CLASSIFYING FUZZY SUBGROUPS OF FINITE NONABELIAN GROUPS

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Abstract. In this paper a first step in classifying the fuzzy subgroups of a finite nonabelian group is made. We develop a general method to count the number of distinct fuzzy subgroups of such groups. Explicit formulas are obtained in the particular case of dihedral groups.

1. **Introduction**

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the particular case of finite abelian groups. Thus, in [12] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [13], [14], [15] and [27] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). Also, recall here the paper [24], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: (arbitrary) finite cyclic groups and finite elementary abelian p-groups. The explicit formula obtained for the first class in [22] leads to an expression of the well-known central Delannoy numbers. Another interesting application has been presented in [25] for the case of finite hamiltonian groups.

In the present paper we extend the above study to finite nonabelian groups G. We shall use the natural equivalence relation introduced in [24] and we shall determine the number and nature of fuzzy subgroups of G with respect to this equivalence. For a different approach for classification see [7] and [8]. In our case the corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups in G. As a principle guide in determining the number of these classes, we first find the number of maximal chains of G. Note that an essential role in solving our counting problem will be played again by the Inclusion-Exclusion Principle. In many situations it leads us to some recurrence relations, whose solutions can be easily found. We shall exemplify our method for the class of finite dihedral groups. These groups are very important in abstract group theory, in combinatorics, as well as in geometry.

The paper is organized as follows. In section 2 we present some preliminary results on fuzzy subgroups and recall the main theorems of [20] and [24]. section

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3 deals with maximal subgroups of dihedral groups D_{2n} and with establishing the recurrence relations satisfied by the numbers of maximal chains of subgroups and distinct fuzzy subgroups of D_{2n}, respectively. These recurrence relations are solved in section 4. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on lattices (respectively on groups) can be found in [2] (respectively in [19]). For subgroup lattice concepts we refer the reader to [17] and [21].

2. **Preliminaries**

Let $(G, \cdot, e)$ be a group (where e denotes the identity of G) and $\mathcal{F}(G)$ is the collection of all fuzzy subsets of G. An element μ of $\mathcal{F}(G)$ is said to be a *fuzzy subgroup* of G if it satisfies the following two conditions:

a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;

b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \sup \mu(G)$. As in the case of subgroups, the set $FL(G)$ consisting of all fuzzy subgroups of G forms a lattice with respect to the usual ordering of fuzzy set inclusion, called the *fuzzy subgroup lattice* of G. For each $\alpha \in [0, 1]$, we define the level subset:

$$_{\mu}G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}.$$

These subsets allow us to characterize the fuzzy subgroups of G, in the next manner: μ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This well-known theorem gives a link between $FL(G)$ and the classical subgroup lattice $L(G)$ of G, and plays an essential role in studying many properties of $FL(G)$ (as the distributivity – see [23]).

The fuzzy subgroups of G can be classified up to some natural equivalence relations on $\mathcal{F}(G)$. One of them (used in [24] and [27], too) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \text{ for all } x, y \in G)$$

and two fuzzy subgroups μ, η of G will be called *distinct* if $\mu \not\sim \eta$. This equivalence relation generalizes that used in Murali's papers [11]-[16]. It is also closely connected to the concept of level subgroup. In this way, suppose that the group G is finite and let $\mu : G \to [0, 1]$ be a fuzzy subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > ... > \alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$_{\mu}G_{\alpha_1} \subset {}_{\mu}G_{\alpha_2} \subset ... \subset {}_{\mu}G_{\alpha_r} = G. \tag{*}$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \Longleftrightarrow i = \max\{j \mid x \in {}_{\mu}G_{\alpha_j}\} \Longleftrightarrow x \in {}_{\mu}G_{\alpha_i} \setminus {}_{\mu}G_{\alpha_{i-1}},$$

