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MORE GENERAL FORMS OF (α, β)-FUZZY IDEALS OF ORDERED SEMIGROUPS

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Abstract. This paper consider the general forms of (α, β)-fuzzy left ideals (right ideals, bi-ideals, interior ideals) of an ordered semigroup, where $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$ and $\alpha \neq \in_\gamma \wedge q_\delta$. Special attention is paid to $(\in_\gamma, \in_\gamma \vee q_\delta)$-left ideals (right ideals, bi-ideals, interior ideals) and some related properties are investigated. The characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals is also investigated.

1. Introduction

The concept of the fuzzy set was introduced by Zadeh [19]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, ect. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. Based on the terminology given by Zadeh, fuzzy sets in ordered groupoids and semigroups have been first considered by Kehayopulu and Tsingelis [5–10, 14, 17]. Using the notion "belongingness ($\in$)" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set introduced by Pu and Liu [12], the concept of (α, β)-fuzzy subgroups where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced by Bhakat and Das [1] in 1992, in which the $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup [13]. A detailed study of $(\in, \in \vee q)$-fuzzy subgroup has been considered in Bhakat and Das [2] and Bhakat [3, 4]. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems with other algebraic structures. Recently, the concept of (α, β)-fuzzy interior ideals in ordered semigroups was studied by Khan and Shabir [11].

In this paper, we consider the general forms of (α, β)-fuzzy left ideals (right ideals, bi-ideals, interior ideals) of an ordered semigroup. Special attention is paid to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals (right ideals, bi-ideals, interior ideals). The rest of this paper is organized as follows. In sections 2 and 3, we summarize some basic concepts which will be used throughout the paper and study a new ordering relation on the set of all fuzzy subsets of a non-empty set. In section 4, we define and investigate (α, β)-fuzzy left ideals (right ideals, bi-ideals, interior ideals). The

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properties and characterizations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals (right ideals, bi-ideals, interior ideals) are investigated in section 5. In section 6, we consider the characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals.

2. Preliminaries

We first summarize some basic concepts of ordered semigroups (see [5–9]), which will be used throughout the paper.

An ordered semigroup is an algebraic system $(S, \cdot, \leq)$ consisting of a non-empty set S together with a binary operation " $+$ " and a compatible ordering " $\leq$ " on S such that $(S, \cdot)$ is a semigroup and $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subset A of S is called a right (resp. left) ideal of S if it satisfies the following conditions:

(1) $AS \subseteq A$ (resp. $SA \subseteq A$),

(2) if $x \in A$ and $S \ni y \leq x$, then $y \in A$.

If A is both a right and a left ideal of S, then A is called an ideal of S.

A subset B of S is called a bi-ideal if it satisfies the following conditions:

(1) $BB \subseteq B$,

(2) $BSB \subseteq B$,

(3) if $x \in B$ and $S \ni y \leq x$, then $y \in B$.

A subset I of S is called an interior ideal if it satisfies the following conditions:

(1) $SIS \subseteq I$,

(2) if $x \in I$ and $S \ni y \leq x$, then $y \in I$.

For $A, B \subseteq S$, denote $(A] := \{x \in S | x \leq y \text{ for some } y \in A\}$ and $AB := \{xy \in S | x \in A, y \in B\}$. For $x \in S$, define $A_x = \{(y, z) \in S \times S | x \leq yz\}$.

3. Fuzzy Sets

A fuzzy subset f of a non-empty set X is defined as a mapping from X to $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The set of all fuzzy subsets of X is denoted by $\mathbb{F}(X)$. For any $A \subseteq X$ and $r \in (0, 1]$, the fuzzy subset r_A of X is defined by

$$r_A(x) = \begin{cases} r & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$. In particular, when $A = \{x\}$, r_A is said to be a fuzzy point with support x and value r and is denoted by x_r.

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point x_r and a fuzzy subset f of X, we say that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(2) $x_r \, q_\delta \, f$ if $f(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r\, q_\delta\, f$.

(4) $x_r \in_\gamma \wedge q_\delta f$ if $x_r \in_\gamma f$ and $x_r\, q_\delta\, f$.

(5) $x_r \overline{\alpha} f$ if $x_r\, \alpha\, f$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

For $A \subseteq X$, define a fuzzy subset $\chi^\delta_{\gamma A}$ of S by $\chi^\delta_{\gamma A}(x) \geq \delta$ for all $x \in A$ and $\chi^\delta_{\gamma A}(x) = \gamma$ otherwise.

Let us now introduce a new ordering relation on $\mathbb{F}(X)$, denoted as " $\subseteq \vee q_{(\gamma,\delta)}$", as follows:

For any $f, g \in \mathbb{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)}\, g$ we mean that $x_r \in_\gamma f \Rightarrow x_r \in_\gamma \vee q_\delta\; g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, f and g are said to be (γ,δ)-*equal*, denoted by $f \approx g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

In the sequel, unless otherwise stated, let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$, $M(r_1, r_2, \cdots, r_n)$, where n is a positive integer, denote $r_1 \wedge r_2 \wedge \cdots \wedge r_n$ for all $r_1, r_2, \cdots, r_n \in [0,1]$, and $\overline{\subseteq \vee q_{(\gamma,\delta)}}$ imply that $\subseteq \vee q_{(\gamma,\delta)}$ is not true.

Lemma 3.1. *Let X be a non-empty set and $f, g \in \mathbb{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)}\, g$ if and only if $g(x) \vee \gamma \geq M(f(x), \delta)$ for all $x \in X$.*

Proof. Assume that $f \subseteq \vee q_{(\gamma,\delta)}\, g$. Let $x \in X$. If $g(x) \vee \gamma < M(f(x), \delta)$, then there exists r such that $g(x) \vee \gamma < r < M(f(x), \delta)$, i.e., $x_r \in_\gamma f$ but $x_r\, \overline{\in_\gamma \vee q_\delta}\, g$, a contradiction. Hence $g(x) \vee \gamma \geq M(f(x), \delta)$.

