THE NUMBER OF FUZZY SUBGROUPS OF SOME NON-ABELIAN GROUPS

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Abstract. In this paper, we compute the number of fuzzy subgroups of some classes of non-abelian groups. Explicit formulas are given for dihedral groups $D_{2n}$, quasi-dihedral groups $QD_{2n}$, generalized quaternion groups $Q_{4n}$ and modular $p$-groups $M_{p^n}$.

Introduction

One of the most important problems of fuzzy group theory is the classification of all the fuzzy subgroups of a finite group. Several papers have treated the problem in the particular cases of finite abelian groups. Laszlo [4] studied the construction of fuzzy subgroups of groups of order at most 6. Zhang and Zou [15] have determined the number of fuzzy subgroups of cyclic groups of order $p^n$, where $p$ is a prime number. Murali and Makamba [7, 8] have considered a similar problem and computed the number of fuzzy subgroups of abelian groups of order $p^m q^n$, where $p$ and $q$ are distinct primes. Tărnăuceanu and Bentea [13] established recurrence relations for the number of fuzzy subgroups of two classes of finite abelian groups; finite cyclic groups and finite elementary abelian $p$-groups. Their result improved the Murali’s works in [7, 8]. Ngcibi, Murali and Makamba [9] computed the number of fuzzy subgroups of abelian $p$-groups of rank two. The first step in classifying the fuzzy subgroups of a finite non-abelian groups is made by Tărnăuceanu [12]. He developed a general method to count the number of distinct fuzzy subgroups of such groups and found the number of fuzzy subgroups of a particular case of dihedral groups. In this paper, we determine the number of fuzzy subgroups of some classes of non-abelian groups including dihedral groups $D_{2n}$, generalized quaternion groups $Q_{4n}$, quasi-dihedral groups $QD_{2n}$ ($n \geq 4$) and modular $p$-groups $M_{p^n}$ ($n \geq 3$). Note that, the groups $D_{2n}$, $Q_{2n}$, $QD_{2n}$ and $M_{p^n}$ together with abelian $p$-groups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ constitute all finite $p$-groups of order $p^n$ having a maximal cyclic subgroup of order $p^{n-1}$.

1. Preliminaries

We begin with recalling some basic notions and results of fuzzy subgroups (see [5, 6, 10] for more details). Let $G$ be a group and $\mu : G \to [0, 1]$ be a fuzzy subset of $G$. Then $\mu$ is a fuzzy subgroup of $G$ if it satisfies the following two conditions:

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(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$,
(ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

Then, we have $\mu(x^{-1}) = \mu(x)$ for any $x \in G$, and $\mu(1) = \max \mu(G)$. For each $\alpha \in [0, 1]$, the level subset corresponding to $\alpha$ is defined as $\mu_\alpha = \{x \in G : \mu(x) \geq \alpha\}$. These subsets are useful in characterization of fuzzy subgroups, in such a way that a fuzzy subset $\mu$ is a fuzzy subgroup of $G$ if and only if its level subsets are subgroups of $G$.

Let $\sim$ be the natural equivalence relation on the set of all fuzzy subsets of $G$.

$\mu \sim \eta$ iff $(\mu(x) > \mu(y) \iff \eta(x) > \eta(y)$ for all $x, y \in G)$.

Utilizing the above equivalence relation, the fuzzy subgroups of $G$ can be classified up to equivalence classes in such a way that two fuzzy subgroups $\mu$ and $\eta$ of $G$ are distinct if $\mu \not\sim \eta$.

The above equivalence relation has a close connection to the concept of level subgroups. In this way, suppose that $G$ is a finite group and $\mu : G \rightarrow [0, 1]$ is a fuzzy subgroup of $G$. Let $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then $\mu$ determines the following chain of subgroups of $G$ ending in $G$.

$$\mu_{\alpha_1} \subseteq \mu_{\alpha_2} \subseteq \cdots \subseteq \mu_{\alpha_n} = G.$$ (1)

Moreover, for any $x \in G$ and $i = 1, 2, \cdots, n$, we have

$$\mu(x) = \alpha_i \iff x \in \mu_{\alpha_i} \setminus \mu_{\alpha_{i-1}},$$

where by convention, we set $\mu_{\alpha_0} = \emptyset$. Volf [14] gives a necessary and sufficient condition for two fuzzy subgroups $\mu, \eta$ of $G$ to be equivalent with respect to $\sim$ in such away that $\mu \sim \eta$ if and only if $\mu$ and $\eta$ have the same set of level subgroups, that is, they determine the same chain of subgroups of type (1). Hence, there exists a bijection between the equivalence classes of fuzzy subgroups of $G$ and the set of chains of subgroups of $G$, which end in $G$. Clearly, in any group with at least two elements there are more distinct fuzzy subgroups than subgroups. Also, the problem of counting all distinct fuzzy subgroups of $G$ can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of $G$, that is computing the number of all chains of subgroups of $G$ that terminate in $G$.

