SOME FIXED POINT THEOREMS IN LOCALLY CONVEX TOPOLOGY GENERATED BY FUZZY N-NORMED SPACES

S. K. ELAGAN AND M. R. SEGI RAHMAT

Abstract. The main purpose of this paper is to study the existence of a fixed point in locally convex topology generated by fuzzy n-normed spaces. We prove our main results, a fixed point theorem for a self mapping and a common fixed point theorem for a pair of weakly compatible mappings in locally convex topology generated by fuzzy n-normed spaces. Also we give some remarks in locally convex topology generated by fuzzy n-normed spaces.

1. Introduction

In [8], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [9, 11, 13, 14]. In [9], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$-norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1, 3, 4, 5, 6, 9, 11, 15, 21]. In [15], A. Narayanan and S. Vijayabalaji have extended n-normed linear space to fuzzy n-normed linear space. In section 2, we quote some basic definitions. In section 3, we show that a fuzzy n-norm is closely related to an ascending system of n-seminorms. In section 4, we introduce a locally convex topology in a fuzzy n-normed space. In section 5, we consider finite dimensional fuzzy n-normed linear spaces. In section 6, we give some fixed point theorems in locally convex topology generated by fuzzy n-normed spaces.

2. Preliminaries

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [15].

Definition 2.1. A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ if $x_1, x_2, \ldots, x_n$ are linearly dependent;
(S2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;
(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ for any real c;
(S4) $\|x_1, \ldots, x_{n-1}, y+z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called an n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

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Definition 2.2. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if :

(F1) For all $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$;
(F2) For all $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
(F3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$;
(F4) For all $t > 0$ and $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|});$$

(F5) For all $s, t \in \mathbb{R}$,

$$N(x_1, \dots, x_{n-1}, y + z, s + t) \geq \min\{N(x_1, \dots, x_{n-1}, y, s), N(x_1, \dots, x_{n-1}z, t)\}.$$

(F6) $N(x_1, x_2, \dots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} N(x_1, x_2, \dots, x_n, t) = 1.$$

3. Ascending Families of n-seminorms

The following two theorems clarify the relationship between Definitions 2.1 and 2.2.

Theorem 3.1. *Let N be a fuzzy n-norm on X. As in* [15] *define for $x_1, x_2, \dots, x_n \in X$ and $\alpha \in (0, 1)$*

$$\|x_1, x_2, \dots, x_n\|_\alpha := \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}. \tag{1}$$

Then the following statements hold.

(A1) *For every $\alpha \in (0, 1)$, $\|\bullet, \bullet, \dots, \bullet\|_\alpha$ is an n-seminorm on X;*
(A2) *If $0 < \alpha < \beta < 1$ and $x_1, \dots, x_n \in X$ then*

$$\|x_1, x_2, \dots, x_n\|_\alpha \leq \|x_1, x_2, \dots, x_n\|_\beta;$$

(A3) *If $x_1, x_2, \dots, x_n \in X$ are linearly independent, then*

$$\lim_{\alpha\to 1-} \|x_1, x_2, \dots, x_n\|_\alpha = \infty.$$

Proof. (A1) and (A2) are shown in [15, Theorem 3.4]. Let $x_1, x_2, \dots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \dots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \dots, x_n\|_\alpha \geq t.$$

This proves (A3). □

We now prove a converse of Theorem 3.1.

Theorem 3.2. *Suppose we are given a family $\|\bullet, \bullet, \dots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, of n-seminorms on X with properties (A2) and (A3). We define*

$$N(x_1, x_2, \dots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \dots, x_n\|_\alpha \geq t\}. \tag{2}$$

where the infimum of the empty set is understood as 1*. Then N is a fuzzy n-norm on X.*

Proof. (F1) holds because the values of an n-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \ldots, x_n$ are linearly dependent, then $N(x_1, \ldots, x_n, t) = 1$ follows from property (S1) of an n-seminorm. If $x_1, \ldots, x_n$ are linearly independent then $N(x_1, \ldots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an n-seminorm.

(F4) is a consequence of property (S3) of an n-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, s)\}. \tag{3}$$

It follows that $\|x_1, \ldots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \ldots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \ldots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \alpha$ and, since α is arbitrary in (3), (F5) follows.

