LANGUAGE OF GENERAL FUZZY RECOGNIZER

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Abstract. In this note first by considering the notion of general fuzzy automata (for simplicity GFA), we define the notions of direct product, restricted direct product and join of two GFA. Also, we introduce some operations on (Fuzzy) sets and then prove some related theorems. Finally we construct the general fuzzy recognizers and recognizable sets and give the notion of (trim) reversal of a given GFA. In particular, we define the notion of the language of a given general fuzzy Σ-recognizer and we show that the language of direct product of two Σ-recognizer is equal to direct product of their languages.

1. **Introduction**

In 1965, L. A. Zadeh introduced the notion of a fuzzy subset of a set as a method for representing uncertainty [11]. His ideas have been applied to a wide range of scientific areas. One such area is automata theory first introduced by W.G. Wee in [10]. Automata have a long history both in theory and application [1, 2, and 7]. Automata are the prime example of general computational systems over discrete spaces. Fuzzy automata (general) and their counterparts fuzzy grammars, combine the capabilities of automata and language theory with fuzzy logic in an elegant way [8]. They have been shown to be very useful for areas which are well-known to be handled by discrete mathematics and probabilistic approaches, e.g. structural matching method and logical design. In general, fuzzy automata (general) provide an attractive systematic way for generalizing discrete applications [3]. Moreover, fuzzy automata (general) are able to create capabilities which are hardly achievable by other tools. On the other hand, the contribution of GFA to neural networks has been considerable, and dynamical fuzzy systems are getting more and more popular and useful [9].

In our work, we sometimes refer to the conventional spectrum of automata. We mean: Deterministic Finite-state Automata (DFA), Non-deterministic Finite-state Automata (NFA), Probabilistic (stochastic) Automata (PA), and Fuzzy Finite-state Automata (FFA). We just note some points which are of our concern to pave the way for our contributions.

(1) NFA have the characteristic of ε-transition. An NFA can transit from one state to another on empty string (ε). Although this property does not make sense practically, it is very useful in developing the theory of formal languages and theorem proving. We will use this theoretically property in a different sense for FFA.

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(2) In NFA, like FFA overlapping of transitions is possible. However, since δ is applied non-deterministically but not probabilistically, and due to ε-transition, there is no way to define deterministic behaviour of a NFA. However, if it had been possible to do that, there would have been no vagueness and fuzziness in NFA operation as all active states, successors, and predecessors would have had an implied mv of 1. This means that the sheer overlapping of transitions does not imply fuzziness. Moreover, any NFA M can be unrolled to DFA M' (which can have considerably more states and is said to be equivalent to M terms of accepting language, i.e. $L(M) = L(M')$) where at any time there is a unique predecessor. But, generally, there is no way to simulate the operation of fuzzy automaton by DFA, or any other type of automata in the conventional spectrum of automata (or even any combination of them). However, it may be possible to derive DFA which are equivalent to some specific and narrowed cases of fuzzy grammars. But, this can not be generalized to a general class of fuzzy automata.

(3) All transitions of a DFA and NFA have an implied weight of 1, while the weights of transitions in a PA and FFA belong to $(0, 1]$. However, in all types of conventional automata, a zero-weight transition means no transition, while in our GFA, a zero-weight transition does not necessarily imply transition. That is why we will use $[0, 1]$ as the fuzzy interval.

(4) although PA and FFA seem to be similar, they have two major differences:
(a) operationally in a PA, at any specific time, there is a single active state which has exactly on predecessor and one successor (although the successor is not predicable, if more than one potential successor exist). The weights are in fact probabilities of transition, which cause δ to be applied randomly. When the PA is put into operation, there is no vagueness in the current state, next state, and extent to which they will be activated. At any time (upon each input symbol), one and exactly one state will be activated with an implied membership value of 1.
(b) In PA, the sum of the weights for the transitions (from the current state) corresponding to a specific input symbol should be one, whilst there is no such requirement for a FFA.

(5) Compared to other automata in the conventional spectrum, fuzzy automata is more general. We have included the function μ_Q (which maps membership values to state) to emphasize this point. Yet, FFA are not general enough to represent completely other automata in the conventional spectrum. That is why, have presented a new and general definition for fuzzy automata [Definition 2.2], which not only encompasses all types of automata, but also several other computational paradigms.