The most important result that we will use frequently, is the following result for the number of fuzzy subgroups of finite cyclic groups.

**Proposition 1.1.** [13] Let $G$ be a finite cyclic group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Then the number of all distinct fuzzy subgroups of $G$ is given by

$$F_G = 2^{2\sum_{i=1}^{k} \alpha_i} \sum_{i_2=0}^{\alpha_2} \cdots \sum_{i_k=0}^{\alpha_k} \left( \frac{1}{2} \prod_{r=2}^{k} \frac{\alpha_r}{i_r} \right) \left( \alpha_1 + \sum_{m=2}^{k} (\alpha_m - i_m) \right),$$

where the above iterated sums are equal to 1 for $k = 1$.

2. Dihedral Groups

Tărnăveneanu [12] uses enumerations on maximal chains of subgroups of dihedral groups to obtain the following results.
Proposition 2.1. [12] The number of all fuzzy subgroups of the dihedral group $D_{2p^m}$ is

$$F_{D_{2p^m}} = \frac{2^m}{p-1} \left(p^{m+1} + p - 2\right).$$

In particular, $F_{D_{2p^m}} = 2^{2m-1}$.

Proposition 2.2. [12] The number of all fuzzy subgroups of the dihedral group $D_{2p^mq}$ is

$$F_{D_{2p^mq}} = \frac{2^m}{(p-1)^3}[(m+2)p^{m+3}q + 2p^{m+3} - (2m+5)p^{m+2}q - 3p^{m+2} + (m+3)p^{m+1}q + p^{m+1} + (m+2)p^3 - p^2q - (4m+9)p^2 + 3pq + (5m+11)p - 2p - (2m+4)]$$

In particular,

$$F(D_{2pq}) = 2(3pq + 2p + 2q + 6).$$

Recall that there is a one-to-one correspondence between the fuzzy subgroups of a group $G$ and its chains of subgroups, which end in $G$. Hence, in what follows, we shall compute the number of all chains of subgroups of dihedral groups $G$ ending in $G$, which results in the number of all fuzzy subgroups of $G$. For this we shall make use of Proposition 1.1 all over the proofs.

Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order $2n$. Then the subgroups of $D_{2n}$ are

- cyclic group $\langle a \rangle$ of order $k$, where $k$ divides $n$,
- cyclic groups $\langle a^ib \rangle$ of order 2, where $i = 0, \ldots, n-1$,
- dihedral groups $\langle a^i, a^b \rangle$ of order $2k$, where $k$ divides $n$ and $i = 0, \ldots, n/k - 1$.

Utilizing the above statements we are able to obtain the number of fuzzy subgroups of dihedral groups. To end this, we need to define a special kind of chains of subgroups. A chain of subgroups of a group is said to be cyclic if all its terms except the whole group are cyclic.

Theorem 2.3. The number of all fuzzy subgroups of the dihedral group $D_{2n}$ is

$$F_{D_{2n}} = \sum_{k|n} \frac{n}{k} F_{Z_k}(k + F_{Z_k}) - (2n - 1) F_{Z_{2n}} + n.$$

Proof. Let $G = D_{2n}$ and $H_n \subset H_{n-1} \subset \cdots \subset H_1 \subset H_0 = G$ be an arbitrary chain of subgroups of $G$, which ends in $G$. Clearly, the group $G$ itself is a chain. Hence we further suppose that the chain contains at least one proper subgroup, that is $n \geq 1$. We proceed in two steps:

Case 1: All of the subgroups in the chain except $G$ are cyclic. Then either $H_i = \langle a^i \rangle$ ($k$ divides $n$) or $H_i = \langle a^ib \rangle$ ($0 \leq i < n$). In the former, we have $F_{Z_k}$ possible chains and in the latter, we have just two possible chains, namely $\langle a^ib \rangle \subset G$ and $1 \subset \langle a^ib \rangle \subset G$.

Case 2: The chain contains a non-cyclic proper subgroup. Let $H$ be the smallest non-cyclic subgroup in the chain. Then $H = \langle a^i, a^ib \rangle$ for some divisor $k$ of $n$ and
i = 0, \ldots, n/k - 1. The number of such chains with H fixed as the smallest non-cyclic subgroup equals the number of cyclic chains of H, which end in H times the number of chains of subgroups of G containing H, which begin and end in 1 and G, respectively. Since H is a dihedral group of order 2k, the number of its cyclic chains is indeed determined in case (1) and equals 1 + 2d + \sum_{d|k} F_{Z_d}. On the other hand, for every subgroup K, such that H \leq K \leq G we have K = \langle a^{k'}, a^b \rangle, where k divides k' and k' divides n. Hence, there exists a bijection between the chain of subgroups of G containing H, which begin in H and end in G, and the number of chains of subgroups of Z, which begin in 1 and end in Z. Therefore, the number of chain of subgroups of containing H, which begin in H and end in G equals \frac{1}{2} F_n. Thus the number of chains of G with H as the smallest non-cyclic subgroup equals

\[ \frac{1}{2} F_n \left( 1 + 2k + \sum_{d|k} F_{Z_d} \right). \]