Conversely, assume that $g(x) \vee \gamma \geq M(f(x), \delta)$ for all $x \in X$. If $f\, \overline{\subseteq \vee q_{(\gamma,\delta)}}\; g$, then there exist $x \in X$ and $r > \gamma$ such that $x_r \in_\gamma f$ but $x_r\, \overline{\in_\gamma \vee q_\delta}\; g$, and so $f(x) \geq r$, $g(x) < r$ and $g(x) + r < 2\delta$. This gives that $g(x) \vee \gamma < M(f(x), \delta)$, a contradiction. Hence $f \subseteq \vee q_{(\gamma,\delta)}\, g$. □

Lemma 3.1 implies that $f \approx g$ if and only if $M(f(x), \delta) \vee \gamma = M(g(x), \delta) \vee \gamma$ for all $x \in X$ and $f, g \in \mathbb{F}(X)$, and that " $\approx$ " is an equivalence relation on $\mathbb{F}(X)$.

Now, let us introduce the product of two fuzzy subsets of an ordered semigroup as follows:

Definition 3.2. Let $(S, \cdot, \leq)$ be an ordered semigroup and $f, g \in \mathbb{F}(S)$. Define the product of f and g, denoted by $f \circ g$, by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} M(f(y), g(z)) & \text{if there exist } y, z \in S \text{ such that } (y,z) \in A_x, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$.

From Lemma 3.1 and Definition 3.2, the following results can be easily deduced.

Lemma 3.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f_1, f_2, g_1, g_2 \in \mathbb{F}(S)$ such that $f_1 \subseteq \vee q_{(\gamma,\delta)}\; f_2$ and $g_1 \subseteq \vee q_{(\gamma,\delta)}\; g_2$. Then*

(1) $f_1 \circ g_1 \subseteq \vee q_{(\gamma,\delta)}\; f_2 \circ g_2$.

(2) $f_1 \cap g_1 \subseteq \vee q_{(\gamma,\delta)}\; f_2 \cap g_2$.

Lemma 3.3 indicates that the equivalence relation " $\approx$ " is a congruence relation on $(\mathbb{F}(S), \circ)$.

Lemma 3.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f, g, h \in \mathbb{F}(S)$. Then*

(1) $f \circ (g \cup h) = f \circ g \cup f \circ h$, $(f \cup g) \circ h = f \circ h \cup g \circ h$.

(2) $f \circ (g \cap h) \subseteq f \circ g \cap f \circ h$, $(f \cap g) \circ h \subseteq f \circ h \cap g \circ h$.

Lemma 3.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $A, B \subseteq S$. Then*

(1) $A \subseteq B$ *if and only if* $\chi^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma B}$.

(2) $\chi^{\delta}_{\gamma A} \cap \chi^{\delta}_{\gamma B} \approx \chi^{\delta}_{\gamma(A \cap B)}$.

(3) $\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B} \approx \chi^{\delta}_{\gamma(AB]}$.

Proof. The proof of (1) and (2) is straightforward. We prove (3). It suffices to show that $M(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B}, \delta) = M(\chi^{\delta}_{\gamma(AB]}, \delta)$. Let $x \in S$. If $x \in (AB]$, then $\chi^{\delta}_{\gamma(AB]} \geq \delta$ and $(a, b) \in A_x$ for some $a \in A$ and $b \in B$. Thus we have

$$(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B})(x) = \bigvee_{(y,z) \in A_x} M(\chi^{\delta}_{\gamma A}(y), \chi^{\delta}_{\gamma B}(z)) \geq M(\chi^{\delta}_{\gamma A}(a), \chi^{\delta}_{\gamma B}(b)) \geq \delta,$$

and so $M(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B}, \delta) = \delta = M(\chi^{\delta}_{\gamma(AB]}, \delta)$.

If $x \notin (AB]$, then $\chi^{\delta}_{\gamma(AB]}(x) = \gamma$. Suppose if $(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B})(x) \neq \gamma$. Then

$$(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B})(x) = \bigvee_{(y,z) \in A_x} M(\chi^{\delta}_{\gamma A}(y), \chi^{\delta}_{\gamma B}(z)) \geq \delta.$$

Hence there exist $y, z \in S$ such that $(y, z) \in A_x$, $\chi^{\delta}_{\gamma A}(y) \geq \delta$ and $\chi^{\delta}_{\gamma B}(z) \geq \delta$, i.e., $x \in (AB]$, a contradiction. Thus we have $M(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B}, \delta) = \gamma = M(\chi^{\delta}_{\gamma(AB]}, \delta)$.

Hence, in both case, we have $M(\chi^{\delta}_{\gamma A} \circ \chi^{\delta}_{\gamma B}, \delta) = M(\chi^{\delta}_{\gamma(AB]}, \delta)$. This completes the proof. □

4. (α, β)-fuzzy Ideals of Ordered Semigroups

In this section, we define and investigate (α, β)-fuzzy ideals of an ordered semigroup, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$ and $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$.

Definition 4.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called an (α, β)-fuzzy left (resp. right) ideal of S if for all $r \in (\gamma, 1]$ and $x, y \in S$,

(F1a) $x_r \, \alpha \, f \Rightarrow (yx)_{M(\delta,r)} \, \beta \, f$ (resp. $(xy)_{M(\delta,r)} \, \beta \, f$),

(F2a) if $y \leq x$, then $x_r \, \alpha \, f \Rightarrow y_r \, \beta \, f$.

If f is both an (α, β)-fuzzy left ideal and an (α, β)-fuzzy right ideal of S, then f is called an (α, β)-fuzzy ideal of S.

Definition 4.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called an (α, β)-fuzzy bi-ideal of S it satisfies conditions (F2a) and

(F3a) $x_r \, \alpha \, f$ and $y_t \, \alpha \, f \Rightarrow (xy)_{M(r,t)} \, \beta \, f$,

(F4a) $x_r \, \alpha \, f$ and $y_t \, \alpha \, f \Rightarrow (xay)_{M(r,t,\delta)} \, \beta \, f$

for all $r, t \in (\gamma, 1]$ and $x, a, y \in S$.