where, by convention, we set $_{\mu}G_{\alpha_0} = \emptyset$. A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [27]: $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of subgroups of type $(*)$. This result shows that *there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which end in G*. So, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G: finding the number of all chains of subgroups of G that terminate in G. Even for some particular classes of finite groups, as finite abelian groups, this problem is very difficult. The largest class of groups for which it was completely solved is constituted by finite cyclic groups (see Corollary 4 of [24]). If G is a finite cyclic group of order n (that is $G \cong \mathbb{Z}_n$) and $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ is the decomposition of n as a product of prime factors, then the number $f(n)$ (which will be also denoted by $f(\mathbb{Z}_n)$) of all distinct fuzzy subgroups of G is given by the equality

$$f(n) = 2^{\sum\limits_{\alpha=1}^{s} m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} ... \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum\limits_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum\limits_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1, for $s = 1$. An important step in order to establish a similar explicit formula for finite elementary abelian p-groups is made in section 3 of [24].

Our next goal is to describe the method that will be used in counting the chains of subgroups of G. Let $M_1, M_2, ..., M_k$ be the maximal subgroups of G and denote by $g(G)$ (respectively by $h(G)$) the number of maximal chains of subgroups in G (respectively the number of chains of subgroups of G ending in G). The technique used in finding $g(G)$ is found on the following simple remark: every maximal chain in G contains a unique maximal subgroup of G. In this way, $g(G)$ and $g(M_i)$, $i = 1, 2, ..., k$, are connected by the equality:

$$g(G) = \sum_{i=1}^{k} g(M_i). \tag{1}$$

The number of maximal chains has been determined for some classes of finite nilpotent groups in [20]. Recall here the explicit formula of this number for finite cyclic groups (see Corollary 2.3 of [20]):

$$g(\mathbb{Z}_n) = \binom{m_1 + m_2 + ... + m_s}{m_1, m_2, ..., m_s} = \frac{(m_1 + m_2 + ... + m_s)!}{m_1! m_2! ... m_s!},$$

where $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ as above.

In order to compute $h(G)$ we shall apply the Inclusion-Exclusion Principle. Let $\mathcal{C}$ be the set of chains in G of type

$$H_1 \subset H_2 \subset ... \subset H_r = G,$$

$\mathcal{C}'$ be the set of chains in G of type

$$H_1 \subset H_2 \subset ... \subset H_r \neq G$$

and $\mathcal{C}_i$ be the set of chains of $\mathcal{C}'$ which are contained in M_i, $i = \overline{1, k}$. Then

$$|\mathcal{C}| = 1 + |\mathcal{C}'| = 1 + \Big| \bigcup_{i=1}^{k} \mathcal{C}_i \Big| =$$

$$= 1 + \sum_{i=1}^{k} |\mathcal{C}_i| - \sum_{1 \le i_1 < i_2 \le k} |\mathcal{C}_{i_1} \cap \mathcal{C}_{i_2}| + ... + (-1)^{k-1} \Big| \bigcap_{i=1}^{k} \mathcal{C}_i \Big|.$$

Clearly, for every $1 \le \ell \le k$ and $1 \le i_1 < i_2 < ... < i_\ell \le k$, the set $\bigcap_{j=1}^{\ell} \mathcal{C}_{i_j}$ consists of all chains of $\mathcal{C}'$ which are included in $\bigcap_{j=1}^{\ell} M_{i_j}$. This shows that

$$\Big| \bigcap_{j=1}^{\ell} \mathcal{C}_{i_j} \Big| = 2h\Big(\bigcap_{j=1}^{\ell} M_{i_j} \Big) - 1$$

and therefore

$$|\mathcal{C}| = 1 + \sum_{i=1}^{k} (2h(M_i) - 1) - \sum_{1 \le i_1 < i_2 \le k} (2h(M_{i_1} \cap M_{i_2}) - 1) + ... + (-1)^{k-1} (2h\Big(\bigcap_{i=1}^{k} M_i \Big) - 1) =$$

$$= 2 \left(\sum_{i=1}^{k} h(M_i) - \sum_{1 \le i_1 < i_2 \le k} h(M_{i_1} \cap M_{i_2}) + ... + (-1)^{k-1} h\Big(\bigcap_{i=1}^{k} M_i \Big) \right) + c,$$

where

$$c = 1 + \sum_{i=1}^{k} (-1) - \sum_{1 \le i_1 < i_2 \le k} (-1) + ... + (-1)^{k-1}(-1) = (1-1)^k = 0.$$

