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L-FUZZIFYING TOPOLOGICAL GROUPS

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Abstract. The main purpose of this paper is to introduce a concept of L-fuzzifying topological groups (here L is a completely distributive lattice) and discuss some of their basic properties and the structures. We prove that its corresponding L-fuzzifying neighborhood structure is translation invariant. A characterization of such topological groups in terms of the corresponding L-fuzzifying neighborhood structure of the unit is given. It is shown that the category of L-fuzzifying topological groups L-**FYTPG** is topological over the category of groups **GRP** with respect to the forgetful functor. As an application, the conclusion that the product of L-fuzzifying topological groups is also an L-fuzzifying topological group is proved. Finally, it is proved the forgetful functor preserves the product.

1. Introduction and Preliminaries

The theory of topological groups is abroad studied and is given the reasonable generalization in the setting of many valued sets. In 1979 the concept of fuzzy topological groups was introduced initially by Foster [2]. Since then, Yu and Ma redefined the notion of fuzzy topological groups in [17] and generalized it to the concept of L-fuzzy topological groups [7, 18] (Here L-topology is stratified, it is slight different from the L-topology discussed in [13]). From then on, many properties of L-fuzzy topological groups were discussed by Yu and Ma. On the other hand, Ying [15] gave a new approach for fuzzy topology from a logical point of view, that is, a concept of fuzzifying topology was given. Influenced by the Ying′s idea, Shen [9] introduced the concept of fuzzifying topological groups based on completely distributive residuated lattice-valued logic in 1992, and discussed the further properties of fuzzifying topological groups in 1994 (see [12]). He also generalized it to the concept of topological groups based on continuous valued logic [0,1] in [10, 11]. Moreover, Höhle [5] and Zhang [19] and Yao [14] introduced the concept of topological L-fuzzifying neighborhood structures, respectively. The equivalences between L-fuzzifying topologies and topological L-fuzzifying neighborhood structures have also been independently established by them. In this paper, we will combine the structure of L-fuzzifying topologies with the structure of groups and introduce the concept of L-fuzzifying topological groups, some basic properties of them are studied. Then we obtain a characterization of L-fuzzifying topological

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groups in terms of the L-fuzzifying neighborhood structure of the unit. Moreover, we will study the property of the forgetful functor which maps the category of L-fuzzifying topological groups L-**FYTPG** to the category of groups **GRP**. As an application, we find that L-fuzzifying topological groups are closed under products. Finally, we prove that the forgetful functor preserves the product.

First we fix some notations throughout this paper. L always denotes a completely distributive lattice if not otherwise stated. 0 and 1 are its bottom element and top element, respectively. $M(L)$ denotes the set of all nonbottom coprimes L. For $a, b \in L$, we say a is wedge below b, in symbols, $a \lhd b$ (see [3]), if for every arbitrary subset $D \subseteq L, \vee D \geq b$ implies that $a \leq d$ for some $d \in D$. For every $x \in X$, $\dot{x}$ denotes the principal filter generated by x, and for all $x \in U \subseteq X$, $\dot{x} \mid U$ denotes the collection $\{V \subseteq U \mid x \in V\}$.

Definition 1.1. [5, 15, 20, 14] An L-fuzzifying topology on the set X is a function $\tau : 2^X \to L$ which fulfills, for all $U, V, U_j \subseteq X (j \in J)$, the following:
(FY1) $\tau(\emptyset) = \tau(X) = 1$;
(FY2) $\tau(U \bigcap V) \geq \tau(U) \bigwedge \tau(V)$;
(FY3) $\tau(\bigcup_{j\in J} U_j) \geq \bigwedge_{j\in J} \tau(U_j)$.

If τ is an L-fuzzifying topology on X, then (X, τ) is an L-fuzzifying topological space.

The value $\tau(U)$ is interpreted as the degree of openness of U. A continuous function between L-fuzzifying topological spaces is a function $f : (X, \tau) \to (Y, \eta)$ such that $\tau(f^{\leftarrow}(U)) \geq \eta(U)$ for all $U \in 2^Y$ (here the notation $f^{\leftarrow}$ may be find in [8]).

Definition 1.2. [5, 6, 19, 14] An L-fuzzifying neighborhood structure on a set X is a family of functions $P = \{p_x : 2^X \to L \mid x \in X\}$ with the following conditions, for all $x \in X, U, V \subseteq X$:

(LN1) $p_x(X) = 1$;
(LN2) $p_x(U) > 0 \Rightarrow x \in U$;
(LN3) $p_x(U \bigcap V) = p_x(U) \bigwedge p_x(V)$.

The pair (X, P) is called an L-fuzzifying neighborhood space, and it will be called topological if it additionally satisfies the following condition, for all $x \in X, U \subseteq X$:

(LN4) $p_x(U) = \bigvee\limits_{V \in \dot{x}|U} \bigwedge\limits_{y \in V} p_y(V)$.

A continuous function between L-fuzzifying neighborhood spaces (X, P) and (Y, Q) is a mapping $f : X \to Y$ such that for all $x \in X, U \subseteq Y, p_x(f^{\leftarrow}(U)) \geq q_{f(x)}(U)$.

Lemma 1.3. [19] *For L a completely distributive lattice, L-fuzzifying topological spaces and topological L-fuzzyifying neighborhood spaces are categorically isomorphic to each other by the transformations* $\tau(U) = \bigwedge\limits_{x\in U} p_x^{\tau}(U)$ *and* $p_x(U) = \bigvee\limits_{V \in \dot{x}|U} \tau_p(V)$.

2. L-fuzzifying Topological Groups

Definition 2.1. Let X be a group and τ an L-fuzzifying topology on X. The pair (X, τ) is said to be an L-fuzzifying topological group iff the following description is satisfied:

Both $f : X \times X \to X,\ (x, y) \to xy$ and $g : X \to X, x \to x^{-1}$ are continuous.$(*)$

Remark 2.2. When L=[0, 1], our definition of L-fuzzifying topological groups is the same as that in [10]. In other words, our definition is a generalization of that in [10]. When L={0, 1}, an L-fuzzifying topological groups is a classical topological groups.

Remark 2.3. By Theorem 3.1 and Proposition 3.2 in [19], it is easy to find that L-fuzzifying topological group must be an L-fuzzy topological group in the sense of Yu and Ma in [18], and an L-fuzzy topological group might be not an L-fuzzifying topological group.

Moreover, by the results obtained by Yu and Ma in [18], it is obvious that the triple $(\mathcal{O}, U, hom)$ is a concrete category, where $\mathcal{O}$ is a set of L-fuzzy topological groups, $U : \mathcal{O} \to \mathcal{U}$ is a set valued function, and hom is a set of L-fuzzy continuous homomorphisms.