In this note we introduce three products (direct product, restricted direct product and join) of two general fuzzy automata (GFA for short). We construct general fuzzy recognizers and investigate certain general recognizable sets. We establish that:
1) The class of (general) recognizable sets is closed with respect to union, intersection, reversal, cross product, left and right quotient sets (by a string and set of stings) [Theorem 4.20] and inverse image of a set by fine homomorphism and right product of a set by a string [Theorem 4.27].

2) If a general fuzzy recognizer M^* is an accessible (coaccessible, trim) then $M^a = M^*$ ($M^b = M^*, M^t = M^*$, respectively), where M^a, M^b and M^t respectively denote the accessible, coaccessible and trim parts of M^*.

2. **Preliminaries**

In this section, we present the required background, addressing some of the shortages of fuzzy automata, and present some conventions and definitions which are necessary.

Definition 2.1. [3] A fuzzy finite-state automaton (FFA) is a six-tuple denoted as $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0, 1]$ and $\omega : Q \to Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in $[0, 1]$. We call this membership value the weight of the transition.

The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refer to the weight of the transition. Otherwise, it specifies the transition itself. Also, Δ shows that a set of all transitions of $\tilde{F}$. The above definition is generally accepted as a formal definition for FFA [6, 8, and 9].

We believe that the current literature and background for fuzzy automata is not established well enough to characterizes the operation of the automaton and thus fulfill the implementation requirements in a well-define manner. There is an important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how assign a membership value to a next state upon the completion of a transition. Secondly, how should we deal with the cases where a state is forced to take several membership values simultaneously via overlapping transition?

During the past few years, several researchers studied the equivalence and isomorphism of fuzzy automata with other types of automata. However, we believe that the role of fuzzy automata goes beyond equivalence. In fact, they can represent not only other types of automata, but also several other computational paradigms. To make the computational generality of fuzzy automata and its generalization capability more systematic and application-friendly, we need a more general definition. In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [3].

Definition 2.2. [3] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where (i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma\{a_1, a_2, \ldots, a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, (iv) Z is a finite set of output symbols, $Z\{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function, (vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$ as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

Which means that membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

There are many options which can be used for the function $F_1(\mu, \delta)$. It can for example be the max $\{\mu, \delta\}$, min $\{\mu, \delta\}$, $\frac{\mu+\delta}{2}$ or any other applicable mathematical function.

(Viii) $F_2 : [0,1]^* \to [0,1]$ is called multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns them a single membership value .

Definition 2.3. [3] Knowing that the entered input to time t has been a_k, active states at time t are those states to which there is at least one transition on the input symbol a_k. Then, the fuzzy set of all active states at t (ordered pairs of states and their mv's) is called active states set at time t, and is denoted as $Q_{act}(t)$.

We let $Q_{act}(t_i)$ be the set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write:

$$q \in Domain(Q_{act}(t_i)) \;\text{ and }\; T \subseteq Domain(Q_{act}(t_i)).$$

Hereafter, we simply denote them as:

$$q \in Q_{act}(t_i) \;\text{ and }\; T \subseteq Q_{act}(t_i).$$

The combination of the operations of function F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 2.4. [3] (Multi-membership resolution)

If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to that:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this ν_i,

$$\nu_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous mv of the active state q_j.

$$\mu^{t+1}(q_j) = \underbrace{F_2^n}_{i=1}[\nu_i] = \underbrace{F_2^n}_{i=1}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

Where,

- n: is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$: is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Definition 2.5. [12, Definition 2.1] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata of the form

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

such that $\tilde{\delta}^* = Q_{act} \times \Sigma^* \times Q \to [0,1]$, where $Q_{act} = Q_{act}(t_0) \cup Q_{act}(t_1) \cup Q_{act}(t_2) \cup \dots$ and let for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & if q = p \\ 0, & otherwise \end{cases}$$

and for every $i, i \geq 1$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p),$$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q')), u_{i+1}, p)$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \dots$$
$$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma$, $1 \leq i \leq n$ and assuming that the entered input at time t_i be u_i, $1 \leq i \leq n-1$.

3. Some Algebraic Operations on General Fuzzy Automata

In this section by considering the notion of general fuzzy automata we define direct product, restricted direct product and join of two GFA. Then we give some examples to illustrate the capabilities achievable from these definitions. Finally, we introduce some operations on fuzzy sets and then prove some related theorems.