Hence $h(G)$ $(= |\mathcal{C}|)$ is given by the following equality:

$$h(G) = 2 \left(\sum_{i=1}^{k} h(M_i) - \sum_{1 \le i_1 < i_2 \le k} h(M_{i_1} \cap M_{i_2}) + ... + (-1)^{k-1} h\Big(\bigcap_{i=1}^{k} M_i \Big) \right). \tag{2}$$

If the maximal subgroup structure of G is known (i.e. if we know the number of maximal subgroups of G, their types and their intersections), in some cases the equalities (1) and (2) will lead to recurrence relations that permit us to determine $g(G)$ and $h(G)$. This fact holds for dihedral groups, which will be minutely studied in the next two sections. Recall that the dihedral group D_{2n} $(n \ge 2)$ is the symmetry group of a regular polygon with n sides and has the order $2n$. The most convenient abstract description of D_{2n} is obtained by using its generators: a rotation x of order n and a reflection y of order 2. Under these notations, we have:

$$D_{2n} = < x, y \mid x^n = y^2 = 1,\ yxy = x^{-1} > .$$

3. The Maximal Subgroup Structure of Dihedral Groups

Since the dihedral group D_{2n} is supersolvable (see [19]), any of its maximal subgroups is of prime index. Let $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ be the decomposition of n as a product of prime factors. Then it is well-known that D_{2n} possesses a unique cyclic maximal subgroup $M_0 = \langle x \rangle \cong \mathbb{Z}_n$ and p_j maximal subgroups of type $D_{2\frac{n}{p_j}}$ (namely $M_{ij} = \left\langle x^{p_j}, x^i y \right\rangle$, $i = 0, 1, ..., p_j - 1$), for all $j = \overline{1, s}$. Then the equality (1) becomes:

$$g(D_{2n}) = g(\mathbb{Z}_n) + \sum_{j=1}^{s} p_j g\left(D_{2\frac{n}{p_j}}\right). \qquad (3)$$

Next we shall focus on computing $h(D_{2n})$. Set $\mathcal{M}_0 = \{M_0\}$ and $\mathcal{M}_j = \{M_{ij} \mid i = 0, 1, ..., p_j - 1\}$, for all $j = \overline{1, s}$. We need to determine the type of an arbitrary intersection of elements in $\bigcup_{j=0}^{s} \mathcal{M}_j$. If one of them is M_0, then we can eliminate it and repeat the following reasoning for the other maximal subgroups. Let $1 \leq \ell \leq k$, $1 \leq i_1 < i_2 < ... < i_\ell \leq k$ and consider the distinct maximal subgroups $M_{i_1}, M_{i_2}, .., M_{i_\ell}$ of D_{2n}. Suppose that they belong to exactly q sets $\mathcal{M}_{j_1}, \mathcal{M}_{j_2}, ..., \mathcal{M}_{j_q}$. Obviously, for $q < \ell$ there are at least two such subgroups M_{i_u} and M_{i_v} contained in the same set $\mathcal{M}_{j_w}$. Then $M_{i_u} \cap M_{i_v} = \langle x^{p_{j_w}} \rangle \cong \mathbb{Z}_{\frac{n}{p_{j_w}}}$ is cyclic and so is $M_{i_1} \cap M_{i_2} \cap ... \cap M_{i_\ell}$. More precisely, we have:

$$M_{i_1} \cap M_{i_2} \cap ... \cap M_{i_\ell} = \langle x^{p_{j_1} p_{j_2} \cdots p_{j_q}} \rangle \cong \mathbb{Z}_{\frac{n}{p_{j_1} p_{j_2} \cdots p_{j_q}}}.$$

