FUZZY INTEGRAL OF MEASURABLE MULTIFUNCTIONS

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Abstract. We study a fuzzy type integral for measurable multifunctions with respect to a fuzzy measure. Some classical properties and convergence theorems are presented.

1. Introduction

Since Zadeh [29] introduced the notion of fuzzy set, various concepts related to fuzzyness were considered (e.g., Merghadi and Aliouche [15], Vaezpour and Karini [26], Wen, Shi and Li [28]).

Due to its applications in economics, probabilities, theory of control artificial intelligence, non-additive (multi)measure theory has been investigated by many authors who introduced and studied different classical aspects (integrability, measurability, continuity, regularity, pseudo-atoms, non-atomicity, Darboux property) in non-additive (or set-valued) case (e.g., Aumann [1], Choquet [2], Croitoru et al. [4], Debreu [5], Drewnowski [6], Gavriluţ [9,10], Gavriluţ and Croitoru [11], Guo and Zhang [12], Jang and Kwon [14], Pap [16], Precupanu [17], Precupanu and Croitoru [18], Precupanu, Gavriluţ and Croitoru [19], Ralescu and Sugeno [21], Roman-Flores, Flores-Franulic and Chalco-Cano [22], Stamate [23], Sugeno [24], Suzuki [25], Wang and Klir [27], Zhang and Guo [30]).

Fuzzy measures (Sugeno [24]) are used in statistics (as imprecise probabilities), in mathematical economics (as games with transferable utility in cooperative game theory), in mathematical finances (as risk measures), in human decision making or in medicine (in prediction of osteoporotic fractures).

Fuzzy integral (Sugeno [24]) is one of the powerful means in representing a subjective evaluation process that can explain a human evaluation process more qualitatively than a linear model.

The integral of a multifunction with respect to a (single-valued) measure was defined by many authors in various ways:

- using the selections method (Aumann [1], Jang and Kwon [14], Zhang and Guo [30]);
- by the Rådström-Hörmander [20] embedding theorem (Debreu [5]);
- using a Dunford [8] type construction with defining sequences (Martellotti and Sambucini [15]).

In this paper we define an integral for measurable multifunctions with respect to a fuzzy measure. The use of the Sugeno manner [24] and the fact that the value of

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this integral is a point (and not usually a set) are the differences from other kinds of integrals. Such an integral may be used in synthetic evaluation of the quality of a given object, when the score function may be set-valued i.e., for each quality factor there exists a multiple score or a set of estimations.

The paper contains three sections. In a first section we give an introduction, in section 2 we present some preliminary definitions. In section 3 we define an integral for measurable multifunctions with respect to a fuzzy measure. Some classical properties and convergence theorems are also presented in this section.

2. Preliminary Definitions and Remarks

For a non-empty set X, $\mathcal{P}_0(X)$ is the family of non-empty subsets of X.
Let (X, d) be a metric space. Then $\mathcal{P}_f(X)$ is the family of non-empty closed subsets of X, $\mathcal{P}_{bf}(X)$ is the family of non-empty closed bounded subsets of X and $\mathcal{P}_k(X)$ is the family of non-empty compact subsets of X. For every $M, N \in \mathcal{P}_0(X)$, we denote $h(M, N) = \max\{e(M, N), e(N, M)\}$, where $e(M, N) = \sup\limits_{x \in M} d(x, N)$ is the excess of M over N and $d(x, N) = \inf\limits_{y \in N} d(x, y)$ is the distance from x to N. It is known that h becomes an extended metric on $\mathcal{P}_f(X)$ (i.e. it is a metric which can also take the value $+\infty$) and h becomes a metric (called the Hausdorff metric) on $\mathcal{P}_{bf}(X)$ (Hu and Papageorgiou [13]).
We denote $\mathbb{R}_+ = [0, +\infty)$.