(F6): Definition (2) shows that N is non-decreasing in t. Moreover, $\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = 1$ because seminorms have finite values. □

It is easy to see that Theorems 3.1 and 3.2 establish a one-to-one correspondence between fuzzy n-norms with the additional property that the function $t \mapsto N(x_1, \ldots, x_n, t)$ is left-continuous for all $x_1, x_2, \ldots, x_n$ and families of n-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \ldots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \ldots, x_n$.

Example 3.3. [15, Example 3.3]. Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Then define $N(x_1, x_2, \ldots, x_n, t) = 0$ if $t \leq 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (3.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1 - \alpha}\|x_1, x_2, \ldots, x_n\|.$$

Example 3.4. Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Then define

$$N(x_1, x_2, \ldots, x_n, t) = \begin{cases} \frac{t}{t+m\|x_1,x_2,\ldots,x_n\|}, & t, m > 0 \\ 0 & ,t \leq 0 \end{cases}.$$

Then (X, N) is a fuzzy $n-$ normed space

4. The Locally Convex Topology Generated by a Fuzzy n-norm

In this section (X, N) is a fuzzy n-normed space, that is, X is real vector space and N is fuzzy n-norm on X. We form the family of n-seminorms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, according to Theorem 3.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1, \ldots, x_{n-1}, \bullet\|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0, 1).$$

The family $\mathcal{F}$ generates a locally convex topology on X; see [16, Def. (37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x \in X : p_i(x) \leq \epsilon_i \text{ for } i = 1, 2, \ldots, n\},$$

where $p_i \in \mathcal{F}$ and $\epsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Theorem 4.1. *The locally convex topology generated by a fuzzy n-norm is Hausdorff.*

Proof. Given $x \in X$, $x \neq 0$, choose $x_1, \ldots, x_{n-1} \in X$ such that $x_1, \ldots, x_{n-1}, x$ are linearly independent. By Theorem 3.1(A3) we find $\alpha \in (0, 1)$ such that $\|x_1, \ldots, x_{n-1}, x\|_\alpha > 0$. The desired statement follows; see [16, Theorem (37.21)]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm N. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy n-normed space as given in [20, Definition 2.2] is meaningless.

Theorem 4.2. *Let $\{x_k\}$ be a sequence in X and $x \in X$. Then $\{x_k\}$ converges to x in the locally convex topology generated by N if and only if*

$$\lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_k - x, t) = 1 \tag{4}$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Proof. Suppose that $\{x_k\}$ converges to x in (X, N). Then, for every $\alpha \in (0, 1)$ and all $a_1, a_2, \ldots, a_{n-1} \in X$, there is K such that, for all $k \geq K$, $\|a_1, a_2, \ldots, a_{n-1}, x_k - x\|_\alpha < \epsilon$. The latter implies

$$N(a_1, a_2, \ldots, a_{n-1}, x_k - x, \epsilon) \geq \alpha.$$

Since $\alpha \in (0, 1)$ and $\epsilon > 0$ are arbitrary we see that (4) holds. The converse is shown in a similar way. □

In a similar way we obtain the following theorem.

Theorem 4.3. *Let $\{x_k\}$ be a sequence in X. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by N if and only if*

$$\lim_{k,m \to \infty} N(a_1, \ldots, a_{n-1}, x_k - x_m, t) = 1 \tag{5}$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

It should be noted that the locally convex topology generated by a fuzzy n-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 4.2 and 4.3 generalize in an obvious way to nets.

Example 4.4. Let $\|\bullet,\bullet,\ldots,\bullet\|$ be a n-norm on X. Then define

$$N(x_1,x_2,\ldots,x_n,t)=\begin{cases}\frac{t}{t+m\|x_1,x_2,\ldots,x_n\|}, & t,m>0\\ 0 & ,t\le 0\end{cases}.$$

Let $\{x_k\}$ be a Cauchy sequence in the locally convex topology generated by N. Then for all $a_1,a_2,\ldots,a_{n-1}\in X$, there is K such that, for all $k,l\ge K$ $\|a_1,a_2,\ldots,a_{n-1},x_k-x_l\|=0$ if and only if $\lim\limits_{k,l} N(a_1,\ldots,a_{n-1},x_k-x_l,t)=\lim\limits_{k,l}\frac{t}{t+m\|x_1,x_2,\ldots,x_k-x_l\|}=1$ for all $t,m>0$.