If $q = \ell$, then we can assume that $M_{i_u} \in \mathcal{M}_{j_u}$, for all $u = \overline{1, \ell}$. In this case, by induction on ℓ, one obtains that:

$$M_{i_1} \cap M_{i_2} \cap ... \cap M_{i_\ell} \cong D_{2\frac{n}{p_{j_1} p_{j_2} \cdots p_{j_\ell}}}.$$

By using the above remarks, we are now able to estimate the right side of the equality (2). The first terms of this can be easily determined:

$$\sum_{i=1}^{k} h(M_i) = h(\mathbb{Z}_n) + \sum_{j=1}^{s} p_j h\left(D_{2\frac{n}{p_j}}\right),$$

$$\sum_{1 \leq i_1 < i_2 \leq k} h(M_{i_1} \cap M_{i_2}) = \sum_{j=1}^{s} \binom{p_j + 1}{2} h\left(\mathbb{Z}_{\frac{n}{p_j}}\right) + \sum_{1 \leq i_1 < i_2 \leq s} p_{i_1} p_{i_2} h\left(D_{2\frac{n}{p_{i_1} p_{i_2}}}\right),$$

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq k} h(M_{i_1} \cap M_{i_2} \cap M_{i_3}) = \sum_{j=1}^{s} \binom{p_j + 1}{3} h\left(\mathbb{Z}_{\frac{n}{p_j}}\right) +$$
$$+ \frac{1}{2} \sum_{1 \leq i_1 < i_2 \leq s} p_{i_1} p_{i_2} (p_{i_1} + p_{i_2}) h\left(\mathbb{Z}_{\frac{n}{p_{i_1} p_{i_2}}}\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq s} p_{i_1} p_{i_2} p_{i_3} h\left(D_{2\frac{n}{p_{i_1} p_{i_2} p_{i_3}}}\right).$$

Let $1 \leq \ell \leq k$. In order to calculate the general term $\sum_{1 \leq i_1 < i_2 < ... < i_\ell \leq k} h(M_{i_1} \cap M_{i_2} \cap ... \cap M_{i_\ell})$, we write first $\binom{k}{\ell} = \frac{k!}{l!(k-l)!}$ as a polynomial in $p_1, p_2, ..., p_s$:

$$\binom{k}{\ell} = \binom{1 + \sum_{j=1}^{s} p_j}{\ell} = \sum_{\alpha_1, \alpha_2, ..., \alpha_s} a(\alpha_1, \alpha_2, ..., \alpha_s) p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s},$$

where all coefficients $a(\alpha_1,\alpha_2,...,\alpha_s)$ are positive integers. We distinguish the following two cases: if $\ell \le s$, then

$$\sum_{1\le i_1<i_2<...<i_\ell\le k} h(M_{i_1}\cap M_{i_2}\cap ...\cap M_{i_\ell}) = \sum_{i=1}^{s}\sum_{\alpha_i} a(0,...,\alpha_i,...,0)p_i^{\alpha_i} h\left(\mathbb{Z}_{\frac{n}{p_i}}\right) +$$
$$+\sum_{1\le i_1<i_2\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}} h\left(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}}}\right) + \quad ... \quad +$$
$$+\sum_{1\le i_1<i_2<...<i_{\ell-1}\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_{\ell-1}}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,\alpha_{i_{\ell-1}},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_{\ell-1}}^{\alpha_{i_{\ell-1}}} h\left(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_{\ell-1}}}}\right)$$
$$+\sum_{1\le i_1<i_2<...<i_\ell\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_\ell}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,\alpha_{i_\ell},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_\ell}^{\alpha_{i_\ell}} h\left(D_{2\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_\ell}}}\right)$$