Using the above results, it follows that

\[ F_G = 1 + 2n + \sum_{d|n} F_{Z_d} + \sum_{k|n, k \neq 1, n} \frac{n}{k} \cdot F_{Z_d} \left( 1 + 2k + \sum_{d|k} F_{Z_d} \right). \]

Since \( F_{Z_m} = 1 + \sum_{h|m} F_{Z_h} \), we may simplify the above formula and obtain

\[ F_G = \sum_{k|n} \frac{n}{k} F_{Z_d} (k + F_{Z_d}) - (2n - 1)F_{Z_n} + n. \]

The proof is complete. \( \Box \)

3. Generalized Quaternion Groups, Quasi-dihedral Groups and Modular \( p \)-groups

In this section, we shall use Theorem 2.3 to obtain the number of fuzzy subgroups of three remained classes of groups, namely generalized quaternion groups, quasi-dihedral groups and modular \( p \)-groups.

Let \( Q_{4n} = \langle a, b : a^{2n} = 1, a^n = b^2, a^b = a^{-1} \rangle \) be the generalized quaternion group of order 4n. Then \( Z(Q_{2n}) = \langle a^n \rangle \) and \( Q_{4n}/Z(Q_{2n}) \cong D_{2n} \). Moreover, if \( H \) is a subgroup of \( G \) such that \( H \cap Z(G) = 1 \), then \( H \leq \langle a^{2n} \rangle \cong Z_m \) is a cyclic group of odd order, in which \( n = 2^k m \), for some odd integer \( m \).

**Theorem 3.1.** The number of all fuzzy subgroups of the generalized quaternion group \( Q_{4n} \) is

\[ F_{Q_{4n}} = F_{D_{2n}} + \sum_{d|m} F_{Z_d} F_{D_{2n}^{Z_d}}, \]

where \( m \) is an odd integer such that \( n = 2^k m \), for some \( k \).

**Proof.** Let \( G = Q_{4n} \). We first observe that the number of chains of subgroups of \( G \) ending in \( G \) equals to the number of chains of subgroups of \( D_{2n} \) ending in \( D_{2n} \). Now consider the following chain of subgroups of \( G \) ending in \( G \) and containing a subgroup \( H \), which does not contain \( Z(G) \).

\[ \cdots \subseteq H \subseteq \cdots \subseteq G. \]
Without loss of generality we may assume that $H$ is a maximal term in the chain that does not contain $Z(G)$. Then $H \leq \langle a^{2^k} \rangle$, where $n = 2^k m$ for some odd integer $m$. Hence $H \cong \mathbb{Z}_d$ for some divisor $d$ of $m$ and the number of these chains with $H$ fixed as the maximal subgroup in the chain that does not contain $Z(G)$ is equal to the number of chains of $H$ ending in $H$ times the number of chains of $G/L \cong D_{2^n}$ ending in $G/L$, where $L = \langle H, a^n \rangle \cong \mathbb{Z}_{2d}$. The proof is complete. \hfill \Box

Let $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1} \rangle$ ($n \geq 4$) be the quasi-dihedral group of order $2^n$. Then $Z(QD_{2^n}) = \langle a^{2^{n-2}} \rangle$ and $QD_{2^n}/Z(QD_{2^n}) \cong D_{2^n-1}$. Moreover, if $H$ is a subgroup of $G$ such that $H \cap Z(G) = 1$, then either $H = 1$ or $H = \langle a^{2^i}b \rangle$, for some $0 \leq i < 2^{n-2}$.

Theorem 3.2. The number of all fuzzy subgroups of the quasi-dihedral 2-group $QD_{2^n}$ ($n \geq 4$) is

$$F_{QD_{2^n}} = 3 \cdot 2^{2n-3}.$$  

Proof. Let $G = QD_{2^n}$. The same as for generalized quaternion groups, the number of chains of subgroups of $G$ ending in $G$ equals to the number of chains of subgroups of $D_{2^n-1}$ ending in $D_{2^n-1}$. It is easy to see that a chain of subgroups of $G$ ending in $G$ and including a subgroup that does not contain $Z(G)$ has the following forms

1. $1 \subseteq \cdots \subseteq G$,  
2. $\langle a^{2^i}b \rangle \subseteq \cdots \subseteq G$,  
3. $1 \subseteq \langle a^{2^i}b \rangle \subseteq \cdots \subseteq G$.