Theorem 2.4. *The mapping f in Definition 2.1 is continuous if and only if for every $x, y \in X$, $W \subseteq X$, $\bigvee\limits_{UV \subseteq W} p_x(U) \bigwedge p_y(V) \geq p_{xy}(W)$. Here $P = \{p_x \mid x \in X\}$ is an L-fuzzifying neighborhood structure determined by τ.*

Proof. Sufficiency. Since $f\colon (X, \tau) \times (X, \tau) \to (X, \tau)$ is continuous, it follows from Lemma 1.3 that the mapping $f\colon (X, P) \times (X, P) \to (X, P)$ is continuous. Thus for every $x, y \in X$, $W \subseteq X$, we have $p_{(x,y)}(f^{\leftarrow}(W)) \geq p_{xy}(W)$. On the other hand, $f^{\leftarrow}(W) = \bigcup_{j \in J}\{U_j \times V_j \mid U_jV_j \subseteq W\}$. So we have

$$\bigvee_{UV \subseteq W} p_x(U) \bigwedge p_y(V) = p_{(x,y)}(f^{\leftarrow}(W)) \geq p_{xy}(W). \tag{1}$$

This means that the sufficiency is proved.

Necessity. From the above proof, for each $x, y \in X$, $W \subseteq X$, we have

$$p_{(x,y)}(f^{\leftarrow}(W)) = \bigvee_{UV \subseteq W} p_x(U) \bigwedge p_y(V) \geq p_{xy}(W). \tag{2}$$

So the mapping $f : (X, P) \times (X, P) \to (X, P)$ is continuous; it follows from Lemma 1.3 that the mapping $f : (X, \tau) \times (X, \tau) \to (X, \tau)$ is continuous. This completes the proof. □

Theorem 2.5. *The mapping g in Definition 2.1 is continuous if and only if for every $x \in X$, $W \subseteq X$, $\bigvee\limits_{U^{-1} \subseteq W} p_x(U) \geq p_{x^{-1}}(W)$. Here $P = \{p_x \mid x \in X\}$ is the same as Theorem 2.4.*

Proof. The proof is similar to that of Theorem 2.4, so we leave it to readers. □

Proposition 2.6. (1) *The mapping f in Definition 2.1 is continuous if and only if the following condition holds, that is, for every $x, y \in X$, $A \subseteq X$ with $xy \in A$,*

$$\bigvee_{BC\subseteq A, B\in\dot{x}, C\in\dot{y}} \tau(B) \bigwedge \tau(C) \geq \tau(A); \tag{3}$$

(2) *The mapping g in Definition 2.1 is continuous if and only if for every $x \in X$, $A \subseteq X$ with $x^{-1} \in A$, we have*

$$\bigvee_{B^{-1}\subseteq A, B\in\dot{x}} \tau(B) \geq \tau(A). \tag{4}$$

Proof. (1) If f is continuous, then for every $x, y \in X$, $A \subseteq X$ with $xy \in A$, by Theorem 2.4, we have

$$\begin{aligned}
\tau(A) &= \bigwedge_{x\in A} p_x(A) \leq p_{xy}(A) \\
&\leq \bigvee_{BC\subseteq A} p_x(B) \bigwedge p_y(C) \\
&= \bigvee_{BC\subseteq A} \Big(\bigvee_{x\in B_1\subseteq B} \tau(B_1) \bigwedge \bigvee_{y\in C_1\subseteq C} \tau(C_1)\Big) \\
&= \bigvee_{BC\subseteq A} \bigvee_{B_1\in\dot{x}|B, C_1\in\dot{y}|C} \tau(B_1) \bigwedge \tau(C_1) \\
&= \bigvee_{BC\subseteq A, x\in B, y\in C} \tau(B) \bigwedge \tau(C).
\end{aligned} \tag{5}$$

Conversely, for all $x, y \in X$, $W \subseteq X$, we have

$$\begin{aligned}
p_{xy}(W) &= \bigvee_{xy\in W_1\subseteq W} \tau(W_1) \\
&\leq \bigvee_{UV\subseteq W_1\subseteq W, x\in U, y\in V} \tau(U) \bigwedge \tau(V) \\
&\leq \bigvee_{UV\subseteq W} p_x(U) \bigwedge p_y(V).
\end{aligned} \tag{6}$$

This means the mapping f is continuous.

(2) If the mapping g is continuous, then for every $x \in X$, $A \subseteq X$ with $x^{-1} \in A$, by Theorem 2.5, we have

$$\begin{aligned}
\tau(A) &= \bigwedge_{y\in A} p_y(A) \leq p_{x^{-1}}(A) \\
&\leq \bigvee_{U^{-1}\subseteq A} p_x(U) \\
&= \bigvee_{U^{-1}\subseteq A} \bigvee_{B\in\dot{x}|U} \tau(B) \\
&\leq \bigvee_{B^{-1}\subseteq A, x\in B} \tau(B).
\end{aligned} \tag{7}$$

On the contrary, for all $x \in X$, $W \subseteq X$, we have

$$\begin{aligned} p_{x^{-1}}(W) &= \bigvee_{x^{-1}\in W_1\subseteq W} \tau(W_1) \\ &\le \bigvee_{x^{-1}\in W_1\subseteq W} \bigvee_{U^{-1}\subseteq W_1, x\in U} \tau(U) \\ &\le \bigvee_{U^{-1}\subseteq W, x\in U} \tau(U) \\ &\le \bigvee_{U^{-1}\subseteq W} p_x(U). \end{aligned} \tag{8}$$

Thus g is continuous. □

Theorem 2.7. *Let X be a group and τ an L-fuzzifying topology on X. The pair (X, τ) is an L-fuzzifying topological group if and only if the following condition holds, that is, for every $x, y \in X$, $A \subseteq X$,*

$$\bigvee_{BC^{-1}\subseteq A} p_x(B) \bigwedge p_y(C) \ge p_{xy^{-1}}(A). \tag{9}$$

Proof. If (X, τ) is an L-fuzzifying topological group, then for all $x, y \in X$, $A \subseteq X$, by Theorem 2.4 and Theorem 2.5, we have

$$\begin{aligned} \bigvee_{BC^{-1}\subseteq A} p_x(B) \bigwedge p_y(C) &\ge \bigvee_{BC^{-1}\subseteq BD\subseteq A} p_x(B) \bigwedge p_y(C) \\ &\ge \bigvee_{BD\subseteq A} \bigvee_{C^{-1}\subseteq D} p_x(B) \bigwedge p_y(C) \\ &= \bigvee_{BD\subseteq A} (p_x(B) \bigwedge \bigvee_{C^{-1}\subseteq D} p_y(C)) \\ &\ge \bigvee_{BD\subseteq A} (p_x(B) \bigwedge p_{y^{-1}}(D)) \\ &\ge p_{xy^{-1}}(A). \end{aligned} \tag{10}$$