and if $\ell > s$, then

$$\sum_{1\le i_1<i_2<...<i_\ell\le k} h(M_{i_1}\cap M_{i_2}\cap ...\cap M_{i_\ell}) = \sum_{i=1}^{s}\sum_{\alpha_i} a(0,...,\alpha_i,...,0)p_i^{\alpha_i} h\left(\mathbb{Z}_{\frac{n}{p_i}}\right) +$$
$$+\sum_{1\le i_1<i_2\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}} h\left(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}}}\right) + \quad ... \quad +$$
$$+\sum_{1\le i_1<i_2<...<i_{s-1}\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_{s-1}}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,\alpha_{i_{s-1}},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_{s-1}}^{\alpha_{i_{s-1}}} h\left(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_{s-1}}}}\right)$$
$$+\left(\binom{k}{\ell} - \sum_{i=1}^{s}\sum_{\alpha_i} a(0,...,\alpha_i,...,0)p_i^{\alpha_i} - \quad ... \quad -\right.$$
$$\left. - \sum_{1\le i_1<i_2<...<i_{s-1}\le s}\ \sum_{\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_{s-1}}} a(0,...,\alpha_{i_1},...,\alpha_{i_2},...,\alpha_{i_{s-1}},...,0)p_{i_1}^{\alpha_{i_1}}p_{i_2}^{\alpha_{i_2}}...p_{i_{s-1}}^{\alpha_{i_{s-1}}}\right) h\left(\mathbb{Z}_{\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_s}}}\right).$$

Clearly, by replacing the previous expressions in the right side of (2) a new equality will be obtained. Because this is too complicated to be written in the general case, we shall focus only on two particular situations: $s = 1$ and $s = 2$ (note that the general situation can be treated in a similar manner).

For $s = 1$ we have

$$\sum_{i=1}^{k} h(M_i) = h(\mathbb{Z}_n) + p_1 h\left(D_{2\frac{n}{p_1}}\right)$$

and

$$h(M_{i_1}\cap M_{i_2}\cap ...\cap M_{i_\ell}) = h\left(\mathbb{Z}_{\frac{n}{p_1}}\right),$$

for any $\ell \ge 2$ and $1 \le i_1 < i_2 < ... < i_\ell \le k$. Then (2) becomes:

$$h(D_{2n}) = 2p_1 h\left(D_{2\frac{n}{p_1}}\right) + 2\left(h(\mathbb{Z}_n) - p_1 h\left(\mathbb{Z}_{\frac{n}{p_1}}\right)\right). \tag{4}$$

For $s=2$ we have

$$\sum_{i=1}^{k} h(M_i) = h(\mathbb{Z}_n) + p_1 h\left(D_{2\frac{n}{p_1}}\right) + p_2 h\left(D_{2\frac{n}{p_2}}\right),$$

$$\sum_{1\le i_1<i_2\le k} h\left(M_{i_1}\cap M_{i_2}\right) = \binom{p_1+1}{2} h\left(\mathbb{Z}_{\frac{n}{p_1}}\right) + \binom{p_2+1}{2} h\left(\mathbb{Z}_{\frac{n}{p_2}}\right) + $$
$$+ p_1p_2 h\left(D_{2\frac{n}{p_1p_2}}\right)$$

and

$$\sum_{1\le i_1<i_2<\ldots<i_\ell\le k} h(M_{i_1}\cap M_{i_2}\cap \ldots \cap M_{i_\ell}) \overset{\ell\ge 3}{=\!=} \binom{p_1+1}{\ell} h\left(\mathbb{Z}_{\frac{n}{p_1}}\right) +$$
$$+\binom{p_2+1}{\ell} h\left(\mathbb{Z}_{\frac{n}{p_2}}\right) + \left(\binom{p_1+p_2+1}{\ell} - \binom{p_1+1}{\ell} - \binom{p_2+1}{\ell}\right) h\left(\mathbb{Z}_{\frac{n}{p_1p_2}}\right),$$

where by convention $\binom{\alpha}{\beta} = 0$ for all $\alpha<\beta$. We infer that $h(D_{2n})$ satisfies:

$$\begin{aligned} h(D_{2n}) &= 2\left(p_1 h\left(D_{2\frac{n}{p_1}}\right) + p_2 h\left(D_{2\frac{n}{p_2}}\right) - p_1p_2 h\left(D_{2\frac{n}{p_1p_2}}\right)\right) + \\ &+ 2\left(h(\mathbb{Z}_n) - p_1 h\left(\mathbb{Z}_{\frac{n}{p_1}}\right) - p_2 h\left(\mathbb{Z}_{\frac{n}{p_2}}\right) + p_1p_2 h\left(\mathbb{Z}_{\frac{n}{p_1p_2}}\right)\right). \end{aligned} \tag{5}$$

Since $g(G)$ and $h(G)=f(G)$ are known for all finite cyclic groups G, it is obvious that the equalities (3), (4) and (5) can be seen as recurrence relations. Their solutions (and therefore explicit formulas of $g(D_{2n})$, and of $h(D_{2n})$ for $s\in\{1,2\}$) will be determined in the next section.

4. **Counting Fuzzy Subgroups of Dihedral Groups**

In the following let $n=p_1^{m_1}p_2^{m_2}...p_s^{m_s}$ be the decomposition of n as a product of prime factors and write $g(m_1,m_2,...,m_s)$ instead of $g(D_{2n})$, respectively $h(m_1,m_2,...,m_s)$ instead of $h(D_{2n})$. Then the equality (3) can be rewritten as:

$$g(m_1,m_2,...,m_s)=\binom{m_1+m_2+...+m_s}{m_1,m_2,...,m_s} + \sum_{j=1}^{s} p_j g(m_1,...,m_j-1,...,m_s). \tag{3'}$$

Clearly, $g(m_1,m_2,...,m_s)$ will be a polynomial in $p_1,p_2,...,p_s$. So, we search for $g(m_1,m_2,...,m_s)$ of type:

$$g(m_1,m_2,...,m_s) = \sum_{\alpha_1=0}^{m_1}\sum_{\alpha_2=0}^{m_2}...\sum_{\alpha_s=0}^{m_s} b_{m_1m_2...m_s}(\alpha_1,\alpha_2,...,\alpha_s)p_1^{\alpha_1}p_2^{\alpha_2}...p_s^{\alpha_s}.$$

In view of $(3)'$, the coefficients $b_{m_1m_2...m_s}(\alpha_1,\alpha_2,...,\alpha_s)$ satisfy the next recurrence relation:

$$b_{m_1m_2...m_s}(\alpha_1,\alpha_2,...,\alpha_s) = b_{m_1-1\,m_2...m_s}(\alpha_1-1,\alpha_2,...,\alpha_s) +$$
$$+ b_{m_1\,m_2-1...m_s}(\alpha_1,\alpha_2-1,...,\alpha_s) + ... + b_{m_1m_2...m_s-1}(\alpha_1,\alpha_2,...,\alpha_s-1).$$

We easily obtain

$$b_{m_1}(\alpha_1) = 1 \text{ and } b_{m_1 m_2}(\alpha_1, \alpha_2) = \binom{\alpha_1 + \alpha_2}{\alpha_1, \alpha_2} \binom{m_1 + m_2 - \alpha_1 - \alpha_2}{m_1 - \alpha_1, m_2 - \alpha_2}.$$

Then a standard induction argument (on s) will show that:

$$b_{m_1 m_2 \ldots m_s}(\alpha_1, \alpha_2, \ldots, \alpha_s) =$$

$$= \binom{\alpha_1 + \alpha_2 + \ldots + \alpha_s}{\alpha_1, \alpha_2, \ldots, \alpha_s} \binom{m_1 + m_2 + \ldots + m_s - \alpha_1 - \alpha_2 - \ldots - \alpha_s}{m_1 - \alpha_1, m_2 - \alpha_2, \ldots, m_s - \alpha_s}.$$

Hence we have proved the following result.