The number of chains of type (2) is the same as the number of chains of subgroups of $G$ that contain $Z(G)$ and ending in $G$ that is $F_{D_{2^n-1}}$. On the other hand, the number of chains of types (3) and (4) is the same and it equals the number of chains of the form $H \subseteq \cdots \subseteq G$ such that $\langle a^{2^i}b \rangle \subseteq H$ plus one for the chain $\{G\}$ with one subgroup. If $\langle a^{2^i}b \rangle \subseteq H$, then $H = \langle a^{2^{n-1-i}}, a^{2i}b \rangle$ for some $1 \leq s \leq n-2$. Also, if $K$ is a subgroup of $G$ such that $H \subseteq K \subseteq G$, then $K = \langle a^{2^{n-1-i}}, a^{2i}b \rangle$ for some $1 \leq t \leq s$. Hence the number of chains $H \subseteq \cdots \subseteq G$ such that $\langle a^{2^i}b \rangle \subseteq H$ equals the number of chains of $\mathbb{Z}_{2^n}$, which begin in 1 and end in $\mathbb{Z}_{2^n}$ that is $2^{n-1}$. Therefore, the number of chains of types (3) and (4), when $i$ ranges over $\{1, \ldots, 2^{n-2}\}$ is equal to

$$2^{n-2} \cdot 2 \cdot \left( 1 + \sum_{s=1}^{n-2} 2^{s-1} \right) = 2^{2n-3},$$

from which the result follows. \hfill \Box

Let $M_{p^n} = \langle a, b : a^{p^{n-1}} = b^p = 1, a^b = a^{p^{n-2}+1} \rangle$ ($n \geq 3$) be the modular $p$-group of order $p^n$. Then $Z(M_{p^n}) = \langle a^{p^{n-2}} \rangle$ and $M_{p^n}/Z(M_{p^n}) \cong \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$. Moreover, if $H$ is a subgroup of $G$ such that $H \cap Z(G) = 1$, then either $H = 1$ or $H = \langle a^{p^{n-2}}b \rangle$, for some $0 \leq i < p$.

Theorem 3.3. The number of all fuzzy subgroups of the modular $p$-group $M_{p^n}$ ($n \geq 3$ and $p^n \neq 8$) is

$$F_{M_{p^n}} = 2^{n-1}(n-1)p + 2^n,$$
**Proof.** Let $G = M_{p^n}$. The same as before, the number of chains of subgroups of $G$ that contain $Z(G)$ and ending in $G$ equals to the number of chains of subgroups of $G/Z(G) = \mathbb{Z}_{p^{n-2}} \times \mathbb{Z}_p$ ending in $G/Z(G)$, which is equal to $2^{n-2}(n-2)p + 2^{n-1}$, by [9, Theorem 2.1]. A simple observation shows that a chain of subgroups of $G$ ending in $G$ and including a subgroup that does not contain $Z(G)$ has the following forms

$$1 \subseteq \cdots \subseteq G,$$

$$\langle a^{i^p^{n-2}}b \rangle \subseteq \cdots \subseteq G,$$

$$1 \subseteq \langle a^{i^p^{n-2}}b \rangle \subseteq \cdots \subseteq G.$$  

The number of chains of type (5) is the same as the number of chains of subgroups of $G$ that contain $Z(G)$ and ending in $G$ that is $2^{n-2}(n-2)p + 2^{n-1}$. Moreover, the number of chains of types (6) and (7) is the same and it equals the number of chains of the form $H \subseteq \cdots \subseteq G$ such that $\langle a^{i^p^{n-2}}b \rangle \subset H$ plus one for the chain $\{G\}$ with one subgroup. If $\langle a^{i^p^{n-2}}b \rangle \subset H$, then $H = \langle a^{i^p^{n-1-t}}, a^{i^p^{n-2}}b \rangle$ for some $1 \leq s \leq n-2$. Also, if $K$ is a subgroup of $G$ such that $H \subseteq K \subseteq G$, then $K = \langle a^{i^p^{n-1-t}}, a^{i^p^{n-2}}b \rangle$ for some $1 \leq t \leq s$. Hence the number of chains of types (6) and (7), when $i$ ranges over $\{0, \ldots, p-1\}$ is equal to

$$2p \left( 1 + \sum_{s=1}^{n-2} 2^{s-1} \right) = 2^{n-1}p,$$

from which the result follows. \qed

We note that modular $p$-groups $M_{p^n}$ have the same maximal subgroup structure as the abelian $p$-group $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$. Hence by the equality (2) in [12],

$$F_{M_{p^n}} = F_{\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p} = 2^{n-1}(n-1)p + 2^n.$$

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**References**


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