Conversely, for all $x, y \in X$, $A \subseteq X$, we have

$$\bigvee_{BC^{-1}\subseteq A} p_x(B) \bigwedge p_y(C) \ge p_{xy^{-1}}(A).$$

Put $x = e$ (e is the unit of group X), then

$$\begin{aligned} p_{y^{-1}}(A) &\le \bigvee_{BC^{-1}\subseteq A} p_e(B) \bigwedge p_y(C) \\ &\le \bigvee_{C^{-1}\subseteq BC^{-1}\subseteq A} p_e(B) \bigwedge p_y(C) \\ &\le \bigvee_{C^{-1}\subseteq A} p_y(C). \end{aligned} \tag{11}$$

In addition, we use y instead of y^{-1} in condition (c), then

$$\begin{aligned}
p_{xy}(A) &\le \bigvee_{BC^{-1}\subseteq A} p_x(B)\bigwedge p_{y^{-1}}(C)\\
&\le \bigvee_{BC^{-1}\subseteq A} (p_x(B)\bigwedge \bigvee_{D^{-1}\subseteq C} p_y(D))\\
&= \bigvee_{BC^{-1}\subseteq A}\bigvee_{D^{-1}\subseteq C} p_x(B)\bigwedge p_y(D)\\
&= \bigvee_{BD\subseteq BC^{-1}\subseteq A} p_x(B)\bigwedge p_y(D)\\
&\le \bigvee_{BD\subseteq A} p_x(B)\bigwedge p_y(D). \qquad (12)
\end{aligned}$$

By Theorem 2.4 and Theorem 2.5, the proof is completed. □

Example 2.8. For real line $\mathbb{R}$, we define an L-fuzzifying topology $\tau : \mathbb{R} \to \mathbb{R}$ as follows: For any $a, b \in \mathbb{R}$,

- for $a < b$, $\tau([a, b)) = \frac{1}{3}$, and $\tau(\{a\}) = \frac{1}{3}$;
- for $a < b$, $\tau((a, b]) = \frac{2}{3}$;
- for $a < b$, $\tau((a, b)) = 1$.

Here $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

For any nonempty set $U \subseteq \mathbb{R}$, we define

$$\tau(U) = \sup_{\bigcup O_i = U, O_i\in\mathcal{B}} \inf_i \tau(O_i),$$

and $\tau(\emptyset) = 1$. Where

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}\bigcup\{[a, b) : a, b \in \mathbb{R}, a < b\}$$
$$\bigcup\{(a, b] : a, b \in \mathbb{R}, a < b\}\bigcup\mathbb{R}$$

Then the set $\mathcal{B}$ defines a base for τ.

Now we can check that $(\mathbb{R}, \tau)$ is exactly an L-fuzzifying topological group, that is to say, we can verify that it satisfies the conditions of Proposition 2.6.

First, we show that $(\mathbb{R}, \tau)$ is an L-fuzzifying topological space, for which we need to verify that τ satisfies the conditions of Definition 1.1.

(i) $\tau(\mathbb{R}) = \sup_{\bigcup_{n=1}^{+\infty}} \inf_n \tau((-n, n)) = 1$, $\tau(\emptyset) = 1$.

(ii) With a basic theorem in real analysis [4], which says that any open set of $\mathbb{R}$ can be represented as a union of at most countable open intervals (a_i, b_i), we hence know that for any classical open set $U \subseteq \mathbb{R}, \tau(U) = 1$. Now we show that for any $U, V, \tau(U\bigcap V) \ge \tau(U)\bigwedge\tau(V)$. Indeed, if both U and V are open sets, then so is $U\bigcap V$, and thus $\tau(U) = \tau(V) = \tau(U\bigcap V) = 1$; for the other cases, one can also check the desired results, and we here omit the details.

(iii) It is similar to (ii) to verify that for any $U_j, j \in J$,

$$\tau(\bigcup_{j\in J} U_j) \ge \bigwedge_{j\in J} \tau(U_j).$$

Finally, to conclude that $(\mathbb{R}, \tau)$ is an L-fuzzifying topological group, we have to check that it satisfies the relations (4) and (5) in Proposition 2.6.

Here we only consider (4), since (5) can be similarly shown. For any $V \subseteq \mathbb{R}$ and $x + y \in V$, suppose $\bigcup_i O_i = V$ for some $O_i \in \mathcal{B}$, and $x + y \in O_{i_0}$ for some i_0. Then clearly there exists $O_{i_1}, O_{i_2} \in \mathcal{B}$ with $x \in O_{i_1}$ and $y \in O_{i_2}$ such that $O_{i_1} + O_{i_2} \subseteq O_{i_0}$ and $\tau(O_{i_0}) \leq \tau(O_{i_1}) \bigwedge \tau(O_{i_2})$, which shows that τ satisfies (4).

So $(\mathbb{R}, \tau)$ has been proven to be an L-fuzzifying topological group.

Example 2.9. Suppose X is a classical compact topological space, and $C(X)$ denotes the set of all real continuous functions on X, $L = \{0, a, b, 1\}$, where $a \bigvee b = 1, a \bigwedge b = 0$. For any $f,\ g \in C(X)$, we define

$$(f + g)(x) = f(x) + g(x),$$

then $(C(X), +)$ is a group. For any $f \in C(X)$ and any $\varepsilon \in (0, 1]$, denote

$$U(f, \varepsilon) = \{g \in C(X) \mid \max_{x \in X} |f(x) - g(x)| < \varepsilon\},$$

Then we define $\tau : 2^{C(X)} \to L$ as

$$\tau(\emptyset) = 1 = \tau(C(X)), \tau(\{f\}) = a,$$

$$\tau(U(f, \varepsilon)) = b\ for any f \in C(X),$$

Similarly to Example 2.8, we can show that $(C(X), \tau)$ defined above is an L-fuzzifying topological group.

Theorem 2.10. *Let (X, τ) be an L-fuzzifying topological group, and $p_x(\cdot)$ is its corresponding L-fuzzifying neighborhood structure of x, then*

(1) $p_x(A^{-1}) = p_{x^{-1}}(A), \forall x \in X, A \subseteq X;$

(2) $p_{xy}(A) = p_y(x^{-1}A) = p_x(Ay^{-1}), \forall x, y \in X, A \subseteq X;$

(3) $\tau(A) = \tau(A^{-1}) = \tau(xA) = \tau(Ax), \forall x \in X, A \subseteq X;$

(4) $\tau(A) \leq \tau(AB) \bigwedge \tau(BA), \forall A, B \subseteq X.$

Proof. (1) By $(LN3)$ in Definition 1.2 and Definition 2.1, we have

$$p_x(A) = \bigvee_{B \subseteq A} p_x(B). \tag{13}$$

$$So\ \ p_x(A^{-1}) = \bigvee_{B \subseteq A^{-1}} p_x(B) = \bigvee_{B^{-1} \subseteq A} p_x(B) \geq p_{x^{-1}}(A).$$

Similarly, we can prove that $p_{x^{-1}}(A) \geq p_x(A^{-1})$.