Now, let X be a real linear space. On $\mathcal{P}_0(X)$ we consider an order relation denoted by " $\leq$ " and we shall write $(\mathcal{P}_0(X), \leq)$. For convenience, the notation $F \geq E$ will be used instead of $E \leq F$, for $E, F \in \mathcal{P}_0(X)$.

Example 2.1. I. The usual set inclusion "$\subseteq$" is an order relation on $\mathcal{P}_0(X)$ and we write $(\mathcal{P}_0(X), \subseteq)$.

II. Let $(X, \leq)$ be a real ordered vector space. For every $E, F \in \mathcal{P}_0(X)$ and $\alpha \in \mathbb{R}$ let

$$E + F = \{x + y | x \in E, y \in F\},$$
$$\alpha E = \{\alpha x | x \in E\},$$
$$E - F = \{x - y | x \in E, y \in F\}.$$

Let P be the cone of positive elements of X, that is, $P = \{x \in X | x \geq 0\}$. For every $E, F \in \mathcal{P}_0(X)$, we set $E \prec F$ if and only if $E \subseteq F - P$ and $F \subseteq E + P$. The relation " $\prec$ " is reflexive and transitive.
For instance, let $E \in \mathcal{P}_0(\mathbb{R})$. Then $E \succ \{0\}$ if and only if $E \subseteq [0, +\infty)$.
If $X = \mathbb{R}$, then the relation "$\prec$" is an order relation on $\mathcal{P}_{kc}(\mathbb{R})$, the family of non-empty compact convex subsets of $\mathbb{R}$.

In [3] we have introduced the concept of (monotone) set-norm. We now recall it for the sequel.

Definition 2.2. Let X be a real linear space.

A function $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is called *a set-norm* on $\mathcal{P}_0(X)$ if it satisfies the conditions:

(i) $|E| = 0 \Leftrightarrow E = \{0\}$, for $E \in \mathcal{P}_0(X)$.

(ii) $|\alpha E| = |\alpha| \cdot |E|$, $\forall \alpha \in \mathbb{R}$, $\forall E \in \mathcal{P}_0(X)$.

(iii) $|E + F| \leq |E| + |F|, \forall E, F \in \mathcal{P}_0(X)$.

Definition 2.3. A set-norm $|\cdot|$ on $(\mathcal{P}_0(X), \leq)$ is called *monotone* if $|E| \leq |F|$ for every sets $E, F \in \mathcal{P}_0(X)$, so that $E \leq F$.

Example 2.4. Let $(X, \|\cdot\|)$ be a real normed space and $\|E\| = \sup_{x \in E} \|x\|$, for every $E \in \mathcal{P}_0(X)$. Then the function $\|\cdot\|$ is a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$, called the usual set-norm on $\mathcal{P}_0(X)$.

Let T be a nonempty set and $\mathcal{C}$ be a σ-algebra of subsets of T.

Definition 2.5. Let X be a real linear space and $|\cdot|$ be a set-norm on $\mathcal{P}_0(X)$. A multifunction $F : T \to \mathcal{P}_0(X)$ is called *sn*-measurable (shortly measurable) if $|F|$ is measurable (where $|F|$ is the function defined by $|F|(t) = |F(t)|$, for all $t \in T$).

Definition 2.6. A set function $\mu : \mathcal{C} \to [0, +\infty]$ is called

(i) *monotone if $\mu(A) \leq \mu(B)$ for every $A, B \in \mathcal{C}$ so that $A \subseteq B$.*

(ii) *a fuzzy measure* if μ is monotone and $\mu(\emptyset) = 0$.

(iii) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$, $\forall A, B \in \mathcal{C}$.

(iv) *null-additive* if $\mu(A \cup B) = \mu(A)$, whenever $A, B \in \mathcal{C}$ and $\mu(B) = 0$.

(v) *continuous from below* if $\mu(\bigcup_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ with $A_n \subseteq A_{n+1}$, for all $n \in \mathbb{N}$.

(vi) *continuous from above* if $\mu(\bigcap_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, for every $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ with $A_n \supseteq A_{n+1}$, for all $n \in \mathbb{N}$.

