AN EXPLICIT FORMULA FOR THE NUMBER OF FUZZY SUBGROUPS OF A FINITE ABELIAN p-GROUP OF RANK TWO

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Abstract. Ngcibi, Murali and Makamba [Fuzzy subgroups of rank two abelian p-group, Iranian J. of Fuzzy Systems **7** (2010), 149-153] considered the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ of rank two, and gave explicit formulas for the cases when m is any positive integer and $n = 1, 2, 3$. Even though their method can be used for the cases when $n = 4, 5, \ldots$ by using inductive arguments, it does not provide an explicit formula for that number for an arbitrarily given positive integer n. In this paper we give a complete answer to this problem. Thus for arbitrarily given positive integers m and n, an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is given.

1. Introduction

Let G be a group with a multiplicative binary operation, and let $\mu : G \to [0, 1]$ be a fuzzy subset of G. Then μ is a *fuzzy subgroup* of G if (1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. The set $\{\mu(x) \mid x \in G\}$ is called the *image* of μ and is denoted by $\mu(G)$. For each $a \in \mu(G)$, the set $\mu_a = \{x \in G \mid \mu(x) \geq a\}$ is called a *level subset* of μ. It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G (see [2]).

Let G be a group. A *chain of subgroups* of G is a set of subgroups of G linearly ordered by set inclusion. A chain of subgroups of G is called *rooted* if it contains G. Otherwise, it is called *unrooted*.

For any fuzzy subgroups μ and ν of a given group G, μ and ν are *equivalent*, written as $\mu \sim \nu$, if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ for all $x, y \in G$. It follows that $\mu \sim \nu$ if and only if μ and ν have the same set of level subgroups (see [8]). Hence there exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G and the collection of rooted chains of subgroups of G (see [8]).

There is another equivalence relation on the set of fuzzy subgroups of a given group G used in [5]. Assume that $\mu(e) = 1$ for any fuzzy subgroup μ of G where e is the identity. For any fuzzy subgroups μ and ν of G, μ and ν are *equivalent*, written as $\mu \approx \nu$, if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ and $\mu(x) = 0 \Leftrightarrow \nu(x) = 0$ for all $x, y \in G$. It is easy to see that the relation $\approx$ is an equivalence relation of

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the set of fuzzy subgroups in which the image of the identity e is 1. There exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G under the relation $\approx$ and the collection of chains of subgroups of G (see Proposition 2.2).

Thus, given a finite group G the number of fuzzy subgroups of G under the relation $\sim$ (or under the relation $\approx$) is equal to the number of rooted chains (or the number of chains) of subgroups in the lattice of subgroups of G. Furthermore, one of the two numbers gives us another (see Proposition 2.2).

In this paper we are concerned with the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ of rank two with order p^{m+n}. S.L. Ngcibi et al. [5] gave an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ under the relation $\approx$ where m is any positive integer and $n = 1, 2, 3$. As commented in [5], their method can be used to find that number when $n = 4, 5, \ldots$ consecutively. But it does not provide an explicit formula for that number for an arbitrarily given positive integer n. In this paper we give a complete answer to this problem. Thus for arbitrarily given positive integers m and n, an explicit formula for the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is given.

This paper is organized as follows. In section 2 we present some notations and results. In section 3 we give an explicit formula for the number of fuzzy subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ of rank two, and then it is proved in section 4.

2. Preliminaries

Given a group G let $\mathcal{C}(G)$, $\mathcal{D}(G)$ and $\mathcal{F}(G)$ be the collections of, respectively, chains of subgroups of G, unrooted chains of subgroups of G and rooted chains of subgroups of G. Let $C(G)$, $D(G)$ and $F(G)$ be the cardinalities of $\mathcal{C}(G)$, $\mathcal{D}(G)$ and $\mathcal{F}(G)$, respectively. Similarly let $\mathcal{F}'(G)$ be the collection of the equivalence classes of fuzzy subgroups of G under the relation $\approx$ and let $F'(G)$ be the cardinality of $\mathcal{F}'(G)$.

The following simple observation is useful for enumerating fuzzy subgroups of a given group. For completeness we give its proof.