(2) By $(LN3)$ in Definition 1.2 and Definition 2.1, we have

$$
\begin{aligned}
p_{xy}(A) &\leq \bigvee_{BC\subseteq A} p_x(B) \bigwedge p_y(C) \\
&\leq \bigvee_{xy\in BC\subseteq A} p_x(B) \bigwedge p_y(C) \\
&\leq \bigvee_{xy\in BC\subseteq A} p_y(C) \\
&\leq \bigvee_{xy\in xC\subseteq A} p_y(C) \\
&= \bigvee_{y\in C\subseteq x^{-1}A} p_y(C) \\
&= p_y(x^{-1}A). \qquad (14)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
p_y(x^{-1}A) = p_{x^{-1}xy}(x^{-1}A) &\leq \bigvee_{BC\subseteq x^{-1}A} p_{x^{-1}}(B) \bigwedge p_{xy}(C) \\
&= \bigvee_{y=x^{-1}xy\in BC\subseteq x^{-1}A} p_{x^{-1}}(B) \bigwedge p_{xy}(C) \\
&\leq \bigvee_{x^{-1}xy\in BC\subseteq x^{-1}A} p_{xy}(C) \\
&\leq \bigvee_{x^{-1}xy\in x^{-1}C\subseteq x^{-1}A} p_{xy}(C) \\
&= \bigvee_{xy\in C\subseteq A} p_{xy}(C) = p_{xy}(A). \qquad (15)
\end{aligned}
$$

Similarly, we can prove that $p_{xy}(A) = p_x(Ay^{-1})$.
(3) From (1),(2) in this theorem, we have

$$
\tau(A) = \bigwedge_{x^{-1}\in A} p_{x^{-1}}(A) = \bigwedge_{x\in A^{-1}} p_x(A^{-1}) = \tau(A^{-1}).
$$

$$
\begin{aligned}
\tau(A) &= \bigwedge_{y\in A} p_y(A) = \bigwedge_{xy\in xA} p_{x^{-1}xy}(A) \\
&= \bigwedge_{xy\in xA} p_{xy}(xA) = \tau(xA).
\end{aligned}
$$

Similarly, it is easy to prove that $\tau(A) = \tau(Ax)$.

(4) By(3), it follows that

$$
\begin{aligned}
\tau(A) &= \bigwedge_{x\in B} \tau(A) = \bigwedge_{x\in B} \tau(xA) \\
&\leq \tau(\bigcup_{x\in B} xA) = \tau(BA).
\end{aligned}
$$

Similarly, we can easily obtain $\tau(A) \leq \tau(AB)$.
So $\tau(A) \leq \tau(AB) \bigwedge \tau(BA)$.
This completes the proof. □

Remark 2.11. Theorem 2.10 shows that corresponding L-fuzzifying neighborhood structure of L-fuzzifying topological groups is translation invariant.

3. Subgroups and Quotient Groups of L-fuzzifying Topological Groups

Definition 3.1. [16] (1) If (X,τ) is an L-fuzzifying topological space, $Y \subseteq X$, then $\tau|_Y$ (where $\tau\mid_Y (U) = \vee\{\tau(V) \mid V \subseteq X, V\mid_Y = U\}$) is called the relative L-fuzzifying topology of τ with respect to Y.

(2) Let $(X,\tau),(Y,\eta)$ be two L-fuzzifying topological spaces. If $Y \subseteq X, \eta = \tau\mid_Y$, then (Y,η) is called a subspace of (X,τ).

Theorem 3.2. *Let (X,τ) be an L-fuzzifying topological group, $P = \{p_x \mid x \in X\}$ is its corresponding L-fuzzifying neighborhood structure. H is a subgroup of X, $\tau\mid_H$ is the induced topology by τ with respect to H. For all $x \in H$, we define a function $p_x^H : 2^H \to L$ by*

$$p_x^H(A) = \bigvee_{B\bigcap H=A} p_x(B). \tag{16}$$

Then we have:

(1) *$P^H = \{p_x^H \mid x \in H\}$ is an L-fuzzifying neighborhood structure of subspace $(H,\tau\mid_H)$, which is called the induced neighborhood structure by $P = \{p_x \mid x \in X\}$ with respect to H.*

(2) *$(H,\tau\mid_H)$ is also an L-fuzzifying topological group, which is called a subgroup of (X,τ).*

Proof. (1) we verify that $P^H = \{p_x^H \mid x \in H\}$ is an L-fuzzifying neighborhood structure of $(H,\tau\mid_H)$, it suffices to show that $p_x^H(A) = \bigvee_{x\in B\subseteq A} \tau\mid_H (B)$.

In fact,

$$\begin{aligned} p_x^H(A) &= \bigvee_{B\bigcap H=A} p_x(B) = \bigvee_{x\in B\bigcap H=A} p_x(B) \\ &= \bigvee_{x\in B\bigcap H=A}\ \bigvee_{x\in C\subseteq B} \tau(C) \\ &= \bigvee_{x\in C\bigcap H\subseteq A} \tau(C) \\ &= \bigvee_{x\in D\subseteq A}\ \bigvee_{C\bigcap H=D} \tau(C) \\ &= \bigvee_{x\in D\subseteq A} \tau\mid_H (D). \end{aligned} \tag{17}$$

(2) Now we prove that $(H,\tau\mid_H)$ is also an L-fuzzifying topological group.
In fact,

$$\begin{aligned} p_{xy^{-1}}^H(A) &= \bigvee_{B\bigcap H=A} p_{xy^{-1}}(B) \\ &\leq \bigvee_{B\bigcap H=A}\ \bigvee_{CD^{-1}\subseteq B} (p_x(C) \bigwedge p_y(D)) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee_{CD^{-1}\bigcap H\subseteq A} p_x(C)\bigwedge p_y(D)\\
&\leq \bigvee_{(C\bigcap H)(D\bigcap H)^{-1}\subseteq A)} p_x(C)\bigwedge p_y(D)\\
&\leq \bigvee_{EF^{-1}\subseteq A}\ \bigvee_{C\bigcap H=E,D\bigcap H=F} p_x(C)\bigwedge p_y(D)\\
&= \bigvee_{EF^{-1}\subseteq A}\Big(\bigvee_{C\bigcap H=E} p_x(C)\bigwedge \bigvee_{D\bigcap H=F} p_y(D)\Big)\\
&= \bigvee_{EF^{-1}\subseteq A} p_x^H(E)\bigwedge p_y^H(F).
\end{aligned}
\tag{18}
$$

This completes the proof. □

Definition 3.3. Let (X,τ) be an L-fuzzifying topological group, H is a normal subgroup of X, $X^*=X/H=\{xH\mid x\in X\}$, and let $f:X\to X^*, x\mapsto xH$, for all $A^*\subseteq X^*$, we define $\tau^*:2^{X^*}\to L$ by

$$
\tau^*(A^*)=\tau(f^{\leftarrow}(A^*)). \tag{19}
$$

then τ^* is an L-fuzzifying topology of X^*, which is called the quotient topology induced with respect to f and τ on X^*, and (X^*,τ^*) is the space of the right accompanying sets of (X,τ). Similarly, we have the definition of the space of the left accompanying sets of (X,τ).