(vii) *continuous* if μ is continuous from below and continuous from above.

(viii) *diffused* if $\mu(\{t\}) = 0$, for every $t \in T$ so that $\{t\} \in \mathcal{C}$.

Remark 2.7. According to Drewnowski [6], a subadditive fuzzy measure is called a submeasure.

3. A Fuzzy Type Integral

In this section we present some properties of an integral for measurable multifunctions with respect to a fuzzy measure.
In the sequel, T is a nonempty set, $\mathcal{C}$ is a σ-algebra of subsets of T and $\mu : \mathcal{C} \to [0, +\infty]$ is a fuzzy measure. Also, X is a real linear space and $|\cdot|$ is a set-norm on $\mathcal{P}_0(X)$.

Definition 3.1. Let $F : T \to \mathcal{P}_0(X)$ be a measurable multifunction. *The fuzzy integral* of F on an arbitrary set $A \in \mathcal{C}$ with respect to μ, denoted by $\int_A F d\mu$, is defined by $\int_A F d\mu = \sup_{\alpha \in [0,+\infty]} \min\{\alpha, \mu(A \cap |F|^{-1}([\alpha, +\infty)))\}$.

Remark 3.2. I. If μ is continuous, then $\int_A F d\mu = (S)\int_A |F| d\mu$, that is the Sugeno integral ([7], [24]) of $|F|$ on A with respect to μ.

II. The integral $\int_A F d\mu$ is in $[0,+\infty]$, for every $A \in \mathcal{C}$ (it can be either finite or infinite as we can see in Example 3.3).

III. If $\|\cdot\|$ is the usual set-norm on $\mathcal{P}_0(\mathbb{R})$ and $F : T \to \mathcal{P}_k(\mathbb{R}_+)$ is a measurable multifunction, then $\int_A F d\mu = \sup\limits_{\alpha\in[0,+\infty]} \min\{\alpha, \mu(A \cap F^{-1}([\alpha,+\infty)))\}$ for every $A \in \mathcal{C}$, where $F^{-1}([\alpha,+\infty)) = \{t \in T | F(t) \cap [\alpha,+\infty) \neq \emptyset\}$. Indeed, we have $\|F(t)\| = \sup F(t)$, for all $t \in T$.

IV. Suppose $X = \mathbb{R}$ and $\|\cdot\|$ is the usual set-norm on $\mathcal{P}_0(\mathbb{R})$. Let $f : T \to [0,+\infty)$ be a measurable function and F, G be the multifunctions defined by $F(t) = \{f(t)\}$ and $G(t) = [0, f(t)]$, for all $t \in T$. Then $\int_A F d\mu = \int_A G d\mu = (S)\int_A f d\mu$, for every $A \in \mathcal{C}$.

Example 3.3. I. Let $T = [0,1]$, $\mathcal{C}$ be the Borel σ-algebra on T and μ be the Lebesgue measure. Let $X = \mathbb{R}^2$ and $\|\cdot\|$ be the usual set-norm on $\mathcal{P}_0(\mathbb{R}^2)$. Let $F : T \to \mathcal{P}_0(\mathbb{R}^2)$ be the multifunction defined by $F(t) = [0, \frac{t\sqrt{2}}{2}] \times [0, \frac{t\sqrt{2}}{2}]$, for all $t \in T$. Then $\|F(t)\| = t$, for every $t \in T$ and $\int_{[0,1]} F d\mu = \frac{1}{2}$.