Proposition 2.1. *Let G be a finite group. Then $F(G) = D(G) + 1$ and $C(G) = F(G) + D(G) = 2F(G) - 1$.*

Proof. For each chain of subgroups in $\mathcal{D}(G)$, we add G in the end of the chain of subgroups. Together with G as considered G-rooted chain of subgroups these chains of subgroups constitute all G-rooted chains of subgroups of G. Hence $F(G) = D(G) + 1$. It is easy to see that $\mathcal{C}(G)$ is the disjoint union of $\mathcal{F}(G)$ and $\mathcal{D}(G)$. □

The following proposition gives us the relation between the two equivalence relations $\sim$ and $\approx$.

Proposition 2.2. *Let G be a finite group with identity e. Then*

$$F'(G) = 2F(G) - 1 = C(G).$$

That is, the number of fuzzy subgroups of G under the relation $\approx$ is equal to the number of chains of subgroups of G.

Proof. Assume that $\mu(e) = 1$ for every fuzzy subgroup μ of G. Each equivalence class of fuzzy subgroups under the relation $\sim$ corresponds to the unique chain of subgroups of G which contains G. Furthermore, except for the equivalence class corresponding to the chain of subgroups $\{G\}$, each equivalence class of fuzzy subgroups under the relation $\sim$ is divided into two equivalence classes under the relation $\approx$. Hence $F'(G) = 2F(G) - 1 = C(G)$ (the last equality follows by Proposition 2.1). $\square$

It immediately follows (by Proposition 2.2) that one of the two numbers $F(G)$ and $F'(G)$ gives the other.

We use a recursive formula for binomial coefficients

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \text{ for all positive integers } n \text{ and } k$$

with initial values $\binom{n}{0} = 1$ for all integers $n \geq 0$ and $\binom{0}{k} = 0$ for all integers $k > 0$.

Let

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \left\langle a, b \mid a^{p^m} = e = b^{p^n}, bab^{-1} = a \right\rangle$$

be a finite abelian p-group of rank two with order p^{m+n} where m and n are positive integers and p is a prime number. Without loss of generality we may assume that $m \geq n$.

By [5] and Proposition 2.2 we have the following.

Proposition 2.3. [5] *Let $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ be the number of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where m and n are positive integers such that $m \geq n$. Then the following holds.*

a) $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_p) = 2^{m+1} + 2^m \binom{m}{1} p.$

b) $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^2}) = 2^{m+2} + 2^{m+1} \binom{m+1}{1} p + 2^m \left[\binom{m}{2} + 2\binom{m-1}{1} \right] p^2.$

c) $$F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^3}) = 2^{m+3} + 2^{m+2} \binom{m+2}{1} p + 2^{m+1} \left[\binom{m+1}{2} + 2\binom{m}{1} \right] p^2 + 2^m \left[\binom{m}{3} + 4\binom{m-1}{2} + 6\binom{m-2}{1} \right] p^3.$$

3. The Number of Chains of Subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$

The Schröder's triangle $T(r,k)$ is defined as follows.

$$T(r,k) = \begin{cases} 0 & \text{if } k < r, \\ T(r,k-1) + T(r-1,k) + T(r-1,k-1) & \text{otherwise} \end{cases}$$

where $T(1,k) = 1$ for any integer k (see [7], A033877). We list some $T(r,k)$ as follows.

$T(1,1) = 1,$

$T(1,2) = 1,\ T(2,2) = 2,$

$T(1,3) = 1,\ T(2,3) = 4,\ T(3,3) = 6,$

$T(1,4) = 1,\ T(2,4) = 6,\ T(3,4) = 16,\ T(4,4) = 22,$

$T(1,5) = 1,\ T(2,5) = 8,\ T(3,5) = 30,\ T(4,5) = 68,\ T(5,5) = 90,$

$T(1,6) = 1,\ T(2,6) = 10,\ T(3,6) = 48,\ T(4,6) = 146,\ T(5,6) = 304,\ T(6,6) = 394.$

The following is the main result of this paper, which will be proved in the next section.

