \bot \), each strong \( \beta_b \)-cover \( U \) of \( G \) with \( b \in \beta(\mathcal{T}_s(U)) \) has a finite subfamily \( V \) which is a \( Q_b \)-cover of \( G \).
3. For any \( b \in \beta(a), \) \( b \neq \bot \), each strong \( \beta_b \)-cover \( U \) of \( G \) with \( b \in \beta(\mathcal{T}_s(U)) \) has a finite subfamily \( V \) which is a strong \( \beta \)-cover of \( G \).
4. For any \( b \in \beta(a), \) \( b \neq \bot \), each strong \( \beta_b \)-cover \( U \) of \( G \) with \( b \in \beta(\mathcal{T}_s(U)) \) has a finite subfamily \( V \) which is a \( \beta \)-cover of \( G \).

4. **Properties of Fuzzy Semicompactness Degrees**

**Theorem 4.1.** Let \((X, \tau)\) be an \( L \)-fuzzy topological space and \( G, H \in L^X \). Then

\[
scd_{\tau}(G \cup H) \geq scd_{\tau}(G) \land scd_{\tau}(H).
\]

**Proof.** By Theorem 3.7 we have

\[
scd_{\tau}(G \cup H) = \bigvee_{V \in \mathcal{E}(\tau)} \{ a \in L : \mathcal{T}_s(U) \land [ (G \cup H) \subset \bigvee V ] \land a \}
\]
\[
\leq \bigvee_{V \in \mathcal{E}(\tau)} \{ a \in L : \mathcal{T}_s(U) \land [ G \subset \bigvee V ] \land [ H \subset \bigvee V ] \land a \}
\]
\[
= \bigvee_{V \in \mathcal{E}(\tau)} \{ a \in L : \mathcal{T}_s(U) \land [ G \subset \bigvee V ] \land [ H \subset \bigvee V ] \land a \}
\]
\[
\leq \bigvee_{V \in \mathcal{E}(\tau)} \{ a \in L : \mathcal{T}_s(U) \land [ H \subset \bigvee V ] \land a \}
\]
\[
\leq \bigvee_{V \in \mathcal{E}(\tau)} \{ a \in L : \mathcal{T}_s(U) \land [ H \subset \bigvee V ] \land a \}
\]
\[
= scd_{\tau}(G) \land scd_{\tau}(H).\]

**Theorem 4.2.** Let \((X, \tau)\) be an \( L \)-fuzzy topological space and \( G, H \in L^X \). Then

\[
scd_{\tau}(G \land H) \geq scd_{\tau}(G) \land \mathcal{T}_s^*(H).
\]
Proof. By Theorem 3.7 we have
\[
\text{scd}_T(G \land H) = \bigvee \{a \in L : T_0(U) \land [G \land H] \subseteq \bigcup U \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [(G \land H) \subseteq \bigcup V] \forall U \subseteq L^X
\]
\[
= \bigvee_{V \in 2(U)} \{a \in L : T_0(U) \land [G \subseteq (H' \lor \bigcup U)] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [G \subseteq (H' \lor \bigcup U)] \forall U \subseteq L^X
\]
\[
\geq \bigvee_{V \in 2(U)} \{a \land T_0' (H) : T_0(U) \land [G \subseteq \bigcup U] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [G \subseteq \bigcup U] \forall U \subseteq L^X = \text{scd}_T(G) \land T_0' (H).
\]
\[\square\]

Corollary 4.3. Let \((X, T)\) be an \(L\)-fuzzy topological space and \(G \in L^X\). Then \(\text{scd}_T (G) \geq \text{scd}_T (\bot) \land T_0^* (G)\).

Theorem 4.4. Let \((X, T_1), (X, T_2)\) be two \(L\)-fuzzy topological spaces and satisfy \(T_1 \leq T_2\), \(G \in L^X\). Then \(\text{scd}_{T_1} (G) \leq \text{scd}_{T_2} (G)\).