5. Fuzzy n-norms on Finite Dimensional Spaces

In this section (X,N) is a fuzzy n-normed space and X has finite dimension at least n. Since the locally convex topology generated by N is Hausdorff by Theorem 4.1 Tihonov's theorem [16, Theorem (23.1)] implies that this locally convex topology is the only one on X. Therefore, all fuzzy n-norms on X are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X=\mathbb{R}^d$ with $d\ge n$.

Lemma 5.1. *Every n-seminorm on $X=\mathbb{R}^d$ is continuous as a function on X^n with the euclidian topology.*

Proof. For every $j=1,2,\ldots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in X converging to $x_j\in X$. Therefore, $\lim\limits_{k\to\infty}\|x_{j,k}-x_j\|=0$, where $\|x\|$ denotes the euclidian norm of x. From property (S4) of an n-seminorm we get

$$|\|x_{1,k},x_{2,k},\ldots,x_{n,k}\|-\|x_1,x_{2,k},\ldots,x_{n,k}\||\le\|x_{1,k}-x_1,x_{2,k},\ldots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant M such that

$$\|y_1,y_2,\ldots,\ y_n\|\le M\,\|y_1\|\ldots\|y_n\|\ \text{ for all } y_j\in X.$$

Therefore,

$$\lim_{k\to\infty}\ \|x_{1,k}-x_1,\ x_{2,k},\ldots,\ x_{n,k}\|=0$$

and so

$$\lim_{k\to\infty}\ |\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|-\|x_1,\ x_{2,k},\ldots,\ x_{n,k}\||=0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty}\|x_{1,k},\ x_{2,k},\ldots,\ x_{n,k}\|=\|x_1,\ x_2,\ldots,\ x_n\|.$$

□

Lemma 5.2. *Let $(\mathbb{R}^d,N)$ be a fuzzy n-normed space. Then $\|x_1,x_2,\ldots,x_n\|_\alpha$ is an n-norm if $\alpha\in(0,1)$ is sufficiently close to 1.*

Proof. We consider the compact set

$$S=\left\{(x_1,\ x_2,\ldots,\ x_n)\in\mathbb{R}^{dn}:x_1,\ x_2,\ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 5.1, S_α is an open subset of S. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{6}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$, then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is β such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (5) is proved. By compactness of S, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Since $\alpha \in (0,1)$ is sufficiently close to 1. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

Theorem 5.3. *Let N be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

(a) $\{x_k\}$ converges to x with respect to N if and only if $\{x_k\}$ converges to x in the euclidian topology.

(b) $\{x_k\}$ is a Cauchy sequence with respect to N if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.

Proof. (a) Suppose $\{x_k\}$ converges to x with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 5.1, for every $\alpha \in (0,1)$,

$$\lim_{k\to\infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. By Lemma 5.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an n-norm. By definition, $\{x_k\}$ converges to x in the n-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from [9, Proposition 3.1] that this implies that $\{x_k\}$ converges to x with respect to euclidian topology.

(b) is proved in a similar way. □

Theorem 5.4. *A finite dimensional fuzzy n-normed space (X, N) is complete.*

Proof. This follows directly from Theorem 3.4. □

6. Some Fixed Point Theorem in Fuzzy $n-$ normed Spaces

In this section we prove some fixed point theorems

Definition 6.1. A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-convergent to $x^* \in X$ and denoted by $x_k \rightsquigarrow x^*$ as $k \to \infty$ if

$$\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and x^* is called the fuzzy n-limit of $\{x_k\}$.

Remark 6.2. It is noted that if (X, N) is a fuzzy n-normed space then the fuzzy n-limit of a fuzzy n-convergent sequence is unique. Indeed, if $\{x_k\}$ is a fuzzy n-convergent sequence and suppose it converges to x^* and y^* in X. Then by definition $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ and $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - y^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By $(N5)$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, x - y, t) &= N(x_1, \cdots, x_{n-1}, x^* - x_k + x_k - y^*, t/2 + t/2) \\ &\geq \min\{N(x_1, \cdots, x_{n-1}, x^* - x_k, t/2), N(x_1, \cdots, x_{n-1}, x_k - y^*, t/2)\}. \end{aligned}$$

By letting $k \to \infty$, we obtain $N(x_1, \cdots, x_{n-1}, x^* - y^*, t) = 1$, which implies that $x^* = y^*$.

Definition 6.3. A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-Cauchy sequence if

$$\lim_{k,m\to\infty} N(x_1, \ldots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$.