Theorem 3.1. *Let m and n be positive integers such that $m \geq n$, and let p be a prime number. Let*

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \left\langle a, b \mid a^{p^m} = e = b^{p^n}, bab^{-1} = a \right\rangle$$

be a finite abelian p-group of rank two with order p^{m+n}. Then the number $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is

$$F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}) = 2^{m+n} + 2^m \sum_{k=1}^{n} 2^{n-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{m+n-k-r+1}{k-r+1} \right] p^k. \qquad (1)$$

Furthermore, the number $C(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ of chains subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is

$$C(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}) = 2F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}) - 1.$$

We remark that $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ is equal to the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ under the equivalence relation defined in [3, 8] and $C(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ is equal to the number of fuzzy subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ under the equivalence relation defined in [4, 5]. A similar formula for $C(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ involving the number $T(r, k)$ has been conjectured by Ngcibi [4].

E. Pergola et al. [6] gave an explicit formula for $T_{r,k}$ as follows.

$$T(r,k) = \frac{k-r+1}{k} \sum_{s=0}^{r-1} \binom{k}{r-s-1} \binom{k+s-1}{s}$$

where r and k are positive integers such that $r \leq k$. Therefore, we have the following.

Corollary 3.2. *Let $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ be a finite abelian p group of rank two with order p^{m+n} where m and n are positive integers such that $m \geq n$ and p is a prime number. Then the number $F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is*

$$F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}) = 2^{m+n} +$$
$$2^m \sum_{k=1}^{n} 2^{n-k} \left[\sum_{r=1}^{k} \frac{k-r+1}{k} \binom{m+n-k-r+1}{k-r+1} \sum_{s=0}^{r-1} \binom{k}{r-s-1} \binom{k+s-1}{s} \right] p^k.$$

4. The Proof Theorem 3.1

The last statement of Theorem 3.1 immediately follows from Proposition 2.1.

As a set

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{a^i b^j \mid i = 0, 1, \ldots, p^m - 1, j = 0, 1, \ldots, p^n - 1\}.$$

We first find all subgroups of index p in $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. Let H be an index p subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. Then $a^p, b^p \in H$ and so

$$H \geq \langle a^p, b^p \rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}}.$$

If $a^{pk+i} \in H$ for some integers k and i such that $1 \leq i \leq p-1$, then

$$H \geq \left\langle a^p, b^p, a^{pk+i} \right\rangle = \left\langle a^p, b^p, a^i \right\rangle = \langle a, b^p \rangle,$$

and hence $H = \langle a, b^p \rangle = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}$. If $b^{pk+j} \in H$ for some integers k and j such that $1 \le j \le p-1$, then

$$H \ge \left\langle a^p, b^p, b^{pk+j} \right\rangle = \left\langle a^p, b^p, b^j \right\rangle = \langle a^p, b \rangle ,$$

and hence $H = \langle a^p, b \rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n}$. If $a^{pk_1+i} b^{pk_2+j} \in H$ for some integers k_1, k_2, i and j such that $1 \le i, j \le p-1$, then $H \ge \left\langle a^p, b^p, a^{pk_1+i} b^{pk_2+j} \right\rangle = \left\langle a^p, b^p, a^i b^j \right\rangle$ and

$$o(\left\langle a^p, b^p, a^i b^j \right\rangle) = \frac{o(\langle a^p, b^p \rangle) \cdot o(\langle a^i b^j \rangle)}{o(\langle a^p, b^p \rangle \cap \langle a^i b^j \rangle)} = \frac{p^{m-1} p^{n-1} \cdot p^m}{p^{m-1}} = p^{m+n-1},$$

which implies that $H = \left\langle a^p, b^p, a^i b^j \right\rangle$, where $i = 1, 2, \ldots, p-1$ and $j = 1, 2, \ldots, p-1$.

Consider the subgroup $H = \left\langle a^p, b^p, a^i b^j \right\rangle$ for all integers i and j such that $1 \le i \le p-1$ and $2 \le j \le p-1$. Since

$$j \in \mathbb{Z}^*_{p^n} := \{0 < t < p^t \mid \gcd(t, p^n) = 1\},$$

there exists $j' \in \mathbb{Z}^*_{p^n}$ such that $jj' \equiv 1 \pmod{p^n}$, and so

$$\left\langle a^p, b^p, a^i b^j \right\rangle = \left\langle a^p, b^p, (a^i b^j)^{j'} = a^{ij'} b \right\rangle .$$