Theorem 4.1. *Let $n = p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s}$ be the decomposition of $n \geq 2$ as a product of prime factors. Then the number $g(D_{2n})$ of all maximal chains of subgroups of the dihedral group D_{2n} is given by the following equality:*

$$g(D_{2n}) = \sum_{\alpha_1=0}^{m_1} \sum_{\alpha_2=0}^{m_2} \ldots \sum_{\alpha_s=0}^{m_s} \binom{\alpha_1 + \alpha_2 + \ldots + \alpha_s}{\alpha_1, \alpha_2, \ldots, \alpha_s}$$

$$\binom{m_1 + m_2 + \ldots + m_s - \alpha_1 - \alpha_2 - \ldots - \alpha_s}{m_1 - \alpha_1, m_2 - \alpha_2, \ldots, m_s - \alpha_s} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}.$$

In the class of dihedral groups the strongest algebraic properties (as the nilpotency) are possessed by the 2-groups D_{2^m}, $m \geq 2$. For these groups the above formula becomes very simple.

Corollary 4.2. *The number $g(D_{2^m})$ of all maximal chains of subgroups of the dihedral group D_{2^m} is given by the following equality:*

$$g(D_{2^m}) = 2^m - 1.$$

Next we shall focus on solving the recurrence relations (4) and (5). For $n = p_1^{m_1}$ we have

$$2\left(h(\mathbb{Z}_n) - p_1 h\left(\mathbb{Z}_{\frac{n}{p_1}}\right)\right) = 2\left(f(n) - p_1 f\left(\frac{n}{p_1}\right)\right) = 2^{m_1}(2 - p_1),$$

and thus (4) becomes

$$h(m_1) = 2p_1 h(m_1 - 1) + 2^{m_1}(2 - p_1). \qquad (4)'$$

Writing $(4)'$ for $m_1 = 1, 2, \ldots$, multiplying each of them by $(2p_1)^{m_1-1}$, $(2p_1)^{m_1-2}$, ..., respectively, and summing up these equalities, we easily find an explicit formula for $h(m_1)$.

Theorem 4.3. *The number $h\left(D_{2p_1^{m_1}}\right)$ of all distinct fuzzy subgroups of the dihedral group $D_{2p_1^{m_1}}$ is given by the following equality:*

$$h\left(D_{2p_1^{m_1}}\right) = \frac{2^{m_1}}{p_1 - 1} \left(p_1^{m_1+1} + p_1 - 2\right).$$

In particular, one obtains:

$$h\left(D_{2^m}\right) = 2^{2m-1}.$$

Our calculation is more difficult for $s = 2$ (that is, when n is of type $p_1^{m_1} p_2^{m_2}$). In this case we have

$$\begin{aligned} f(n) &= f(m_1, m_2) = 2^{m_1+m_2} \sum_{i=0}^{m_2} \left(-\frac{1}{2}\right)^i \binom{m_2}{i} \binom{m_1 + m_2 - i}{m_2} = \\ &= 2^{m_1+m_2} \sum_{i=0}^{m_2} \frac{1}{2^i} \binom{m_1}{i} \binom{m_2}{i}, \end{aligned}$$

according to Corollary 5 and Remark 6 of [24]. The recurrence relation (5) can be rewritten as

$$\begin{aligned} h(m_1, m_2) &= 2(p_1 h(m_1 - 1, m_2) + p_2 h(m_1, m_2 - 1) - \\ &\quad - p_1 p_2 h(m_1 - 1, m_2 - 1)) + f'(m_1, m_2), \end{aligned} \tag{5'}$$

where by $f'(m_1, m_2)$ we have denoted the quantity

$$2(f(m_1, m_2) - p_1 f(m_1 - 1, m_2) - p_2 f(m_1, m_2 - 1) + p_1 p_2 f(m_1 - 1, m_2 - 1)).$$