Proposition 6.4. *In a fuzzy n-normed space (X, N), every fuzzy n-convergent sequence is a fuzzy n-Cauchy sequence.*

Proof. Let $\{x_k\}$ be a fuzzy n-convergent sequence in X converging to $x^* \in X$. Then $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By (N5),

$$\begin{aligned} &N(x_1, \cdots, x_{n-1}, x_k - x_m, t) \\ &\quad = N(x_1, \cdots, x_{n-1}, x_k - x^* + x^* - x_m, t/2 + t/2) \\ &\quad \geq \min\{N(x_1, \cdots, x_{n-1}, x_k - x^*, t/2), N(x_1, \cdots, x_{n-1}, x^* - x_m, t/2)\}. \end{aligned}$$

By letting $n, m \to \infty$, we get,

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. □

If every fuzzy n-Cauchy sequence in X converges to an $x^* \in X$, then (X, N) is called a complete fuzzy n-normed space. A complete fuzzy n-normed space is then called a fuzzy n-Banach space.

Theorem 6.5. *Let (X, N) be a fuzzy n-normed space. Let $f : X \to X$ be a map satisfies the condition:*
There exists a $\lambda \in (0,1)$ such that for all $x, x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$,one has

$$N(x_1, \cdots, x_{n-1}, x, t) > 1 - t \quad \Rightarrow \quad N(x_1, \cdots, x_{n-1}, f(x), \lambda t) > 1 - \lambda t. \tag{7}$$

Then

(i) *For any real number $\epsilon > 0$ there exists $k_0(\epsilon) \in \mathbb{N}$ such that $f^k(x) \rightsquigarrow 0$.*

(ii) *f has at most a fixed point, that is the null vector of X. Moreover, if f is a linear mapping, f has exactly one fixed point.*

Proof. (i) Note that if f satisfies the condition (1), then for every $\epsilon \in (0, 1)$, there exists a $k_0 = k_0(\epsilon)$ such that, for all $k \geq k_0$, and for every $x, x_1, \cdots, x_{n-1} \in X$

$$N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) > 1 - \epsilon$$

holds. Indeed, one has easily that

$$N(x_1, \cdots, x_{n-1}, x, 1 + \epsilon) > 1 - (1 + \epsilon).$$

Then by (5.1), for all $x, x_1, \cdots, x_{n-1} \in X$ and $k \geq 1$,

$$N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) > 1 - \lambda^k(1 + \epsilon)$$

holds. Indeed, for each $\epsilon > 0$ there exists a $k = k_0$ implies that $\lambda^n(1 + \epsilon) \leq \epsilon$, from which, because of condition (N6), there exists a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$,

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) &\geq N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) \\ &> 1 - \lambda^k(1 + \epsilon) \\ &\geq 1 - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have $f^k(x) \rightsquigarrow 0$ as required.

(ii) Assume that $f(x) = x$. By applying part (i), for all $\epsilon \in (0, 1)$ one has

$$N(x_1, \cdots, x_{n-1}, x, \epsilon) > 1 - \epsilon$$

for every $x_1, \cdots, x_{n-1} \in X$. This implies that

$$N(x_1, \cdots, x_{n-1}, x, 0+) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$, i.e., $x = 0$ □

Lemma 6.6. *Let $\{x_k\}$ be a sequence in a fuzzy n-normed space (X, M). If for every $t > 0$, there exists a constant $\lambda \in (0, 1)$ such that*

$$N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, t) \geq N(x_1, \ldots, x_{n-1}, x_{k-1} - x_k, t/\lambda) \tag{8}$$

for all $x_1, \cdots, x_{n-1} \in X$, then $\{x_k\}$ is a fuzzy n-Cauchy sequence in X.

Proof. Let $t > 0$ and $\lambda \in (0,1)$. Then for $m \geq k$, by using (N5) and the inequality (1), we have

$$\begin{aligned} & N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, (1-\lambda)t), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\} \\ & \qquad \cdots \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\} \end{aligned}$$

Also,

$$\begin{aligned} & N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t) \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_{k+1} - x_{k+2}, (1-\lambda)\lambda t), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\} \\ & \qquad \cdots \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\} \end{aligned}$$

By repeating these argument, we get

$$\begin{aligned} & N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_{m-1} - x_m, \lambda^{m-n-1} t)\} \\ & \qquad \cdots \\ & \quad \geq \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}), \\ & \qquad N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{t}{\lambda^k})\} \end{aligned}$$

Since $(1-\lambda)\frac{t}{\lambda^k} \leq \frac{t}{\lambda^k}$ and the property (F6), we conclude that

$$N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \geq N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}).$$

Therefore, by letting $m \geq k \to \infty$, we get

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. □

Definition 6.7. A pair of maps (f, g) is called weakly compatible if they commute at coincidence point, i.e., $fx = gx$ implies $fgx = gfx$.