We set $ij' = pk + i'$ where k and i' are integers such that $0 \le i' \le p-1$. Then we have $\left\langle a^p, b^p, a^i b^j \right\rangle = \left\langle a^p, b^p, a^{i'} b \right\rangle$. Therefore, the subgroups of the form $\left\langle a^p, b^p, a^i b^j \right\rangle$, where $i = 1, 2, \ldots, p-1$ and $j = 1, 2, \ldots, p-1$, constitute $(p-1)$ different subgroups as follows.

$$\langle a^p, b^p, ab \rangle = \langle b^p, ab \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}, \left\langle a^p, b^p, a^2 b \right\rangle = \left\langle b^p, a^2 b \right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}, \ldots,$$
$$\left\langle a^p, b^p, a^{p-1} b \right\rangle = \left\langle b^p, a^{p-1} b \right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}.$$

Consequently we have proved the following.

Lemma 4.1. *The group* $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ *has* $(p+1)$ *index* p *subgroups* $\langle a^p, b \rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n}$, $\left\langle b^p, a^i b \right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}$, $i = 1, 2, \ldots, p-1$, *and* $\langle a, b^p \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}$.

Let $a_{m,n} := F(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n})$ be the number of rooted chains of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

In the following we obtain recurrence relations for $a_{m,n}$.

Lemma 4.2. *Let* m *and* n *be non-negative integers such that* $m \ge n$.

a) *If* $m > n$, *then*

$$a_{m,n} = 2a_{m-1,n} + 2pa_{m,n-1} - 2pa_{m-1,n-1}. \tag{2}$$

b) *If* $m = n$, *then*

$$a_{m,m} = 2(1+p)a_{m,m-1} - 2pa_{m-1,m-1}. \tag{3}$$

Remark 4.3. Note that $a_{m,0} = F(\mathbb{Z}_{p^m}) = 2^m$ for any positive integer m. By using a general theory of solving a recurrence relation (for example see [1, 9]), one may find an explicit formula for the number $a_{m,n}$ when $n = 1, 2, \ldots$ consecutively much more easily than that using the method described in [5]. But from those results it seems not easy to derive an explicit formula for the number $a_{m,n}$ when positive integers m and n are arbitrarily given. In fact, one may guess equation (1) from Proposition 2.3.

Proof. We only give the proof of equation (2), equation (3) can be proved by a similar method.

By Lemma 4.1

$$\mathcal{D}(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}) = \mathcal{C}(\langle a^p, b\rangle \cong \mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}) \bigcup_{i=1}^{p-1} \mathcal{C}(\left\langle b^p, a^i b\right\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}) \bigcup \mathcal{C}(\langle a, b^p\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}).$$

Note that every t-intersection ($t \geq 2$) of elements of the set

$$\{\mathcal{C}(\langle a^p, b\rangle \cong \mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}), \mathcal{C}(\left\langle b^p, a^i b\right\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}), \mathcal{C}(\langle a, b^p\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}) \mid i = 1, 2, \ldots, p-1\}$$

is $\mathcal{C}(\langle a^p, b^p\rangle \cong \mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^{n-1}})$. Now, using the inclusion-exclusion principle we have

$$\begin{aligned} D(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}) &= C(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}) + pC(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}) - \binom{p+1}{2}C(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^{n-1}}) \\ &\quad + \binom{p+1}{3}C(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^{n-1}}) + \cdots + (-1)^{p+2}\binom{p+1}{p+1}C(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^{n-1}}) \\ &= C(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}) + pC(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}) - pC(\mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^{n-1}}). \end{aligned}$$

Thus, we have equation (2) by Proposition 2.1. □

Now it is enough to show that equation (1) satisfies the two recurrence relations of Lemma 4.2. To do it, the following lemma is crucial.

Lemma 4.4. *For any positive integers ℓ and k such that $k \geq 2$ and $\ell \geq 2k$, let*

$$\Delta := -\sum_{r=1}^{k} T_{r,k}\binom{\ell-k-r}{k-r} + 2\sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r+1}{k-r} - \sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r}{k-r}.$$