Definition 4.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called an (α, β)-fuzzy interior ideal of S, it satisfies conditions (F2a) and

(F5a) $x_r \, \alpha \, f \Rightarrow (axb)_{M(r,\delta)} \, \beta \, f$

for all $r \in (\gamma, 1]$ and $a, x, b \in S$.

Note that an (α, β)-fuzzy interior ideal of S with $\gamma = 0$ and $\delta = 0.5$ is the (α, β)-fuzzy interior ideal of S studied by Khan and Shabir [11].

The case $\alpha = \in_\gamma \wedge q_\delta$ must be omitted since for a fuzzy subset f of S such that $f(x) \leq \delta$ for any $x \in S$ in the case $x_r \in \wedge q f$ we have $f(x) \geq r$ and $f(x) + r > 2\delta$. Thus $f(x) + f(x) > f(x) + r > 2\delta$, which implies $f(x) > \delta$. This means that $\{x_r : x_r \in \wedge q f\} = \emptyset$.

As it is not difficult to see, each (α, β)-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S. So, in the theory of (α, β)-fuzzy ideals the central role is played by $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy ideals and we only need to investigate the properties of $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy ideals.

Lemma 4.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then any $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) f of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.*

Proof. This is straightforward since $x_r \in_\gamma f$ implies that $x_r \in_\gamma \vee q_\delta f$ for all $x \in S$ and $r \in (\gamma, 1]$. □

Lemma 4.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then any $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) f of S is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.*

Proof. This is straightforward since $x_r \, q_\delta \, f$ implies that $x_r \in_\gamma \vee q_\delta f$ for all $x \in S$ and $r \in (\gamma, 1]$. □

Theorem 4.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup, f an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S and $2\delta = 1 + \gamma$. Then the set $f_{\hat{\gamma}}$ is a left ideal (right ideal, bi-ideal, interior ideal) of S, where $f_{\hat{\gamma}} = \{x \in S | f(x) > \gamma\}$.*

Proof. Assume that f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x, y \in f_{\hat{\gamma}}$ and $v \leq u$ such that $u \in f_{\hat{\gamma}}$. Then $f(x) > \gamma$, $f(y) > \gamma$ and $f(u) > \gamma$. We consider the following two cases.

Case 1: $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then $x_{f(x)} \, \alpha \, f$, $y_{f(y)} \, \alpha \, f$ and $u_{f(u)} \, \alpha \, f$. By (F3a), $(xy)_{M(f(x),f(y))} \in_\gamma \vee q_\delta f$, i.e., $(xy)_{M(f(x),f(y))} \in_\gamma f$ or $(xy)_{M(f(x),f(y))} \, q_\delta \, f$. It follows that $f(xy) \geq M(f(x), f(y))$ or $f(xy) + M(f(x), f(y)) > 2\delta$, and so $f(xy) \geq M(f(x), f(y)) > \gamma$ or $f(xy) > 2\delta - M(f(x), f(y)) \geq 2\delta - 1 = \gamma$. Hence $f(xy) > \gamma$ and so $xy \in f_{\hat{\gamma}}$. Similarly, we may show that $v \in f_{\hat{\gamma}}$ and $xay \in f_{\hat{\gamma}}$ for all $a \in S$. Therefore, $f_{\hat{\gamma}}$ is a bi-ideal of S.

Case 2: $\alpha = q_\delta$. Then $x_1 \, \alpha \, f$, $y_1 \, \alpha \, f$ and $u_1 \, \alpha \, f$ since $2\delta = 1 + \gamma$. Analogous to proof of Case 1, we may prove that $f_{\hat{\gamma}}$ is a bi-ideal of S.

The cases for $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left ideals (right ideals, interior ideals) of S can be similarly proved. □

Theorem 4.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup, A a non-empty subset of S and $2\delta = 1 + \gamma$. Then A is a left ideal (right ideal, bi-ideal, interior ideal) of S if and only if $\chi_{\gamma A}^{\delta}$ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.*

Proof. Assume that A is a bi-ideal of S and let $f = \chi_{\gamma A}^{\delta}$. Now let $x, y \in S$ and $r, t \in (\gamma, 1]$ be such that $x_r \,\alpha\, f$ and $y_t \,\alpha\, f$. Then we have the following four cases.

Case 1: $x_r \in_\gamma f$ and $y_t \in_\gamma f$. Then $f(x) \geq r > \gamma$ and $f(y) \geq t > \gamma$. Thus, $f(x) \geq \delta$ and $f(y) \geq \delta$, i.e., $x, y \in A$.

Case 2: $x_r \, q_\delta \, f$ and $y_t \, q_\delta \, f$. Then $f(x) + r > 2\delta$ and $f(y) + t > 2\delta$, and so $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and $f(y) > 2\delta - t \geq 2\delta - 1 = \gamma$. It follows that $f(x) \geq \delta$ and $f(y) \geq \delta$, i.e., $x, y \in A$.

Case 3: $x_r \in_\gamma f$ and $y_t \, q_\delta \, f$. Then $f(x) \geq r > \gamma$ and $f(y)+t > 2\delta$. Analogous to the proof of Case 2, $f(x) \geq \delta$ and $f(y) \geq \delta$, i.e., $x, y \in A$.

Case 4: $x_r \, q_\delta \, f$ and $y_t \in_\gamma f$. Then $f(x)+r > 2\delta$ and $f(y) \geq t > \gamma$. Analogous to the proof of Case 2, $f(x) \geq \delta$ and $f(y) \geq \delta$, i.e., $x, y \in A$.

Thus, in any case, $x, y \in A$. Hence $xy \in A$, which implies that $f(xy) \geq \delta$. If $M(r,t) \leq \delta$, then $f(xy) \geq \delta \geq M(r,t) > \gamma$, i.e., $(xy)_{M(r,t)} \in_\gamma f$. If $M(r,t) > \delta$, then $f(xy) + M(r,t) > \delta + \delta = 2\delta$, i.e., $(xy)_{M(r,t)} \, q_\delta \, f$. Therefore, $(xy)_{M(r,t)} \in_\gamma \vee q_\delta \, f$. Analogous to the above proof, we have $(xay)_{M(r,t,\delta)} \in_\gamma \vee q_\delta \, f$ for all $a \in S$.