Theorem 6.8. *Let (X, M) be a fuzzy n-normed space and let $f, g \colon X \to X$ satisfy the following conditions:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *any of $f(X)$ or $g(X)$ is complete;*

(iii) *$N(x_1, \ldots, x_{n-1}, f(x) - f(y), t) \geq N(x_1, \ldots, x_{n-1}, g(x) - g(y), t/\lambda)$, for all $x, y, x_1, \cdots, x_{n-1} \in X$, $t > 0$, $\lambda \in (0, 1)$.*

Then f and g have a unique common fixed point provided f and g are weakly compatible on X

Proof. Let $x_0 \in X$. By condition (i), we can find $x_1 \in X$ such that $f(x_0) = g(x_1) = y_1$. By induction, we can define a sequence y_k in X such that

$$y_{k+1} = f(x_k) = g(x_{k+1}),$$

$n = 0, 1, 2, \cdots$. We consider two cases:

Case I: If $y_r = y_{r+1}$ for some $r \in \mathbb{N}$, then

$$y_r = f(x_{r-1}) = f(x_r) = g(x_r) = g(x_{r+1}) = y_{r+1} = z$$

for some $z \in X$. Since $f(x_r) = g(x_r)$ and f, g are weakly compatible, we have $f(z) = fg(x_r) = gf(x_r) = g(z)$. By condition (iii), for all $x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(z) - z, t) &= N(x_1, \cdots, x_{n-1}, f(z) - f(x_r), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda) \\ &\geq \cdots \geq N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda^k). \end{aligned}$$

Clearly, the righthand side of the inequality approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$. Hence, $N(x_1, \cdots, x_{n-1}, f(z) - z, t) = 1$. This implies that $f(z) = z = g(z)$, i.e., z is a common fixed point of f and g.

Case II: $y_k \neq y_{k+1}$, for each $k = 0, 1, 2, \cdots$. Then, by condition (ii) again, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, y_k - y_{k+1}, t) &= N(x_1, \cdots, x_{n-1}, g(x_k) - g(x_{k+1}), t) \\ &= N(x_1, \cdots, x_{n-1}, f(x_{k-1}) - f(x_k), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(x_{k-1}) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, y_{k-1} - y_k, t) \end{aligned}$$

Then, by Lemma 5.1, $\{y_k\}$ is a Cauchy sequence (with respect to fuzzy n-norm) in X. Since $g(X)$ is complete, there exists $w \in g(X)$ such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} g(x_k) = w.$$

Also, since $w \in g(X)$, we can find a $p \in X$ such that $g(p) = w$. Note that

$$w = g(p) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k).$$

Thus, by (iii), we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(p) - g(p), t) &= \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, f(p) - f(x_k), t) \\ &\geq \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, g(p) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, g(p) - w, t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, w - w, t/\lambda), \end{aligned}$$

which implies that $w = f(p) = g(p)$ is a common fixed point of f and g. Furthermore, f and g are weakly compatible maps, we have

$$f(w) = fg(w) = gf(w) = g(w).$$

But then, by (iii),

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(w) - w, t) &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t/\lambda) \\ &\geq \cdots \geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda^k). \end{aligned}$$

Clearly, the expression on the righthand side approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$, which implies that $f(w) = w$. Therefore, w is a common fixed point of f and g. The uniqueness of fixed point is immediate from condition (iii) □

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S. K. Elagan*, Mathematics and statistics Department, Faculty of Science, Taif University (P.O.888), Zip Code 21974, Kingdom of Saudi Arabia (KSA) and Department of Mathematics, Faculty of Science, Menofiya University, Shebin Elkom, Egypt
E-mail address: sayed_khalil2000@yahoo.com

M. R. Segi Rahmat, School of Applied Mathematics, University of Nottingham Malaysia Campus, Jalan Broga, 43500, Semenyih, Selangor D.E, Malaysia
E-mail address: Mohd.Rafi@nottingham.edu.my

*Corresponding author