Let $P^*=\{p^*_{x^*}\mid x^*\in X^*\}$ be an L-fuzzifying neighborhood structure on X^*, i.e., for any $x^*\in X^*, A^*\subseteq X^*$,

$$
p^*_{x^*}(A^*)=\bigvee_{x^*\in B^*\subseteq A^*}\tau^*(B^*). \tag{20}
$$

Theorem 3.4. *Let (X,τ) be an L-fuzzifying topological group, H is a normal subgroup of X, $X^*=X/H$, and let $f:X\to X^*$ be the natural homomorphism, (X^*,τ^*) is the space of the right accompanying sets of (X,τ), and $P^*=\{p^*_{x^*}\mid x^*\in X^*\}$ is the L-fuzzifying neighborhood structure on X^*. Then (X^*,τ^*) is also an L-fuzzifying topological group.*

Proof. Since $P^*=\{p^*_{x^*}\mid x^*\in X^*\}$ is the L-fuzzifying neighborhood structure on X^*, we shall now prove (X^*,P^*) is an L-fuzzifying topological group.

In fact,

$$
\begin{aligned}
p^*_{x^*}(A^*) &= \bigvee_{x^*\in B^*\subseteq A^*}\tau^*(B^*)=\bigvee_{x^*\in B^*\subseteq A^*}\tau(f^{\leftarrow}(B^*))\\
&\leq \bigvee_{x^*\in B^*\subseteq A^*} p_x(f^{\leftarrow}(B^*))\leq \bigvee_{x^*\in B^*\subseteq A^*} p_x(f^{\leftarrow}(A^*))\\
&= p_x(f^{\leftarrow}(A^*)),\quad x\in f^{\leftarrow}(A^*).
\end{aligned}
\tag{21}
$$

On the other hand, if $x\in f^{\leftarrow}(A^*)$, then

$$\begin{aligned} p_x(f^{\leftarrow}(A^*)) &= \bigvee_{x\in B\subseteq f^{\leftarrow}(A^*)} \tau(B) \leq \bigvee_{x^*\in B^*\subseteq A^*} \tau(B) \\ &\leq \bigvee_{x^*\in B^*\subseteq A^*} \tau(BH) = \bigvee_{x^*\in B^*\subseteq A^*} \tau(f^{\leftarrow}(B^*)) \\ &= \bigvee_{x^*\in B^*\subseteq A^*} \tau^*(B^*) = p^*_{x^*}(A^*). \end{aligned} \tag{22}$$

Thus, for all $A^* \subseteq X^*$, we have

$$p^*_{x^*}(A^*) = p_x(f^{\leftarrow}(A^*)), \quad x \in f^{\leftarrow}(A^*). \tag{23}$$

In addition, it is not difficult to verify that for all $x^*, y^* \in X^*$, we have $f^{\leftarrow}(x^*y^{*-1}) = \{xy^{-1} \mid x \in f^{\leftarrow}(x^*), y \in f^{\leftarrow}(y^*)\}$.

So, for all $x^*, y^* \in X^*, A^* \subseteq X^*$, we have

$$\begin{aligned} p^*_{x^*y^{*-1}}(A^*) &= p_{xy^{-1}}(f^{\leftarrow}(A^*)) \\ &\leq \bigvee_{BC^{-1}\subseteq f^{\leftarrow}(A^*)} p_x(B) \bigwedge p_y(C) \\ &\leq \bigvee_{B^*C^{*-1}\subseteq A^*} p_x(B) \bigwedge p_y(C) \\ &\leq \bigvee_{B^*C^{*-1}\subseteq A^*} p_x(f^{\leftarrow}(B^*)) \bigwedge p_y(f^{\leftarrow}(C^*)) \\ &= \bigvee_{B^*C^{*-1}\subseteq A^*} p^*_{x^*}(B^*) \bigwedge p^*_{y^*}(C^*). \end{aligned} \tag{24}$$

This completes the proof. □

4. L-fuzzifying Neighborhood Structure of the Unit in L-fuzzifying Topological Groups

Theorem 4.1. *Let (X, τ) be an L-fuzzifying topological group and $p_e(\cdot)$ its corresponding L-fuzzifying neighborhood structure of the unit, then it has the following properties:*

(1) $p_e(X) = 1$;

(2) *for all* $U \subseteq X, p_e(U) > 0 \Rightarrow e \in U$;

(3) *for all* $U, V \subseteq X, p_e(U \bigcap V) = p_e(U) \bigwedge p_e(V)$;

(4) *for all* $W \subseteq X, p_e(W) \leq \bigvee_{UV^{-1}\subseteq W} p_e(U) \bigwedge p_e(V)$;

(5) *for all* $x \in X, U \subseteq X, p_e(U) \leq \bigvee_{xVx^{-1}\subseteq U} p_e(V)$.

Conversely, let X be a group and let a mapping $p(\cdot) : 2^X \to L$ satisfy conditions (1)-(5), then there exists an L-fuzzifying topology τ_p on X such that (X, τ_p) is an

L-fuzzifying topological group, and $p(\cdot)$ is an L-fuzzifying neighborhood structure of the unit with respect to τ_p.

Proof. Necessity. Conditions (1), (2) and (3) follow directly from the definition of the L-fuzzifying neighborhood structure. Condition (4) is obvious by Theorem 2.4. It remains to prove (5) only.

(5) For each $x \in X, U \subseteq X$, from Theorem 2.9, we have

$$\bigvee_{xVx^{-1}\subseteq U} p_e(V) = \bigvee_{W\subseteq U} p_e(x^{-1}Wx) = \bigvee_{W\subseteq U} p_x(Wx) = \bigvee_{W\subseteq U} p_e(W) = p_e(U). \tag{25}$$

Sufficiency. Suppose that $p(\cdot)$ is a mapping from 2^X to L which satisfies conditions (1)-(5). Let $p_x(U) = p(x^{-1}U)$, $\tau_p(U) = \bigwedge_{x\in U} p_x(U)$, First we prove that τ_p satisfies conditions (FY1), (FY2),and (FY3) in Definition 1.1.