Now let $x, y \in S$ and $r \in (\gamma, 1]$ be such that $y \leq x$ and $x_r \,\alpha\, f$. Then we have the following two cases.

Case 1: $x_r \in_\gamma f$. Then $f(x) \geq r > \gamma$, which implies that $x \in A$.

Case 2: $x_r \, q_\delta \, f$. Then $f(x) + r > 2\delta$, and so $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$. It follows that $x \in A$.

Thus, in both case, $x \in A$, and so $y \in A$. If $r \leq \delta$, then $f(y) \geq r > \gamma$, i.e., $y_r \in_\gamma f$. If $r > \delta$, then $f(y)+r \geq r+r > \delta+\delta = 2\delta$, i.e., $y_r \, q_\delta \, f$. Hence, $y_r \in_\gamma \vee q_\delta \, f$.

Therefore, f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.

Conversely, assume that f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. It is easy to see that $A = f_{\hat{\gamma}}$. Hence it follows from Theorem 4.6 that A is a bi-ideal of S.

The cases for $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left ideals (right ideals, interior ideals) of S can be similarly proved. □

Example 4.8. Define on the set $S = \{a, b, c, d\}$ an ordered relation by $a \leq b$ and $a \leq c \leq d$ and a multiplication operation "$\cdot$" by the table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then $(S, \cdot, \leq)$ is an ordered semigroup. Define a fuzzy subset f of S by

$$f(a) = 0.8,\ f(b) = 0.6,\ f(c) = 0.2 \text{ and } f(d) = 0.2.$$

Then f is an $(\alpha; \in_{0.2} \vee q_{0.6})$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.

5. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of an Ordered Semigroup

In this section, using the new ordering relation defined on $\mathbb{F}(S)$, we investigate the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of an ordered semigroup. In what follows, we take $\alpha = \in_\gamma$ and $\beta = \in_\gamma \vee q_\delta$ in Definitions 4.1-4.3. Before proceeding, we first present some characterizations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

Lemma 5.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$ such that f satisfies condition* (F2a). *Then* (F1a) *holds if and only if one of the following conditions holds:*

(F1b) *$f(xy) \vee \gamma \geq M(f(y), \delta)$ (resp. $f(xy) \vee \gamma \geq M(f(x), \delta)$) for all $x, y \in S$.*
(F1c) *$\chi^\delta_{\gamma S} \circ f \subseteq \vee q_{(\gamma,\delta)} f$ (resp. $f \circ \chi^\delta_{\gamma S} \subseteq \vee q_{(\gamma,\delta)} f$).*

Proof. (F1a)$\Rightarrow$(F1b) Let $x, y \in S$. If $f(xy) \vee \gamma < r = M(f(y), \delta)$, then $y_r \in_\gamma f$, $(xy)_r \overline{\in_\gamma} f$ and $f(xy) + r < r + r \leq 2\delta$, i.e., $(xy)_r \overline{\in_\gamma \vee q_\delta}\, f$, a contradiction. Hence (F1b) is valid.

(F1b)$\Rightarrow$(F1c) For $x_r \in_\gamma \chi^\delta_{\gamma S} \circ f$, suppose if possible, let $x_r \overline{\in_\gamma \vee q_\delta}\, f$. Then $x_r \overline{\in_\gamma} f$ and $x_r \overline{q_\delta} f$, i.e., $f(x) < r$ and $f(x) + r \leq 2\delta$ which implies that $f(x) < \delta$. For any $y, z \in S$ such that $x \leq yz$, by (F2a), we have

$$\delta > f(x) \vee \gamma \geq M(f(yz), \delta) \vee \gamma = M(f(yz) \vee \gamma, \delta) \geq M(M(f(z), \delta), \delta) = M(f(z), \delta),$$

it follows that $f(x) \vee \gamma \geq f(z)$. Hence we have

$$r \leq (\chi^\delta_{\gamma S} \circ f)(x) = \bigvee_{(y,z) \in A_x} M(\chi^\delta_{\gamma S}(y), f(z)) \leq \bigvee_{(y,z) \in A_x} f(z) \leq \bigvee_{(y,z) \in A_x} f(x) \vee \gamma = f(x) \vee \gamma,$$

a contradiction. Hence (F1c) is satisfied.

(F1c)$\Rightarrow$(F1a) Let $x, y \in S$ and $r \in (\gamma, 1]$ be such that $y_r \in_\gamma f$. Then we have

$$(\chi^\delta_{\gamma S} \circ f)(xy) = \bigvee_{(a,b) \in A_{xy}} M(\chi^\delta_{\gamma S}(a), f(b)) \geq M(\delta, f(y)) \geq M(\delta, r) > \gamma.$$

Hence $(xy)_{M(\delta,r)} \in_\gamma \chi^\delta_{\gamma S} \circ f$ and so (F1a) holds. □

Lemma 5.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$. Then* (F2a) *holds if and only if the following condition holds:*

(F2b) *$f(y) \vee \gamma \geq M(f(x), \delta)$ for all $x, y \in S$ such that $y \leq x$.*

Proof. (F2a)$\Rightarrow$(F2b) Let $x, y \in S$ be such that $y \leq x$. If $f(y) \vee \gamma < r = M(f(x), \delta)$. Then $x_r \in_\gamma f$, $x_r \overline{\in_\gamma} f$ and $f(y) + r < r + r \leq 2\delta$, i.e., $y_r \overline{\in_\gamma \vee q_\delta}\, f$, a contradiction. Hence (F2b) is valid.

(F2b)$\Rightarrow$(F2a) Let $x, y \in S$ be such that $y \leq x$ and $x_r \in_\gamma f$. If $y_r \overline{\in_\gamma \vee q_\delta}\, f$, then $y_r \overline{\in_\gamma} f$ and $y_r \overline{q_\delta} f$, i.e., $f(y) < r$ and $f(y) + r \leq 2\delta$, which implies that $f(y) < \delta$. It follows that $f(y) \vee \gamma < M(f(x), \delta)$, a contradiction. Hence (F2a) is satisfied. □

Lemma 5.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$ such that f satisfies condition (F2a). Then* (F3a) *holds if and only if one of the following conditions holds:*

(F3b) $f(xy) \vee \gamma \geq M(f(x), f(y), \delta)$ *for all* $x, y \in S$.