Corollary 4.5. Let \((X, T)\) be an \(L\)-fuzzy topological space and let \(B\) be a base or subbase \([7, 8, 34]\) of \(T\), \(G \in L^X\). Then \(\text{scd}_G (G) \leq \text{scd}_B (G)\).

Theorem 4.6. Let \(f : X \rightarrow Y\) be a set mapping, \(T_1\) be an \(L\)-fuzzy topology on \(X\), \(T_2\) be an \(L\)-fuzzy topology on \(Y\), and \(f : (X, T_1) \rightarrow (Y, T_2)\) be an \(L\)-fuzzy strong irresolute mapping. Then for any \(G \in L^X\), \(\text{cd}_{T_1} (G) \leq \text{scd}_{T_2} (f_{L^+} (G))\).

Proof. For any \(G \in L^X\), we have
\[
\text{scd}_{T_2} (f_{L^+} (G)) = \bigvee \{a \in L : (T_2)_a \land [f_{L^+} (G) \subseteq \bigcup V] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [f_{L^+} (G) \subseteq \bigcup V] \forall U \subseteq L^X
\]
\[
\geq \bigvee_{V \in 2(U)} \{a \in L : T_1 (f_{L^+} (U)) \land [G \subseteq \bigcup f_{L^+} (V)] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [G \subseteq \bigcup f_{L^+} (V)] \forall U \subseteq L^X \geq \text{cd}_{T_1} (G).
\]
\[\square\]

Theorem 4.7. Let \(f : X \rightarrow Y\) be a set mapping, \(T_1\) be an \(L\)-fuzzy topology on \(X\), \(T_2\) be an \(L\)-fuzzy topology on \(Y\), and \(f : (X, T_1) \rightarrow (Y, T_2)\) be an \(L\)-fuzzy irresolute mapping. Then for any \(G \in L^X\), \(\text{scd}_{T_1} (G) \leq \text{scd}_{T_2} (f_{L^+} (G))\).

Proof. For any \(G \in L^X\), we have
\[
\text{scd}_{T_2} (f_{L^+} (G)) = \bigvee \{a \in L : (T_2)_a \land [f_{L^+} (G) \subseteq \bigcup V] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [f_{L^+} (G) \subseteq \bigcup V] \forall U \subseteq L^X
\]
\[
\geq \bigvee_{V \in 2(U)} \{a \in L : (T_2)_a \land [G \subseteq \bigcup f_{L^+} (U)] \land a \}
\]
\[
\leq \bigvee_{V \in 2(U)} [G \subseteq \bigcup f_{L^+} (U)] \forall U \subseteq L^X \geq \text{scd}_{T_1} (G).
\]
\[\square\]
Theorem 4.8. Let \( f : X \rightarrow Y \) be a set mapping, \( T_1 \) be an \( L \)-fuzzy topology on \( X \), \( T_2 \) be an \( L \)-fuzzy topology on \( Y \), and \( f : (X, T_1) \rightarrow (Y, T_2) \) be an \( L \)-fuzzy semicontinuous mapping. Then for any \( G \in L_X \), \( \text{scd}_{T_1}(G) \leq \text{cd}_{T_2}(f \rightarrow L(G)) \).

Proof. For any \( G \in L_X \), we have
\[
\text{cd}_{T_2}(f \rightarrow L(G)) = \bigvee \{a \in L : T_2(U) \land [f \rightarrow L(G) \in U] \land a \}
\leq \bigvee_{V \in \mathcal{I}(U)} [f \rightarrow L(G) \in V] \land a
\geq \bigvee \{a \in L : \langle T_1 \rangle(f \rightarrow L(U)) \land [G \in f \rightarrow L(V)] \land a \}
\leq \bigvee_{V \in \mathcal{I}(U)} [G \in f \rightarrow L(V)] \land a
\geq \text{scd}_{T_1}(G).
\]

\( \square \)

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