Then

$$\Delta = 0.$$

Proof. Since $T(k, k-1) = 0$ and $T_{r,k} = T_{r,k-1} + T_{r-1,k} + T_{r-1,k-1}$, we have

$$\begin{aligned} \sum_{r=1}^{k} T_{r,k}\binom{\ell-k-r}{k-r} &= \sum_{r=1}^{k} T_{r,k-1}\binom{\ell-k-r}{k-r} + \sum_{r=1}^{k} T_{r-1,k}\binom{\ell-k-r}{k-r} \\ &\quad + \sum_{r=1}^{k} T_{r-1,k-1}\binom{\ell-k-r}{k-r} = \sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r}{k-r} \\ &\quad + \sum_{r=1}^{k-1} T_{r,k}\binom{\ell-k-r-1}{k-r-1} + \sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r-1}{k-r-1}. \end{aligned}$$

Thus

$$\begin{aligned} \Delta &= 2\sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r+1}{k-r} - 2\sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r}{k-r} \\ &\quad - \sum_{r=1}^{k-1} T_{r,k}\binom{\ell-k-r-1}{k-r-1} - \sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r-1}{k-r-1}. \end{aligned}$$

Furthermore, since $\binom{\ell-k-r+1}{k-r} = \binom{\ell-k-r}{k-r} + \binom{\ell-k-r}{k-r-1}$, we have

$$\begin{aligned} \Delta &= -\sum_{r=1}^{k-1} T_{r,k}\binom{\ell-k-r-1}{k-r-1} + 2\sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r}{k-r-1} \\ &\quad - \sum_{r=1}^{k-1} T_{r,k-1}\binom{\ell-k-r-1}{k-r-1}. \end{aligned}$$

We claim that

$$\Delta = -\sum_{r=1}^{k-j} T_{r,k}\binom{\ell-k-r-j}{k-r-j} + 2\sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-(j-1)}{k-r-j} - \sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-j}{k-r-j} \tag{4}$$

for any $j = 1, 2, \ldots, k-1$. We prove it by induction on j. We have already shown that equation (4) holds for $j = 1$. Assume now that equation (4) holds for j. Since $T_{r,k} = T_{r,k-1} + T_{r-1,k} + T_{r-1,k-1}$, we have

$$\begin{aligned}\sum_{r=1}^{k-j} T_{r,k}\binom{\ell-k-r-j}{k-r-j} &= \sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-j}{k-r-j} + \sum_{r=1}^{k-j} T_{r-1,k}\binom{\ell-k-r-j}{k-r-j}\\ &\quad + \sum_{r=1}^{k-j} T_{r-1,k-1}\binom{\ell-k-r-j}{k-r-j} = \sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-j}{k-r-j}\\ &\quad + \sum_{r=1}^{k-(j+1)} T_{r,k}\binom{\ell-k-r-(j+1)}{k-r-(j+1)} + \sum_{r=1}^{k-(j+1)} T_{r,k-1}\binom{\ell-k-r-(j+1)}{k-r-(j+1)}.\end{aligned}$$

Thus equation (4) becomes

$$\begin{aligned}\Delta &= 2\sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-(j-1)}{k-r-j} - 2\sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-j}{k-r-j}\\ &\quad - \sum_{r=1}^{k-(j+1)} T_{r,k}\binom{\ell-k-r-(j+1)}{k-r-(j+1)} - \sum_{r=1}^{k-(j+1)} T_{r,k-1}\binom{\ell-k-r-(j+1)}{k-r-(j+1)}.\end{aligned}$$

Furthermore, since

$$\binom{\ell-k-r-(j-1)}{k-r-j} = \binom{\ell-k-r-j}{k-r-j} + \binom{\ell-k-r-j}{k-r-(j+1)},$$

we have

$$\begin{aligned}\Delta &= 2\sum_{r=1}^{k-j} T_{r,k-1}\binom{\ell-k-r-j}{k-r-(j+1)} - \sum_{r=1}^{k-(j+1)} T_{r,k}\binom{\ell-k-r-(j+1)}{k-r-(j+1)}\\ &\quad - \sum_{r=1}^{k-(j+1)} T_{r,k-1}\binom{\ell-k-r-(j+1)}{k-r-(j+1)} = -\sum_{r=1}^{k-(j+1)} T_{r,k}\binom{\ell-k-r-(j+1)}{k-r-(j+1)}\\ &\quad + 2\sum_{r=1}^{k-(j+1)} T_{r,k-1}\binom{\ell-k-r-j}{k-r-(j+1)} - \sum_{r=1}^{k-(j+1)} T_{r,k-1}\binom{\ell-k-r-(j+1)}{k-r-(j+1)}.\end{aligned}$$