(FY1) $\tau_p(\emptyset) = 1$, $\tau_p(X) = \bigwedge_{x\in X} p_x(X) = \bigwedge_{x\in X} p(x^{-1}X) = \bigwedge_{x\in X} p(X) = 1.$

(FY2)
$$\begin{aligned}\tau_p(U\bigcap V) &= \bigwedge_{x\in U\bigcap V} p_x(U\bigcap V)\\ &= \bigwedge_{x\in U\bigcap V} p[x^{-1}(U\bigcap V)]\\ &= \bigwedge_{x\in U\bigcap V} p[(x^{-1}U)\bigcap(x^{-1}V))]\\ &= \bigwedge_{x\in U\bigcap V} [p(x^{-1}U)\bigwedge p(x^{-1}V)]\\ &= \bigwedge_{x\in U\bigcap V} [p_x(U)\bigwedge p_x(V)]\\ &\geq [\bigwedge_{x\in U} p_x(U)]\bigwedge[\bigwedge_{x\in V} p_x(V)]\\ &= \tau_p(U)\bigwedge \tau_p(V).\end{aligned}$$

(FY3) For any $a \lhd \bigwedge_{j\in J} \tau_p(U_j)$, then for any $j \in J$, we have $a \lhd \tau_p(U_j)$. Hence for any $x \in U_j$, we have $a \lhd p_x(U_j)$. For any $y \in \bigcup_{j\in J} U_j$, there exists $j_0 \in J$ such that $y \in U_{j_0}$.

Then we have $a \lhd p_y(U_{j_0}) \leq p_y(\bigcup_{j\in J} U_j)$. So $a \lhd \bigwedge_{x\in \bigcup_{j\in J} U_j} p_x(\bigcup_{j\in J} U_j) = \tau_p(\bigcup_{j\in J} U_j)$.

By the arbitrariness of a, we have $\bigwedge_{j\in J} \tau_p(U_j) \leq \tau_p(\bigcup_{j\in J} U_j)$.

Therefore (X, τ_p) is an L-fuzzifying topological space.

Next we prove that $p_x(\cdot)$ is a topological L-fuzzifying neighborhood structure of x with respect to τ_p. Specially, $p(\cdot)$ is an L-fuzzifying neighborhood structure of the unit in X.

In fact, from Lemma 1.3, we can see that $q_x^{\tau_p}(U) = \bigvee\limits_{V\in\dot{x}|U} \tau_p(V)$ is a topological L-fuzzifying neighborhood structure of x with respect to τ_p, Now it remains to prove that $q_x^{\tau_p}(U) = p_x(U)$, $\forall U \subseteq X$. For any $a \lhd\ q_x^{\tau_p}(U)$, there exists $V \in \dot{x}|U$ such that $a \lhd \tau_p(V)$, By $x \in V$, we can obtain $a \lhd p_x(U)$, So, $\ q_x^{\tau_p}(U) \le p_x(U)$.

On the other hand, for any $a \lhd p_x(U) = p(x^{-1}U)$, Let

$$G_U^a = \{z \mid \text{ there exists } V \subseteq X, \text{ such that } a \lhd p(V) \text{ and } zV \subseteq x^{-1}U\}, \tag{26}$$

Since $e \in G_U^a$, we have $G_U^a \neq \emptyset$, for any $z \in G_U^a$, there exists $V \subseteq X$ such that $a \lhd p(V)$ and $zV \subseteq x^{-1}U$, by condition (2), we have $e \in V$, then $z \in x^{-1}U$, thus $G_U^a \subseteq x^{-1}U$.

Moreover, by (26), $\forall\ z \in G_U^a$, there exists $V \subseteq X$ such that $a \lhd p(V)$ and $zV \subseteq x^{-1}U$, For V, by condition (4), there exists $W \subseteq X$ such that $a \lhd p(W)$ and $WW \subseteq V$, then we have $zWW \subseteq x^{-1}U$. So we can easily obtain $zW \subseteq G_U^a$, thus $a \lhd p(W) = p_z(zW) \le p_z(G_U^a)$. Let $G_U^* = xG_U^a$, we can easily know that $x \in G_U^*$ and $G_U^* \subseteq U$. Also,

$$\begin{aligned} a &\le \bigwedge_{z\in G_U^a} p_z(G_U^a) = \bigwedge_{z\in x^{-1}G_U^*} p_z(x^{-1}G_U^*) = \bigwedge_{xz\in G_U^*} p_{xz}(G_U^*) \\ &= \bigwedge_{y\in G_U^*} p_y(G_U^*) = \tau_p(G_U^*) \le q_x^{\tau_p}(U). \end{aligned}$$

So, $q_x^{\tau_p}(U) \ge p_x(U)$, furthermore we have $q_x^{\tau_p}(U) = p_x(U)$, $\forall U \subseteq X$, This means that $p_x(\cdot)$ is a topological L-fuzzifying neighborhood structure of x with respect to τ_p. Specially, $p(\cdot)$ is an L-fuzzifying neighborhood structure of the unit in X.

Finally we prove that (X, τ_p) is an L-fuzzifying topological group.

$$\begin{aligned} p_{xy^{-1}}(A) = p(yx^{-1}A) &\le \bigvee_{UV^{-1}\subseteq yx^{-1}A} p(U) \bigwedge p(V) \\ &\le \bigvee_{UV^{-1}\subseteq yx^{-1}A} ((\bigvee_{yBy^{-1}\subseteq U} p(B)) \bigwedge (\bigvee_{yCy^{-1}\subseteq V} p(C))) \\ &= \bigvee_{UV^{-1}\subseteq yx^{-1}A} \bigvee_{yBy^{-1}\subseteq U} \bigvee_{yCy^{-1}\subseteq V} (p(B) \bigwedge p(C)) \\ &\le \bigvee_{(xB)(yC)^{-1}\subseteq A} p(B) \bigwedge p(C) \qquad (27) \\ &= \bigvee_{BC^{-1}\subseteq A} p(x^{-1}B) \bigwedge p(y^{-1}C) \\ &= \bigvee_{BC^{-1}\subseteq A} p_x(B) \bigwedge p_y(C), \qquad (28) \end{aligned}$$

Where (27) holds because of

$$(xB)(yC)^{-1} = xBC^{-1}y^{-1} \subseteq xB^{-1}y^{-1}V^{-1} \subseteq xy^{-1}UV^{-1} \subseteq xy^{-1}yx^{-1}A = A.$$

Therefore (X, τ_p) is an L-fuzzifying topological group, and $p(\cdot)$ is an L-fuzzifying neighborhood structure of the unit. □

5. Category of L-fuzzifying Topological Groups and Category of Groups

In this section, we will use L-**FYTPG** denotes the category of L-fuzzifying topological groups, where morphisms are continuous homomorphisms. The category of classical groups and homomorphisms is denoted by **GRP**.