II. Let $T = \mathbb{N}$, $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and let the fuzzy measure $\mu : \mathcal{C} \to [0,+\infty]$ be defined by

$$\mu(A) = \begin{cases} \operatorname{card} A, & A \text{ is finite} \\ +\infty, & \text{otherwise} \end{cases}$$

for every $A \in \mathcal{C}$, where by card$A$ we mean the cardinality of A. Let $X = \mathbb{R}^2$ and let $\|\cdot\|$ be the usual set-norm on $\mathcal{P}_0(\mathbb{R}^2)$. Let $F : T \to \mathcal{P}_0(\mathbb{R}^2)$ be the multifunction defined by $F(t) = \{(x,y) | (x,y) \in \mathbb{R}^2, \|(x,y) - (t,t)\| = \sqrt{(x-t)^2 + (y-t)^2} < 1\}$. Then $\|F(t)\| = \sup\limits_{x\in F(t)} \|x\| = 1 + t\sqrt{2}$, for every $t \in T$ and $\int_{\mathbb{N}} F d\mu = +\infty$.

By the properties of the Sugeno integral (see for instance [27]), the following results are obtained.

Theorem 3.4. I. *If $A \in \mathcal{C}$ and $\mu(A) = 0$, then $\int_A F d\mu = 0$, for every measurable multifunction $F : T \to \mathcal{P}_0(X)$.*

II. *If F is constant, $F(t) = E \in \mathcal{P}_0(X)$, for all $t \in T$, then $\int_A F d\mu = \min\{|E|, \mu(A)\}$.*

III. *Let "$\leq$" be an order relation on $\mathcal{P}_0(X)$ such that the set-norm $|\cdot|$ is monotone on $(\mathcal{P}_0(X), \leq)$. If $F, G : T \to \mathcal{P}_0(X)$ are measurable multifunctions so that $F(t) \leq G(t)$, for all $t \in T$, then*

$$\int_A F d\mu \leq \int_A G d\mu, \quad \forall A \in \mathcal{C}.$$

IV. *Suppose μ is subadditive. If $F : T \to \mathcal{P}_0(X)$ is a measurable multifunction, then for every $A, B \in \mathcal{C}$, it holds:*

$$\int_{A\cup B} F d\mu \leq \int_A F d\mu + \int_B F d\mu.$$

V. *Let* $F : T \to \mathcal{P}_0(X)$ *be a measurable multifunction and let* $A \in \mathcal{C}$ *be an arbitrary set. Then* $\int_A F d\mu = \int_T F \cdot \chi_A d\mu$, *where* χ_A *is the characteristic function of* A *and* $F \cdot \chi_A$ *is the multifunction defined by*

$$(F \cdot \chi_A)(t) = \begin{cases} F(t), & \text{if } t \in A \\ \{0\}, & \text{if } t \notin A. \end{cases}$$

VI. *Let* $F : T \to \mathcal{P}_0(X)$ *be a measurable multifunction and* $\nu : \mathcal{C} \to [0, +\infty]$ *be the set function defined by* $\nu(A) = \int_A F d\mu$, *for every* $A \in \mathcal{C}$. *Then* ν *is a fuzzy measure. Moreover, if* μ *is diffused, then* ν *is diffused.*

We now recall the following definition.

Definition 3.5. Let $A \in \mathcal{C}$ and P be a proposition with respect to the points in A. If there is $E \in \mathcal{C}$, $E \subseteq A$ with $\mu(E) = 0$ such that P is true on $A \backslash E$, then we say "P is almost everywhere true on A". We denote "almost everywhere" by "a.e.".

In classical measure theory, the following property is well-known:

$$\begin{array}{l}\text{if the measurable functions } f \text{ and } g \text{ are} \\ \text{equal a.e., then their integrals are equal.}\end{array} \tag{1}$$

As we can see in the next example, this result is no longer valid for the fuzzy integral defined in this paper.