Thus equation (4) holds for $j+1$. Hence the claim is proved. Now equation (4) with $j = k-1$ gives us that

$$\begin{aligned}\Delta &= -T_{1,k}\binom{\ell-2k}{0} + 2T_{1,k-1}\binom{\ell-2k+1}{0} - T_{1,k-1}\binom{\ell-2k}{0}\\ &= 0.\end{aligned}$$

□

Assume first that $m > n$. In this case we prove that equation (1) satisfies the recurrence relation (2), which is equivalent to show that

$$2^{m+n}+2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r+1}{k-r+1}\right]p^k=$$
$$2\left[2^{m+n-1}+2^{m-1}\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^k\right]$$
$$+2p\left[2^{m+n-1}+2^m\sum_{k=1}^{n-1}2^{n-k-1}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^k\right]$$
$$-2p\left[2^{m+n-2}+2^{m-1}\sum_{k=1}^{n-1}2^{n-k-1}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r-1}{k-r+1}\right]p^k\right].$$

That is,

$$2^{m+n}+2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r+1}{k-r+1}\right]p^k=2^{m+n}$$
$$+2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^k+2^{m+n}p$$
$$+2^m\sum_{k=1}^{n-1}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^{k+1}-2^{m+n-1}p$$
$$-2^{m-1}\sum_{k=1}^{n-1}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r-1}{k-r+1}\right]p^{k+1}. \tag{5}$$

To show that equation (5) holds, we consider it as the equation of the two polynomials

$$2^{m+n}+2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r+1}{k-r+1}\right]p^k$$

of variable p and

$$2^{m+n}+2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^k$$
$$+2^{m+n}p+2^m\sum_{k=1}^{n-1}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r}{k-r+1}\right]p^{k+1}$$
$$-2^{m+n-1}p-2^{m-1}\sum_{k=1}^{n-1}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r-1}{k-r+1}\right]p^{k+1}$$

of the variable p, and then prove that coefficients of p^k of both sides are equal for any $k=0,1,\dots,n$.

Clearly the constant terms of both sides are equal to 2^{m+n}. The coefficient of p of the left hand side of equation (5) is

$$2^m2^{n-1}T_{1,1}\binom{m+n-1}{1}=2^{m+n-1}(m+n-1).$$

On the other hand, the coefficient of p of the right hand side of equation (5) is

$$2^m2^{n-1}T_{1,1}\binom{m+n-2}{1}+2^{m+n}-2^{m+n-1}=2^{m+n-1}(m+n-1).$$

Thus the coefficients of p of both sides of equation (5) are equal. Now it remains to show that for any positive integer k such that $2\le k\le n$, the coefficients of p^k of both sides of equation (5) are equal. Note that the coefficient of p^k of the right hand side of equation (5) is

$$2^m2^{n-k}\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r+1}{k-r+1}=2^{m+n-k}\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r+1}{k-r+1}$$

and the coefficient of p^k of the left hand side of equation (5) is

$$
\begin{aligned}
&2^m 2^{n-k} \sum_{r=1}^{k} T_{r,k} \binom{m+n-k-r}{k-r+1} + 2^m 2^{n-k+1} \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r+1}{k-r} \\
&\quad - 2^{m-1} 2^{n-k+1} \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r}{k-r} = 2^{m+n-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{m+n-k-r}{k-r+1} \right. \\
&\qquad \left. +2 \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r+1}{k-r} - \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r}{k-r} \right] \\
&\qquad = 2^{m+n-k} \left[\sum_{r=1}^{k} T_{r,k} \left[\binom{m+n-k-r+1}{k-r+1} - \binom{m+n-k-r}{k-r} \right] \right. \\
&\qquad \left. +2 \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r+1}{k-r} - \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r}{k-r} \right] \\
&\qquad = 2^{m+n-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{m+n-k-r+1}{k-r+1} - \sum_{r=1}^{k} T_{r,k} \binom{m+n-k-r}{k-r} \right. \\
&\qquad\qquad \left. +2 \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r+1}{k-r} - \sum_{r=1}^{k-1} T_{r,k-1} \binom{m+n-k-r}{k-r} \right],
\end{aligned}
$$

which are equal by Lemma 4.4. Therefore, we have shown that equation (1) satisfies the recurrence relation (2).