Theorem 5.1. *Let $U : L$- **FYTPG** $\to$ **GRP** be a forgetful functor. Then U is topological.*

Proof. Let X be an object in **GRP**, (X_i, τ_i) be an object in L-**FYTPG** and $f_i : X \to U(X_i)$ be a morphism in **GRP** for all $i \in \Delta$. By Definition 21.1 and Definition 21.9 in [1], it follows to prove that there exists an L-fuzzifying topology τ on X with $(X, \tau) \in L$-**FYTPG** such that τ is the weakest topology with respect to which each mapping $f_i : (X, \tau) \to (X_i, \tau_i)$ is continuous. Suppose that $p_i(\cdot) : 2^{X_i} \to L$ is an L-fuzzifying neighborhood structure of zero element in (X_i, τ_i) and $\ll \Delta \gg$ denotes the set of all finite subsets of Δ. Put

$$p(A) = \bigvee_{V \in \dot{e}|A} \bigwedge \{\, p_i(V_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V,\ J_0 \in \ll \Delta \gg \,\}, \quad \forall\, A \subseteq X.$$

First we prove that the mapping $p(\cdot) : 2^X \to L$ satisfies the conditions (1)- (5) in Theorem 4.1.

It is easy to find the conditions (1) and (2) hold.

(3) By the definition of $p(\cdot)$, the mapping $p(\cdot)$ preserves order. Then for all $A,\ B \in 2^X$, $p(A \bigcap B) \leq p(A) \wedge p(B)$. Conversely, for all $a \lhd p(A) \wedge p(B)$, we have $a \lhd p(A)$ and $a \lhd p(B)$. Then there exist $V_1 \in \dot{e}|A$, $V_2 \in \dot{e}|B$ such that $a \lhd \bigwedge \{\, p_i(A_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(A_i) = V_1,\ J_0 \in \ll \Delta \gg \}$ and $a \lhd \bigwedge \{\, p_j(B_j) \mid \bigcap_{j \in J_1} f_j^{\leftarrow}(B_j) = V_2,\ J_1 \in \ll \Delta \gg \,\}$. Let $V = V_1 \cap V_2$, it is clear $V \in \dot{e}|(A \cap B)$ and $\Big(\bigcap_{i \in J_0} f_i^{\leftarrow}(A_i)\Big) \cap \Big(\bigcap_{j \in J_1} f_j^{\leftarrow}(B_j)\Big) = V$. Hence $a \leq p(A \cap B)$. This means (3) holds.

(4) For each $W \subseteq X$ and every $a \lhd p(W)$, there exists $V \in \dot{e}|W$ such that $a \lhd \bigwedge \{p_i(V_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V,\ J_0 \in \ll \Delta \gg\}$. Then $a \lhd p_i(V_i)$ for all $i \in J_0$. Since $(X_i, \tau_i) \in L$-**FYTPG**, there exist $A_i,\ B_i \in 2^{X_i}$ with $A_i B_i^{-1} \subseteq V_i$ such that $a \leq p_i(A_i) \wedge p(B_i)$. Put $A = \bigcap_{i \in J_0} f_i^{\leftarrow}(A_i)$, $B = \bigcap_{i \in J_0} f_i^{\leftarrow}(B_i)$, then $AB^{-1} \subseteq V$ and $a \leq p(A) \wedge p(B)$. Hence $p(W) \leq \bigvee_{AB^{-1} \subseteq W} p(A) \wedge p(B)$.

(5) For all $A \subseteq X,\ x \in X$, and each $a \lhd p(A)$, then there exists $V \in \dot{e}|A$ such that $a \lhd \bigwedge \{p_i(V_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V,\ J_0 \in \ll \Delta \gg\}$. Thus $a \lhd p_i(V_i)$ for each $i \in J_0$.

Since $(X_i, \tau_i) \in L$-**FYTPG**, there is $W_i \subseteq X_i$ with $f_i(x)W_i(f_i(x))^{-1} \subseteq V_i$ such that $a \leq p_i(W_i)$. Let $W = \bigcap_{i \in J_0} f_i^{\leftarrow}(W_i)$, clearly $e \in W$, and

$$\begin{aligned} xWx^{-1} &= x \bigcap_{i \in J_0} f_i^{\leftarrow}(W_i)x^{-1} = \bigcap_{i \in J_0} xf_i^{\leftarrow}(W_i)x^{-1} \\ &\subseteq \bigcap_{i \in J_0} f_i^{\leftarrow}(f_i(x)W_i(f_i(x))^{-1}) \subseteq \bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V \subseteq A. \end{aligned}$$

So we have $a \leq \bigwedge\{p_i(W_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(W_i) = W,\ J_0 \in \ll \Delta \gg\} \leq p(W)$. Hence the following relation holds.

$$p(A) \leq \bigvee_{xWx^{-1} \subseteq A} p(W).$$

Hence there exists an L-fuzzifying topology τ on X such that $(X, \tau) \in L$-**FYTPG**.

Second we prove that τ is the weakest L-fuzzifying topology such that f_i is continuous for all $i \in \Delta$.

For each $V_i \subseteq X_i$ and each $x \in X$,

$$\begin{aligned} &p_x(f_i^{\leftarrow}(V_i)) = p(f_i^{\leftarrow}(V_i)x^{-1}) \\ &= \bigvee_{A \in \dot{e}|(f_i^{\leftarrow}(V_i)x^{-1})} \bigwedge\{\ p_j(W_j) \mid \bigcap_{j \in J_0} f_j^{\leftarrow}(W_j) = A,\ J_0 \in \ll \Delta \gg \} \\ &\geq p_i(V_i(f_i(x))^{-1}) = p_{i,f_i(x)}(V_i). \end{aligned}$$

This means $f_i : (X, \tau) \to (X_i, \tau_i)$ is continuous.

In the following we will prove that τ is coarser than each L-fuzzifying topology $\widetilde{\tau}$ which makes f_i continuous for all $i \in \Delta$. In fact, suppose that $\widetilde{p}$ is an L-fuzzifying neighborhood structure of zero element with respect to $\widetilde{\tau}$. By Lemma 1.3, we prove the next relation $p(A) \leq \widetilde{p}(A)$ for all $A \subseteq X$. For each $a \lhd p(A)$, there exists $V \in \dot{e}|A$ such that $a \lhd \bigwedge\{\ p_i(V_i) \mid \bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V,\ J_0 \in \ll \Delta \gg \}$. Then $a \lhd p_i(V_i)$ for all $i \in J_0$ with $\bigcap_{i \in J_0} f_i^{\leftarrow}(V_i) = V$. Since the mapping $f_i : (X, \widetilde{\tau}) \to (X_i, \tau_i)$ is continuous, we obtain $p_i(V_i) \leq \widetilde{p}(f_i^{\leftarrow}(V_i))$ for each $i \in J_0$. So

$$a \lhd \bigwedge_{i \in J_0} \widetilde{p}(f_i^{\leftarrow}(V_i)) = \widetilde{p}(\bigcap_{i \in J_0} f_i^{\leftarrow}(V_i)) = \widetilde{p}(V) \leq \widetilde{p}(A).$$

Hence $\tau \subseteq \tilde{\tau}$. This means each U-structured source $(X \to U(X_i))_{i \in \Delta}$ has a unique U-initial lift $(X \to X_i)_{i \in \Delta}$. This completes the proof. □

From the proof of Theorem 5.1, we may obtain the following Corollary easily.