(F3c) $f \circ f \subseteq \vee q_{(\gamma,\delta)} f$.

Proof. The proof is analogous to that of Lemma 5.1. □

Lemma 5.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$ such that f satisfies condition (F2a). Then* (F4a) *holds if and only if one of the following conditions holds:*

(F4b) $f(xay) \vee \gamma \geq M(f(x), f(y), \delta)$ *for all* $x, a, y \in S$.

(F4c) $f \circ \chi^{\delta}_{\gamma S} \circ f \subseteq \vee q_{(\gamma,\delta)} f$.

Proof. (F4a)⇒(F4b) Let $x, a, y \in S$. If $f(xay) \vee \gamma < r = M(f(x), f(y), \delta)$, then $x_r \in_\gamma f$, $y_r \in_\gamma f$, but $f(xay)+r < r+r \leq 2\delta$, i.e., $(xay)_r \overline{\in_\gamma \vee q_\delta}\, f$, a contradiction. Hence (F4b) is valid.

(F4b)⇒(F4c) For $x_r \in_\gamma f \circ \chi^{\delta}_{\gamma S} \circ f$, suppose if possible, let $x_r \overline{\in_\gamma \vee q_\delta}\, f$. Then $x_r \overline{\in_\gamma} f$ and $x_r \overline{q_\delta} f$, i.e., $f(x) < r$ and $f(x) + r \leq 2\delta$, which implies that $f(x) < \delta$. For any $a, b, z \in S$ such that $x \leq abz$, by (F2a), we have

$$\delta > f(x) \vee \gamma \geq M(f(abz), \delta) \vee \gamma = M(f(abz) \vee \gamma, \delta) \geq M(M(f(a), f(z), \delta), \delta) = M(f(a), f(z), \delta),$$

it follows that $f(x) \vee \gamma \geq M(f(a), f(z))$. Hence we have

$$\begin{aligned} r \leq (f \circ \chi^{\delta}_{\gamma S} \circ f)(x) &= \bigvee_{(y,z)\in A_x} M((f \circ \chi^{\delta}_{\gamma S})(y), f(z)) \\ &= \bigvee_{(y,z)\in A_x} M\left(\bigvee_{(a,b)\in A_y} M(f(a), \chi^{\delta}_{\gamma S}(b)), f(z) \right) \\ &= \bigvee_{(y,z)\in A_x,(a,b)\in A_y} M\left(f(a), \chi^{\delta}_{\gamma S}(b), f(z)\right) \\ &\leq \bigvee_{(y,z)\in A_x,(a,b)\in A_y} M(f(a), f(z)) \\ &\leq \bigvee_{(y,z)\in A_x,(a,b)\in A_y} f(x) \vee \gamma = f(x) \vee \gamma, \end{aligned}$$

a contradiction. Hence (F4c) is satisfied.

(F4c)⇒(F4a) Let $x, a, y \in S$ and $r, t \in (\gamma, 1]$ be such that $x_r, y_t \in_\gamma f$. Then we have

$$(f \circ \chi^{\delta}_{\gamma S} \circ f)(xay) = \bigvee_{xay \leq bcd} M\left(f(b), \chi^{\delta}_{\gamma S}(c), f(d)\right) \geq M(f(x), \delta, f(y)) \geq M(r, t, \delta) > \gamma.$$

Hence $(xay)_{M(r,t,\delta)} \in_\gamma f \circ \chi^{\delta}_{\gamma S} \circ f$ and so (F4a) holds. □

Lemma 5.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup and let $f \in \mathbb{F}(S)$ satisfies condition (F2a). Then* (F5a) *holds if and only if one of the following conditions holds:*

(F5b) $f(axb) \vee \gamma \geq M(f(x), \delta)$,*for all* $a, x, b \in S$.

(F5c) $\chi^{\delta}_{\gamma S} \circ f \circ \chi^{\delta}_{\gamma S} \subseteq \vee q_{(\gamma,\delta)} f$.

Proof. The proof is analogous to that of Lemma 5.4. □

From Lemmas 5.1-5.5, we obtain the following results.

Theorem 5.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right) ideal of S if and only if it satisfies conditions* (F1b) *and* (F2b)*, or conditions* (F1c) *and* (F2b).

Theorem 5.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal ideal of S if and only if it satisfies conditions* (F2b)-(F4b)*, or conditions* (F2b), (F3c) *and* (F4c).

Theorem 5.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if it satisfies conditions* (F2b) *and* (F5b)*, or conditions* (F2b) *and* (F5c).

For any $f \in \mathbb{F}(S)$, define $f_r = \{x \in S | x_r \in_\gamma f\}$, $f^\delta_r = \{x \in S | x_r \, q_\delta \, f\}$ and $[f]^\delta_r = \{x \in S | x_r \in_\gamma \vee q_\delta \, f\}$ for all $r \in [0, 1]$. It is clear that $[f]^\delta_r = f_r \cup f^\delta_r$.

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of S and crisp ideals of S.

Theorem 5.9. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $f \in \mathbb{F}(S)$. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S if and only if one of the following conditions holds:*

(1) *non-empty set f_r is a left ideal (right ideal, bi-ideal, interior ideal) of S for all $r \in (\gamma, \delta]$.*

(2) *non-empty set f^δ_r is a left ideal (right ideal, bi-ideal, interior ideal) of S for all $r \in (\delta, 1]$ provided that $2\delta = 1 + \gamma$.*

(3) *non-empty set $[f]^\delta_r$ is a left ideal (right ideal, bi-ideal, interior ideal) of S for all $r \in (\gamma, 1]$ provided that $2\delta = 1 + \gamma$.*

Proof. We only prove that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S if and only if condition (3) holds. The other cases can be similarly proved.

Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, $a \in S$ and $x, y \in [f]^\delta_r$ for some $r \in (\gamma, 1]$. Then $x_r \in_\gamma \vee q_\delta \, f$ and $y_r \in_\gamma \vee q_\delta \, f$, i.e., $f(x) \geq r$ or $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$, and $f(y) \geq r$ or $f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, we have $f(xy) \geq M(f(x), f(y), \delta)$ and $f(xay) \geq M(f(x), f(y), \delta)$. We consider the following two cases:

Case 1: $r \in (\gamma, \delta]$. Since $r \in (\gamma, \delta]$, we have $2\delta - r \geq \delta \geq r$, and so $f(xy) \geq M(r, r, \delta) = r$ or $f(xy) \geq M(r, 2\delta - r, \delta) = r$ or $f(xy) \geq M(2\delta - r, 2\delta - r, \delta) = \delta \geq r$. Hence $(xy)_r \in_\gamma f$. Similarly, $(xay)_r \in_\gamma f$.

Case 2: $r \in (\delta, 1]$. Since $r \in (\delta, 1]$, we have $2\delta - r < \delta < r$, and so $f(xy) \geq M(r, r, \delta) = \delta > 2\delta - r$ or $f(xy) > M(r, 2\delta - r, \delta) = 2\delta - r$ or $f(xy) > M(2\delta - r, 2\delta - r, \delta) = 2\delta - r$. Hence $(xy)_r \, q_\delta \, f$. Similarly, Hence $(xay)_r \, q_\delta \, f$.

Thus, in both case, $(xy)_r\, q_\delta\, f$ and $(xay)_r\, q_\delta\, f$, i.e., $xy \in [f]_r^\delta$ and $xay \in [f]_r^\delta$.

Now let $x, y \in S$ be such that $y \leq x$ and $x \in [f]_r^\delta$. Then $x_r \in_\gamma \vee q_\delta\, f$, i.e., $f(x) \geq r$ or $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$, and $f(y) \geq M(f(x), \delta)$. If $r \in (\gamma, \delta]$, then $f(y) \geq M(r, \delta) = r$. Hence $y_r \in_\gamma f$. If $r \in (\delta, 1]$, then $f(y) \geq M(r, \delta) = \delta > 2\delta - r$. Hence $y_r\, q_\delta\, f$.

Summing up the above arguments, $[f]_r^\delta$ is a bi-ideal of S.

Conversely, assume that the given condition holds. Let $x, y \in S$. If $f(xy) \vee \gamma < r = M(f(x), f(y), \delta)$, then $x, y \in [f]_r^\delta$, but $xy \notin [f]_r^\delta$, a contradiction. Therefore, $f(xy) \vee \gamma \geq M(f(x), f(y), \delta)$. Similarly we can show that the conditions (F2b) and (F4b) are valid. Therefore, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S by Theorem 5.6. □

As a direct of consequence of Theorem 5.9, we have the following result.

Corollary 5.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\gamma, \gamma', \delta, \delta' \in [0, 1]$ such hat $\gamma < \delta$, $\gamma' < \delta'$, $\gamma < \gamma'$ and $\delta' < \delta$. Then any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.*

The following example shows that the converse of Corollary 5.10 is not true in general.

Example 5.11. Consider Example 4.8. Define fuzzy subset f of S by

$$f(a) = 0.7,\ f(b) = 0.1,\ f(c) = 0.2 \text{ and } f(d) = 0.2.$$

Then f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S but it is not an $(\in_{0.1}, \in_{0.1} \vee q_{0.7})$-fuzzy left ideal (right ideal, bi-ideal, interior ideal) of S.

Theorem 5.12. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then:*

(1) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior of S.*

(2) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal (right ideal) of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. It is straightforward by Lemmas 5.1-5.5. □

The following examples show that the converse of Theorems 5.12(1)-(2) is not true in general.

Example 5.13. Consider Example 4.8. Define a subset f of S by

$$f(a) = 0.8,\ f(b) = 0.3,\ f(c) = 0.5 \text{ and } f(d) = 0.5.$$

Then f is $(\in_{0.4}, \in_{0.4} \vee q_{0.8})$-fuzzy interior ideal of S but it is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.8})$-fuzzy ideal of S since $M(f(c), 0.8) = 0.5 > 0.4 = f(b) \vee 0.4 = f(dc) \vee 0.4$.

Example 5.14. Define on the set $S = \{a, b, c, d\}$ an ordered relation by $a \leq b \leq c \leq d$ and a multiplication operation " $\cdot$ " by the table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	b	a
d	a	d	b	b

Then $(S, \cdot, \leq)$ is an ordered semigroup. Define a fuzzy subset f of S by

$$f(a) = 0.8,\ f(b) = 0.6,\ f(c) = 0.2 \text{ and } f(d) = 0.2.$$

Then f is an $(\in_{0.2}; \in_{0.2} \vee q_{0.6})$-fuzzy bi-ideal of S but not an $(\in_{0.2}; \in_{0.2} \vee q_{0.6})$-fuzzy left ideal (right ideal) of S since $M(f(b), 0.6) = 0.6 > 0.4 = f(c) \vee 0.4 = f(bc) \vee 0.4$ and $M(f(b), 0.6) = 0.6 > 0.4 = f(d) \vee 0.4 = f(db) \vee 0.4$.

6. The Characterization of Regular Ordered Semigroups in Terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals

In this section, we will concentrate our study on the characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals. In what follows, denote by $\mathbb{GFLI}(S)$, $\mathbb{GFRI}(S)$, $\mathbb{GFI}(S)$, $\mathbb{GFBI}(S)$ and $\mathbb{GFII}(S)$ the set of all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals, respectively.

An ordered semigroup $(S, \cdot, \leq)$ is called regular if for every $x \in S$ there exists $a \in S$ such that $x \leq xax$. Equivalent definitions: (1) $x \in (xSx]$ $\forall x \in S$, (2) $A \subseteq (ASA]$ $\forall A \subseteq S$.