Definition 3.1. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ and

$$\tilde{F}' = (Q', \Sigma', \tilde{R}', Z', \tilde{\delta}', \omega', F_1', F_2')$$

be general fuzzy automata. Define $\tilde{\delta} \times \tilde{\delta}' : ((Q \times Q') \times [0,1]) \times (\Sigma \times \Sigma') \times (Q \times Q') \to [0,1]$ as follows:

$$\begin{aligned} \mu^{t_{i+j+1}}(q_j, q_j') &= \tilde{\delta} \times \tilde{\delta}'(((q_i, q_i'), (\mu^{t_{i+j}}(q_i, q_i'))), (a_k, a_k'), (q_j, q_j')) \\ &= \tilde{\delta}((q_i, \mu^{t_i}(q_i)), a_k, q_j) \wedge \tilde{\delta}'((q_i', \mu^{t_j}(q_i')), a_k', q_j') \\ &= F_1(\mu^{t_i}(q_i), \tilde{\delta}(q_i, a_k, q_j)) \wedge F_1'(\mu^{t_j}(q_i'), \tilde{\delta}'(q_i', a_k', q_j')) \\ &= \wedge(F_1(\mu^{t_i}(q_i), \tilde{\delta}(q_i, a_k, q_j)), F_1'(\mu^{t_j}(q_i'), \tilde{\delta}'(q_i', a_k', q_j'))) \end{aligned}$$

and $\omega \times \omega' : Q \times Q' \to Z \times Z'$ as follows: $(\omega \times \omega')(q, q') = (\omega(q), \omega'(q'))$.

Recall that the general fuzzy automaton

$$\tilde{F}\times\tilde{F}'=(Q\times Q',\Sigma\times\Sigma',\tilde{R}\times\tilde{R}',Z\times Z',\tilde{\delta}\times\tilde{\delta}',\omega\times\omega',F_1'',F_2'')$$

is called the direct product of $\tilde{F}$ and $\tilde{F}'$, where $F_1'':[0,1]\times[0,1]\to[0,1]$ is defined by $F_1''=Min(F_1,F_1')$ and $F_2'':[0,1]^*\to[0,1]$ is any other applicable mathematical function, in other words we have:

$$\mu^{t_{i+j+1}}(q_j,q_j')=\underbrace{F_2''^{n}}_{i=1}[\tilde{\delta}\times\tilde{\delta}'(((q_i,q_i'),(\mu^{t_{i+j}}(q_i,q_i'))),(a_k,a'_{k'}),(q_j,q_j'))]$$

and $\tilde{R}\times\tilde{R}'\subseteq\tilde{P}(Q\times Q')$ such that:

$$\tilde{R}\times\tilde{R}'=\{\mu_Q\times\mu_{Q'}|\mu_Q\times\mu_{Q'}:Q\times Q'\to[0,1];\mu_Q\in\tilde{R},\mu_{Q'}\in\tilde{R}'\}$$

where $(\mu_Q\times\mu_{Q'})(q,q')=\mu_Q(q)\wedge\mu_{Q'}(q')$.

Let $\tilde{F}=(Q,\Sigma,\tilde{R},Z,\tilde{\delta},\omega,F_1,F_2)$ and $\tilde{F}'=(Q',\Sigma,\tilde{R}',Z,\tilde{\delta}',\omega',F_1',F_2')$ be general fuzzy automata. Define $\tilde{\delta}\cap\tilde{\delta}':((Q\times Q')\times[0,1])\times\Sigma\times(Q\times Q')\to[0,1]$ as follows:

$$\begin{aligned}\mu^{t_{i+j+1}}(q_j,q_j') &= \tilde{\delta}\cap\tilde{\delta}'(((q_i,q_i'),(\mu^{t_{i+j}}(q_i,q_i'))),a_k,(q_j,q_j'))\\ &= \tilde{\delta}((q_i,\mu^{t_i}(q_i)),a_k,q_j)\wedge\tilde{\delta}'((q_i',\mu^{t_j}(q_i')),a_k,q_j'),\end{aligned}$$

and $\omega\cap\omega':Q\times Q'\to Z$ as follows:

$$(\omega\cap\omega')(q,q')=\begin{cases} b_k \text{ if } q\in Q_{act},q'\in Q'_{act},\ \text{where } b_k\in\{\omega(q)=b_i,\omega'(q')=b_j;\\ \text{for some} 1\le i,j\le k\\ \text{and } k=Min\{i,j\}\\ \omega'(q'),\ \text{otherwise}\end{cases}$$