Example 3.6. Let $T = \{a, b\}$, $\mathcal{C} = \mathcal{P}(T)$ and let μ be defined for every $A \in \mathcal{C}$ by:

$$\mu(A) = \begin{cases} 0, & A \neq T \\ 1, & A = T. \end{cases}$$

Let $X = \mathbb{R}^2$ and $\| \cdot \|$ be the usual set-norm on $\mathcal{P}_0(\mathbb{R}^2)$. Let $F, G : T \to \mathcal{P}_0(\mathbb{R}^2)$ be the multifunctions defined for every $t \in T$ by:

$$F(t) = \begin{cases} \{0\}, & t = a \\ \left[0, \frac{\sqrt{2}}{2}\right] \times \left[0, \frac{\sqrt{2}}{2}\right], & t = b \end{cases} \text{ and } G(t) = \left[0, \frac{\sqrt{2}}{2}\right] \times \left[0, \frac{\sqrt{2}}{2}\right].$$

Then $F = G$ μ-a.e., but $\int_T F d\mu = 0$ and $\int_T G d\mu = 1$.
The next result gives an equivalent condition for the validity of (1).

Theorem 3.7. *Suppose* $|\{x\}| = |x|$ *for every* $x \in X$ *and there exists* $x \in X$ *such that* $0 < |x| < +\infty$. *Let* $\mu : \mathcal{C} \to [0, +\infty]$ *be a fuzzy measure. Then* μ *is null-additive if and only if the following property holds:*

$$\begin{array}{l}\textit{for every measurable multifunctions} \\ F, G : T \to \mathcal{P}_0(\mathbb{R}_+) \textit{ s.t. } F = G \ \mu\textit{-a.e.}, \\ \textit{it results } \int_T F d\mu = \int_T G d\mu.\end{array} \tag{2}$$

Proof. I. Suppose μ is null-additive and let $F, G : T \to \mathcal{P}_0(\mathbb{R}_+)$ be measurable multifunctions such that $F = G$ μ-a.e. Denoting $A = \{t \in T; F(t) \neq G(t)\}$, we have $\mu(A) = 0$. Let $B = \{t \in T; |F(t)| \neq |G(t)|\}$. Since μ is monotone and $B \subseteq A$, it follows $\mu(B) = 0$. So, $|F| = |G|$ μ a.e. According to the corresponding result on the Sugeno integrals (Theorem 7.3-[27]), we have $(S)\int_T |F|d\mu = (S)\int_T |G|d\mu$, that is, $\int_T Fd\mu = \int_T Gd\mu$.

II. Suppose (2) holds and let $A, B \in \mathcal{C}$ so that $\mu(B) = 0$. We have two situations:

(i) If $\mu(A) = +\infty$, by the monotonicity of μ, it follows:

$$\mu(A \cup B) = +\infty = \mu(A).$$

(ii) If $\mu(A) < +\infty$, suppose on the contrary that $\mu(A) < \mu(A \cup B)$ and let $a \in (0, +\infty)$ such that $\mu(A) < a < \mu(A \cup B)$. Let $x_0 \in X$ so that $0 < |x_0| < +\infty$ and $F, G : T \to \mathcal{P}_0(X)$ be defined for every $t \in T$ by

$$F(t) = \begin{cases} \{\frac{a}{|x_0|}x_0\}, & t \in A \\ \{0\}, & t \notin A \end{cases} \text{ and}$$

$$G(t) = \begin{cases} \{\frac{a}{|x_0|}x_0\}, & t \in A \cup B \\ \{0\}, & t \notin A \cup B. \end{cases}$$

Then $\{t \in T | F(t) \neq G(t)\} = B \backslash A$. By the monotonicity of μ, we have:

$$\mu(\{t \in T | F(t) \neq G(t)\}) = \mu(B \backslash A) \leq \mu(B) = 0,$$

which shows that $F = G$ μ-a.e. But

$$\int_T Fd\mu = \min\{a, \mu(A)\} = \mu(A)$$

and

$$\int_T Gd\mu = \min\{a, \mu(A \cup B)\} = a.$$

So $\int_T Fd\mu \neq \int_T Gd\mu$, that contradicts (2). □

Corollary 3.8. *In the hypothesis of Theorem* 3.7, *let* $\mu : \mathcal{C} \to [0, +\infty]$ *be a null-additive fuzzy measure,* $F, G : T \to \mathcal{P}_0(X)$ *be measurable multifunctions and* $A \in \mathcal{C}$ *such that* $F = G$ μ*-a.e. on* A*. Then* $\int_A Fd\mu = \int_A Gd\mu$*.*