Lemma 6.1. [15, 18] *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *$R \cap L = (RL]$ for any right ideal R and any left ideal L of S.*
(2) *$B \cap I = (BIB]$ for any bi-ideal B and any ideal I of S.*

Theorem 6.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is regular.*
(2) *$f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$ for any $f \in \mathbb{GFRI}(S)$ and $g \in \mathbb{GFBI}(S)$.*
(3) *$f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$ for any $f \in \mathbb{GFBI}(S)$ and $g \in \mathbb{GFLI}(S)$.*
(4) *$f \cap g \approx f \circ g$ for any $f \in \mathbb{GFRI}(S)$ and $g \in \mathbb{GFLI}(S)$.*

Proof. Assume that (1) holds. Let $f \in \mathbb{GFRI}(S)$ and $g \in \mathbb{GFBI}(S)$. For $x \in S$, since S is regular, there exists $a \in S$ such that $x \leq xax$. Thus we have

$$\begin{aligned}(f \circ g)(x) \vee \gamma = & \bigvee_{(y,z) \in A_x} M(f(y), g(z)) \vee \gamma \\ \geq & M(f(xa), g(x)) \vee \gamma \geq M(f(xa) \vee \gamma, g(x)) \\ \geq & M(f(x), g(x), \delta) = M((f \cap g)(x), \delta).\end{aligned}$$

It follows that $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$. Hence (2) holds. And (3) can be similarly proved. On the other hand, it is clear that (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) by Theorem 5.12. Now, assume that (4) holds, let R and L be any right and left ideal of S, respectively. It follows from Theorem 4.7 that $\chi^\delta_{\gamma R}$ and $\chi^\delta_{\gamma L}$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. Now, by the assumption and Lemma 3.5, we have

$$\chi^\delta_{\gamma(RL]} \approx \chi^\delta_{\gamma R} \circ \chi^\delta_{\gamma L} \approx \chi^\delta_{\gamma R} \cap \chi^\delta_{\gamma R} \approx \chi^\delta_{\gamma(R \cap L)}.$$

Hence $(RL] = R \cap L$ by Lemma 3.5. Therefore, S is regular by Lemma 6.1. □

Theorem 6.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is regular.*
(2) *$f \cap g \approx f \circ g \circ f$ for any $f \in \mathbb{GFBI}(S)$ and $g \in \mathbb{GFII}(S)$.*
(3) *$f \cap g \approx f \circ g \circ f$ for any $f \in \mathbb{GFBI}(S)$ and $g \in \mathbb{GFI}(S)$.*
(4) *$f \approx f \circ \chi^{\delta}_{\gamma S} \circ f$ for any $f \in \mathbb{GFBI}(S)$.*

Proof. (1)⇒(2) Assume that (1) holds. Let $f \in \mathbb{GFBI}(S)$ and $g \in \mathbb{GFII}(S)$. For $x \in S$, since S is regular, there exists $a \in S$ such that $x \leq xax$. Thus we have

$$\begin{aligned}(f \circ g \circ f)(x) \vee \gamma = & \bigvee_{(y,z)\in A_x} M((f \circ g)(y), f(z)) \vee \gamma \\ \geq & M((f \circ g)(xa), f(x)) \vee \gamma \\ = & M\left(\bigvee_{(b,c)\in A_{xa}} M(f(b), g(c)), f(x)\right) \vee \gamma \\ \geq & M(f(xax), g(axa), f(x)) \vee \gamma \\ & \text{(since } xa \leq xaxa \leq xaxaxa) \\ \geq & M(f(xax) \vee \gamma, g(axa) \vee \gamma, f(x)) \\ \geq & M(f(x), g(x), \delta) = M((f \cap g)(x), \delta).\end{aligned}$$

It follows that $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g \circ f$. On the other hand, since $f \in \mathbb{GFBI}(S)$ and $g \in \mathbb{GFII}(S)$, by Lemmas 5.1 and 5.4, we have

$$f \circ g \circ f \subseteq \vee q_{(\gamma,\delta)} f \circ \chi^{\delta}_{\gamma S} \circ f \subseteq \vee q_{(\gamma,\delta)} f$$

and

$$f \circ g \circ f \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma S} \circ f \circ \chi^{\delta}_{\gamma S} \subseteq \vee q_{(\gamma,\delta)} g.$$

Hence $f \circ g \circ f \subseteq \vee q_{(\gamma,\delta)} f \cap g$. Therefore $f \cap g \approx f \circ g \circ f$.

(2)⇒(3) This is straightforward by Theorem 6.3.

(3)⇒(4) This is straightforward since $\chi^{\delta}_{\gamma S}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.

(4)⇒(1) Assume that (4) holds, let B be any bi-ideal of S. It follows from Theorem 4.7 that $\chi^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Now, by the assumption and Lemma 3.5, we have

$$\chi^{\delta}_{\gamma B} \approx \chi^{\delta}_{\gamma B} \circ \chi^{\delta}_{\gamma S} \circ \chi^{\delta}_{\gamma B} \approx \chi^{\delta}_{\gamma (BSB]}.$$

Hence $B = (BSB]$ by Lemma 3.5. Therefore S is regular by Lemma 6.1. □

Theorem 6.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is regular.*
(2) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)} f \circ g \circ h$ for any $f \in \mathbb{GFBI}(S)$, $g \in \mathbb{GFII}(S)$ and $h \in \mathbb{GFBI}(S)$.*

(3) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFII}(S)$ *and* $h\in\mathbb{GFBI}(S)$.

(4) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFBI}(S)$, $g\in\mathbb{GFII}(S)$ *and* $h\in\mathbb{GFLI}(S)$.

(5) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFII}(S)$ *and* $h\in\mathbb{GFBI}(S)$.

(6) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFBI}(S)$, $g\in\mathbb{GFI}(S)$ *and* $h\in\mathbb{GFBI}(S)$.

(7)$f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFI}(S)$ *and* $h\in\mathbb{GFBI}(S)$.

(8) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFBI}(S)$, $g\in\mathbb{GFI}(S)$ *and* $h\in\mathbb{GFLI}(S)$.

(9) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFI}(S)$ *and* $h\in\mathbb{GFLI}(S)$.