Unfortunately, we are not able to solve $(5)'$ without use of a computer program. By a direct calculation we can obtain an explicit formula only for $h(m_1, 1)$. We have

$$h(m_1, 1) = 2p_1 h(m_1 - 1, 1) + f''(m_1, 1),$$

where

$$\begin{aligned} f''(m_1, 1) &= 2p_2 h(m_1, 0) - 2p_1 p_2 h(m_1 - 1, 0) + f'(m_1, 1) = \\ &= 2p_2 h(m_1) - 2p_1 p_2 h(m_1 - 1) + 2(f(m_1, 1) - p_1 f(m_1 - 1, 1) - \\ &\quad - p_2 f(m_1, 0) + p_1 p_2 f(m_1 - 1, 0)) = \\ &= 2^{m_1+1} \frac{p_2}{p_1 - 1} \left(p_1^{m_1+1} + p_1 - 2\right) - 2^{m_1} \frac{p_1 p_2}{p_1 - 1} (p_1^{m_1} + p_1 - 2) + \\ &\quad + 2\left[2^{m_1}(m_1 + 2) - 2^{m_1-1}(m_1 + 1)p_1 - 2^{m_1} p_2 + 2^{m_1-1} p_1 p_2\right] = \\ &= \frac{2^{m_1}}{p_1 - 1}\left[p_1^{m_1+1} p_2 - (m_1+1)p_1^2 + p_1 p_2 + (3m_1+5)p_1 - 2p_2 - (2m_1+4)\right]. \end{aligned}$$

Now, it is clear that $h(m_1, 1)$ can completely be determined by using the same method as for $(4)'$.

Theorem 4.4. *The number $h\left(D_{2p_1^{m_1} p_2}\right)$ of all distinct fuzzy subgroups of the dihedral group $D_{2p_1^{m_1} p_2}$ is given by the following equality:*

$$\begin{aligned} h\left(D_{2p_1^{m_1} p_2}\right) &= \frac{2^{m_1}}{(p_1 - 1)^3} \left[(m_1 + 2)p_1^{m_1+3} p_2 + 2p_1^{m_1+3} - (2m_1 + 5)p_1^{m_1+2} p_2 - \right. \\ &- 3p_1^{m_1+2} + (m_1 + 3)p_1^{m_1+1} p_2 + p_1^{m_1+1} + (m_1 + 2)p_1^3 - p_1^2 p_2 - \\ &- \left. (4m_1 + 9)p_1^2 + 3p_1 p_2 + (5m_1 + 11)p_1 - 2p_2 - (2m_1 + 4)\right]. \end{aligned}$$

In particular, one obtains:

$$h\left(D_{2p_1 p_2}\right) = 2(3p_1 p_2 + 2p_1 + 2p_2 + 6).$$

5. **Conclusions and Further Research**

All our previous results show that the study concerning the classification of fuzzy subgroups, started in [11]-[16], [24] and [27] for finite abelian groups and continued in this paper for finite dihedral groups, is a significant aspect of fuzzy group theory. Clearly, it can successfully be extended to other classes of finite nonabelian groups. This will surely constitute the subject of some further research.

Two open problems with respect to this topic are the following.

Problem 5.1. In [24] an explicit formula for the number of distinct fuzzy subgroups of a finite cyclic group is given. Find a similar formula for an *arbitrary* finite abelian group.

Problem 5.2. Write the general recurrence relation satisfied by the numbers $h(m_1, m_2, ..., m_s)$ defined in section 4 and solve it.

Finally, recall that for a given group G a remarkable sublattice of $FL(G)$ is the *fuzzy normal subgroup lattice* $FN(G)$, which consists of all fuzzy normal subgroups of G. These particular fuzzy subgroups can be also classified by using the same equivalence relation $\sim$ defined in section 2. Again, there exists a bijection between the equivalence classes of fuzzy normal subgroups of G and the set of chains of normal subgroups of G which end in G. Hence the problem of classifying fuzzy normal subgroups can be studied for several classes of finite groups whose normal subgroup structure is known.

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