Proof. If $F = G$ μ-a.e. on A, then $F \cdot \chi_A = G \cdot \chi_A$ μ-a.e. According to Theorem 3.7, we have $\int_T F \cdot \chi_A d\mu = \int_T G \cdot \chi_A d\mu$. Now, from Theorem 3.4-V, it results $\int_A Fd\mu = \int_A Gd\mu$. □

Theorem 3.9. *(Monotone Convergence) Let "$\leq$" be an order relation on* $\mathcal{P}_0(X)$ *so that the set-norm* $|\cdot|$ *is monotone on* $(\mathcal{P}_0(X), \leq)$*.*

I. *Suppose* μ *is a continuous fuzzy measure. Let* $F_n : T \to \mathcal{P}_0(X)$ *be measurable multifunctions such that* $F_n(t) \geq F_{n+1}(t)$*, for all* $t \in T$*,* $n \in \mathbb{N}$*,* $F_n \xrightarrow{h} F$ *(i.e.* $\lim\limits_{n\to\infty} h(F_n(t), F(t)) = 0$*, for all* $t \in T$*),* F *is measurable and there exists* $n_0 \in \mathbb{N}$ *so that* $\mu(\{t; |F_{n_0}(t)| > \int_T |F|d\mu\}) < +\infty$*. Then* $\lim\limits_{n\to\infty} \int_T F_n d\mu = \int_T Fd\mu$*.*

II. *Suppose μ is continuous from below. Let $F_n : T \to \mathcal{P}_0(X)$ be measurable multifunctions such that $F_n(t) \leq F_{n+1}(t)$, for all $t \in T$, $n \in \mathbb{N}$, $F_n \xrightarrow{h} F$ and F is measurable. Then $\int_A F d\mu = \lim\limits_{n\to\infty} \int_A F_n d\mu$, for every $A \in \mathcal{C}$.*

Proof. We have $||F_n(t)| - |F(t)|| \leq h(F_n(t), F(t))$, for all $t \in T$, $n \in \mathbb{N}$. Since $F_n \xrightarrow{h} F$, it follows $\lim\limits_{n\to\infty} |F_n(t)| = |F(t)|$, for every $t \in T$. The theorem now follows from the corresponding result (Theorem 7.5-[27]) on the Sugeno integrals of the sequence $(|F_n|)_{n\in\mathbb{N}}$. □

Theorem 3.10. *(Uniform Convergence) Let $F, F_n : T \to \mathcal{P}_0(X)$, $n \in \mathbb{N}$, be measurable multifunctions such that $F_n \xrightarrow{h} F$ uniformly on T. Then $\lim\limits_{n\to\infty} \int_T F_n d\mu = \int_T F d\mu$.*

Proof. Since $F_n \xrightarrow{h} F$ uniformly on T, for any $\varepsilon > 0$, there exists $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that $h(F_n(t), F(t)) < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N}$, $n \geq n_0$ and $t \in T$. By the properties of h, we have

$$||F_n(t)| - |F(t)|| \leq h(F_n(t), F(t)) < \frac{\varepsilon}{2}, \quad \forall t \in T.$$

According to Lemma 7.2-[27], we obtain

$$\left|(S)\int_T |F_n| d\mu - (S)\int_T |F| d\mu\right| = \left|\int_T F_n d\mu - \int_T F d\mu\right| \leq \frac{\varepsilon}{2} < \varepsilon$$

for every $n \in \mathbb{N}$, $n \geq n_0$ and this proves the theorem. □

4. **Conclusion**

We established some classical properties and convergence theorems concerning an integral for measurable multifunctions with respect to a fuzzy measure. It would be interesting to compare this integral with other types of known integrals, such as the Choquet, Aumann, Debreu ones.

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