Assume now that $m = n$. In this case we prove that equation (1) satisfies the recurrence relation (3), which is equivalent to show that

$$
\begin{aligned}
&2^{2m} + 2^m \sum_{k=1}^{m} 2^{m-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r+1}{k-r+1} \right] p^k = \\
&\quad 2(1+p) \left[2^{2m-1} + 2^m \sum_{k=1}^{m-1} 2^{m-1-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r}{k-r+1} \right] p^k \right] \\
&\qquad - 2p \left[2^{2m-2} + 2^{m-1} \sum_{k=1}^{m-1} 2^{m-1-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r-1}{k-r+1} \right] p^k \right].
\end{aligned}
$$

That is,

$$
\begin{aligned}
&2^{2m} + 2^m \sum_{k=1}^{m} 2^{m-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r+1}{k-r+1} \right] p^k = 2^{2m} + 2^{2m-1} p \\
&\quad + 2^{m+1} \sum_{k=1}^{m-1} 2^{m-1-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r}{k-r+1} \right] p^k + 2^{m+1} \sum_{k=1}^{m-1} 2^{m-1-k} \times \\
&\quad \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r}{k-r+1} \right] p^{k+1} - 2^m \sum_{k=1}^{m-1} 2^{m-1-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r-1}{k-r+1} \right] p^{k+1}.
\end{aligned} \tag{6}
$$

We consider equation (6) as the equation of the two polynomials

$$
2^{2m} + 2^m \sum_{k=1}^{m} 2^{m-k} \left[\sum_{r=1}^{k} T_{r,k} \binom{2m-k-r+1}{k-r+1} \right] p^k
$$

of variable p and

$$2^{2m}+2^{2m-1}p+2^{m+1}\sum_{k=1}^{m-1}2^{m-1-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r}{k-r+1}\right]p^k$$
$$+2^{m+1}\sum_{k=1}^{m-1}2^{m-1-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r}{k-r+1}\right]p^{k+1}$$
$$-2^m\sum_{k=1}^{m-1}2^{m-1-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r-1}{k-r+1}\right]p^{k+1}$$

of the variable p, and then prove that the coefficients of p^k of both sides are equal for any $k=0,1,\ldots,m$.

Clearly the constant terms of both sides are equal to 2^{2m}. The coefficient of p of the left side of equation (6) is

$$2^m2^{m-1}T_{1,1}\binom{2m-1}{1}=2^{2m-1}(2m-1).$$

On the other hand, the coefficient of p of the right side of equation (6) is

$$2^{2m-1}+2^{m+1}2^{m-2}T_{1,1}\binom{2m-2}{1}=2^{2m-1}(2m-1).$$

Thus the coefficients of p of both sides of equation (6) are equal. Now it remains to show that for any positive integer k such that $2\le k\le m$, the coefficients of p^k of both sides of equation (6) are equal. Note that the coefficient of p^k of the right side of equation (6) is

$$2^m2^{m-k}\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r+1}{k-r+1}=2^{2m-k}\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r+1}{k-r+1}$$

and the coefficient of p^k of the right and side of equation (6) is

$$2^{m+1}2^{m-1-k}\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r}{k-r+1}+2^{m+1}2^{m-1-k+1}\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r+1}{k-r}$$
$$-2^m2^{m-1-k+1}\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r}{k-r}=2^{2m-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r}{k-r+1}\right.$$
$$\left.+2\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r+1}{k-r}-\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r}{k-r}\right]$$
$$=2^{2m-k}\left[\sum_{r=1}^{k}T_{r,k}\left[\binom{2m-k-r+1}{k-r+1}-\binom{2m-k-r}{k-r}\right]\right.$$
$$\left.+2\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r+1}{k-r}-\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r}{k-r}\right]$$
$$=2^{2m-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r+1}{k-r+1}-\sum_{r=1}^{k}T_{r,k}\binom{2m-k-r}{k-r}\right.$$
$$\left.+2\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r+1}{k-r}-\sum_{r=1}^{k-1}T_{r,k-1}\binom{2m-k-r}{k-r}\right],$$

which are equal by Lemma 4.4. Therefore, we have shown that equation (1) satisfies the recurrence relation (3). This completes the proof of Theorem 3.1.

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