Proof. Assume that (1) holds. Let $f\in\mathbb{GFBI}(S)$, $g\in\mathbb{GFII}(S)$ and $h\in\mathbb{GFBI}(S)$. For $x\in S$, since S is regular, there exists $a\in S$ such that $x\le xax$, and so $x\le xax\le xaxax\le xaxaxax\le xaxaxaxax$. Thus we have

$$\begin{aligned}(f\circ g\circ h)(x)\vee\gamma=&\bigvee_{(y,z)\in A_x,(b,c)\in A_y}M(f(b),g(c),h(z))\vee\gamma\\ \ge&M(f(xax),g(axa),h(xax))\vee\gamma\\ \ge&M(f(xax)\vee\gamma,g(axa)\vee\gamma,h(xax)\vee\gamma)\\ \ge&M(f(x),g(x),h(x),\delta)=M((f\cap g\cap h)(x),\delta).\end{aligned}$$

It follows that $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$. Hence (2) holds. On the other hand, it is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (9), (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9), (2) $\Rightarrow$ (5) $\Rightarrow$ (9) and (2) $\Rightarrow$ (6) $\Rightarrow$ (9) by Theorem 5.12 and Lemma 6.3. Now, assume that (9) holds, let $f\in\mathbb{GFRI}(S)$ and $h\in\mathbb{GFLI}(S)$. Since $\chi^{\delta}_{\gamma S}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideal of S, by the assumption, we have

$$f\cap h\approx f\circ\chi^{\delta}_{\gamma S}\circ h\subseteq\vee q_{(\gamma,\delta)}f\circ h\subseteq\vee q_{(\gamma,\delta)}f\circ\chi^{\delta}_{\gamma S}\cap\chi^{\delta}_{\gamma S}\circ h\subseteq\vee q_{(\gamma,\delta)}f\circ h.$$

Hence $f\cap h\approx f\circ h$. Therefore, S is regular by Lemma 6.2. □

Theorem 6.5. *Let $(S,\cdot,\le)$ be an ordered semigroup. Then the following conditions are equivalent.*

(1) *S is regular.*

(2) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}f\circ g\circ h$ *for any* $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFBI}(S)$ *and* $h\in\mathbb{GFLI}(S)$.

Proof. (1)⇒(2) Assume that (1) holds. Let $f\in\mathbb{GFRI}(S)$, $g\in\mathbb{GFBI}(S)$ and $h\in\mathbb{GFLI}(S)$. For $x\in S$, since S is regular, there exists $a\in S$ such that $x\le xax$. Thus we have

$$\begin{aligned}(f\circ g\circ h)(x)\vee\gamma=&\bigvee_{(y,z)\in A_x}M((f\circ g)(y),h(z))\vee\gamma\\ \ge&M((f\circ g)(x),h(ax))\vee\gamma\\ =&M\left(\bigvee_{(y,z)\in A_x}M(f(y),g(z)),h(ax)\right)\vee\gamma\\ \ge&M(f(xa),g(x),h(ax))\vee\gamma\\ \ge&M(f(xa)\vee\gamma,g(x),h(ax)\vee\gamma)\\ \ge&M(f(x),g(x),h(x),\delta)=M((f\cap g\cap h)(x),\delta).\end{aligned}$$

It follows that $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)} f \circ g \circ h$.

(2)⇒(1) It is analogous to that of Theorem 6.5. □

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References

[1] S. K. Bhakat and P. Das, *On the definition of a fuzzy subgroup*, Fuzzy Sets and Systems, **51** (1992), 235-241.

[2] S. K. Bhakat and P. Das, $(\in, \in \vee q)$*-fuzzy subgroups*, Fuzzy Sets and Systems, **80** (1996), 359-368.

[3] S. K. Bhakat, $(\in, \in \vee q)$*-fuzzy normal, quasinormal and maximal subgroups*, Fuzzy Sets and Systems, **112** (2000), 299-312.

[4] S. K. Bhakat, $(\in, \in \vee q)$*-level subsets*, Fuzzy Sets and Systems, **103** (1999), 529-533.

[5] N. Kehayopulu and M. Tsingelis, *Fuzzy sets in ordered groupoids*, Semigroup Forum, **65** (2002), 128-132.

[6] N. Kehayopulu and M. Tsingelis, *Fuzzy interior ideals in ordered semigroups*, Lobachevskii J. Math., **21** (2006), 65-71.

[7] N. Kehayopulu and M. Tsingelis, *Fuzzy bi-ideals in orderd semigroups*, Information Sciences, **171** (2005), 13-28.

[8] N. Kehayopulu and M. Tsingelis, *Regular ordered semigroups in terms of fuzzy subset*, Information Sciences, **176** (2006), 3675-3693.

[9] N. Kehayopulu and M. Tsingelis, *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math., **29** (2008), 14-20.

[10] A. Khan, Y. B. Jun and M. Shabir, *Ordered semigroups characterized by their intuitionistic fuzzy bi-ideals*, Iranian Journal of Fuzzy Systems, **7** (2010), 55-69.

[11] A. Khan and M. Shabir, (α, β)*-fuzzy interior ideals in ordered semigroups*, Lobachevskii J. Math., **30** (2009), 30-39.

[12] P. M. Pu and Y. M. Liu, *Fuzzy topology I: neighbourhood structure of a fuzzy point and MooreCSmith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.

[13] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl, **35** (1971), 512-517.

[14] M. Shabir, Y. B. Jun and M. Bano, *On prime fuzzy bi-ideals of semigroups*, Iranian Journal of Fuzzy Systems, **7** (2010), 115-128.

[15] M. Shabir and A. Khan, *Characterizations of ordered semigroups by the properties of generalized fuzzy bi-ideals*, New Math. Nat. Comput., **4(2)** (2008), 237-250.

[16] M. Shabir and A. Khan, *Characterizations of ordered semigroups by the properties of their fuzzy ideals*, Comput. Math. Appl., **59** (2010), 539-549.

[17] X. Y. Xie and J. Tang, *Regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy subsets*, Iranian Journal of Fuzzy Systems, **7** (2010), 121-140.

[18] Y. Yin and H. Li, *The characterization of regular and intra-regular ordered semigroups*, Ital. J. Pure Appl. Math., **24** (2008), 41-52.

[19] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-358.

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