Corollary 5.2. *If $(X_j, \tau_j)_{j \in \Delta}$ is a family of L-fuzzifying topological groups, then the product $(\prod_{j \in \Delta} X_j, \prod_{j \in \Delta} \tau_j)$ is also an L-fuzzifying topological group.*

Theorem 5.3. *Let $U : L$-**FYTPG** $\to$ **GRP** be a forgetful functor. Then it preserves the product.*

Proof. First we prove that there exists a product in L- **FYTPG**. For each family of objects $(X_j,\tau_j)_{j\in\Delta}$ in L-**FYTPG**, by Corollary 5.2, $(\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j)$ is an object in L-**FYTPG**. In the following, we prove that $\{f_j : (\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j) \to (X_j,\tau_j) \mid j\in\Delta\}$ is a product of $(X_j,\tau_j)_{j\in\Delta}$ in the category of L-**FYTPG**, here the mapping $f_j : \prod_{j\in\Delta} X_j \to X_j (\forall\, j\in\Delta)$ is an ordinary projection. Let $(Q,\hat{\tau})$ be an object in L-**FYTPG** and any family of morphisms $\{q_j : (Q,\hat{\tau}) \to (X_j,\tau_j) \mid j\in\Delta\}$, we define a mapping $r : Q \to \prod_{j\in\Delta} X_j$ as follows: for each $x\in Q$ and any $j\in\Delta$, $(r(x))_j = q_j(x)$. Clearly $r(x) = (r(x)_j)_{j\in\Delta} \in \prod_{j\in\Delta} X_j$ and $q_j = f_j\circ r$ for each $j\in\Delta$. In addition, the mapping $r : (Q,\hat{\tau}) \to (\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j)$ is a morphism. In fact, for each $A\subseteq \prod_{j\in\Delta} X_j$, from the proof of Theorem 5.1 and Corollary 5.2, $p(A) = \bigvee\limits_{V\in\dot{e}|A} \bigwedge\{\, p_j(V_j) \mid \bigcap\limits_{j\in J_0} f_j^{\leftarrow}(V_j) = V,\ J_0 \in\ll \Delta \gg \}$. Then for any $a \lhd p(A)$, there exists $V\in \dot{e}|A$ such that $a\lhd \bigwedge\{p_j(V_j) \mid \bigcap\limits_{j\in J_0} f_j^{\leftarrow}(V_j) = V,\ J_0 \in\ll \Delta \gg\}$. Thus $a\lhd p_j(V_j)$ for each $j\in J_0$. Since the mapping $q_j : (Q,\hat{\tau}) \to (X_j,\tau_j)$ is a morphism, it implies $q_j : (Q,\hat{\tau}) \to (X_j,\tau_j)$ is continuous. So, $a \le p_j(V_j) \le p^Q(q_j^{\leftarrow}(V_j))$ for all $j\in J_0$, here p^Q denotes the L-fuzzifying neighborhood structure of the unit e in $(Q,\hat{\tau})$. Furthermore,

$$\begin{aligned} a &\le \bigwedge_{j\in J_0} p^Q(q_j^{\leftarrow}(V_j)) = p^Q(\bigcap_{j\in J_0} q_j^{\leftarrow}(V_j)) = p^Q(\bigcap_{j\in J_0} (f_j\circ r)^{\leftarrow}(V_j)) \\ &= p^Q(\bigcap_{j\in J_0} [r^{\leftarrow}(f_j^{\leftarrow}(V_j))]) = p^Q(r^{\leftarrow}[\bigcap_{j\in J_0} (f_j^{\leftarrow}(V_j))]) = p^Q(r^{\leftarrow}(V)) \\ &\le p^Q(r^{\leftarrow}(A)). \end{aligned}$$

This means $r : (Q,\hat{\tau}) \to (\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j)$ is continuous, that is to say $r : (Q,\hat{\tau}) \to (\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j)$ is a morphism. As for the uniqueness of r, from the definition of r, it is obvious. Thus there exists a product in L-**FYTPG**.

Secondly, from the classical theory of the category, we know there exists a product in **GRP**. By the above proof, $((\prod_{j\in\Delta} X_j, \prod_{j\in\Delta}\tau_j), (f_j)_{j\in\Delta})$ is the product of the objects $(X_j,\tau_j)_{j\in\Delta}$ in L-**FYTPG**. From the definition of the forgetful functor $U : L$-**FYTPG** $\to$ **GRP**, $U(X_j,\tau_j) = X_j \in$ **GRP** for each $j\in\Delta$. In the following, we will prove that $(\prod_{j\in\Delta} X_j, (U(f_j))_{j\in\Delta})$ is the product of groups $\{X_j\}_{j\in\Delta}$. In fact, by the classical theory of groups, $\prod_{j\in\Delta} X_j$ is a group under the following operation $*$,

$$x*y = (x_j y_j)_{j\in\Delta},\quad \forall x = (x_j)_{j\in\Delta},\ y = (y_j)_{j\in\Delta} \in \textstyle\prod_{j\in\Delta} X_j.$$

Then for each object $C\in$ **GRP**, and each family of homomorphisms $\{q_j : C\to X_j \mid j\in\Delta\}$, we define a mapping $r : C\to \prod_{j\in\Delta} X_j$ as follows: for each $z\in C$ and any $j\in\Delta$, $(r(z))_j = q_j(z)$. Clearly $r(z) = (r(z)_j)_{j\in\Delta} \in \prod_{j\in\Delta} X_j$ and $q_j = f_j\circ r = U(f_j)\circ r$ for each $j\in\Delta$. it remains to prove that the mapping r is a homomorphism. For all $z, g\in C$,

$$r(zg) = (r(zg)_j)_{j\in\Delta} = (q_j(zg))_{j\in\Delta} = (q_j(z)q_j(g))_{j\in\Delta},$$

and

$$r(z) * r(g) = \Big((r(z))_j (r(g))_j\Big)_{j\in\Delta} = (q_j(z)q_j(g))_{j\in\Delta}.$$

Thus $r(zg) = r(z) * r(g)$. This means that the mapping r is a homomorphism and $(\prod_{j\in\Delta} X_j, (U(f_j))_{j\in\Delta})$ is the product of groups $\{X_j\}_{j\in\Delta}$. The proof is completed. □

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