Recall that the general fuzzy automaton,

$$\tilde{F}\cap\tilde{F}'=(Q\times Q',\Sigma,\tilde{R}\cap\tilde{R}',Z,\tilde{\delta}\cap\tilde{\delta}',\omega\cap\omega',F_1'',F_2'')$$

is called the restricted direct product of $\tilde{F}$ and $\tilde{F}'$, where $F_1'':[0,1]\times[0,1]\to[0,1]$ is defined by $F_1''=Min(F_1,F_1')$ and $F_2'':[0,1]^*\to[0,1]$ is any other applicable mathematical function, in other words we have:

$$\mu^{t_{i+j+1}}(q_j,q_j')=\underbrace{F_2''^{n}}_{i=1}[\tilde{\delta}\cap\tilde{\delta}'(((q_i,q_i'),(\mu^{t_{i+j}}(q_i,q_i'))),a_k,(q_j,q_j'))]$$

and $\tilde{R}\cap\tilde{R}'\subseteq\tilde{P}(Q\times Q')$ such that:

$$\tilde{R}\cap\tilde{R}'=\{\mu_Q\cap\mu_{Q'}|\mu_Q\cap\mu_{Q'}:Q\times Q'\to[0,1];\mu_Q\in\tilde{R},\mu_{Q'}\in\tilde{R}'\}$$

where $(\mu_Q\times\mu_{Q'})(q,q')=\mu_Q(q)\wedge\mu_{Q'}(q')$.

Definition 3.2. Let $\tilde{F}=(Q,\Sigma,\tilde{R},Z,\tilde{\delta},\omega,F_1,F_2)$ and

$$\tilde{F}'=(Q',\Sigma,\tilde{R}',Z,\tilde{\delta}',\omega',F_1',F_2')$$

be general fuzzy automata such that $Q\cap Q'=\emptyset$. Define

$$\tilde{\delta}\cup\tilde{\delta}':((Q\cup Q')\times[0,1])\times\Sigma\times(Q\cup Q')\to[0,1]$$

as follows:

$$\tilde{\delta} \cup \tilde{\delta}'((q_i, \mu^t(q_i)), a_k, q_j) = \begin{cases} \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j), & if q_i, q_j \in Q \\ \tilde{\delta}'((q_i, \mu^t(q_i)), a_k, q_j), & if q_i, q_j \in Q' \\ 0, otherwise \end{cases}$$

and $\omega \cup \omega' : Q \cup Q' \to Z$ as follows:

$$\omega \cup \omega'(q) = \begin{cases} \omega(q), & if q \in Q \\ \omega'(q), & if q \in Q'. \end{cases}$$

Then we define the general fuzzy automaton $\tilde{F} \cup \tilde{F}' = (Q \cup Q', \Sigma, \tilde{R} \cup \tilde{R}', Z, \tilde{\delta} \cup \tilde{\delta}', \omega \cup \omega', F_1'', F_2'')$ and it is called the join of $\tilde{F}$ and $\tilde{F}'$.

By definition of $\tilde{\delta} \cup \tilde{\delta}'$ we have $F_1'' = F_1$, $F_2'' = F_2$ and $F_1'' = F_1'$, $F_2'' = F_2'$ when $q_i, q_j \in Q$, for all i, j. Also $\tilde{R} \cup \tilde{R}' \subseteq \tilde{P}(Q \cup Q')$ such that:

$$\tilde{R} \cup \tilde{R}' = \{\mu_Q \cup \mu_{Q'} | \mu_Q \cup \mu_{Q'} : Q \cup Q' \to [0, 1]\}$$

where $(\mu_Q \cup \mu_{Q'})(q) = \begin{cases} \mu_Q(q) & if q \in Q \\ \mu_{Q'}(q) & if q \in Q'. \end{cases}$

Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton. A general fuzzy automaton $\tilde{F}^s = (Q^s, \Sigma^s, \tilde{R}^s, Z^s, \tilde{\delta}^s, \omega^s, F_1, F_2)$ is called a sub-general fuzzy automaton of $\tilde{F}$, written by $\tilde{F}^s \subseteq \tilde{F}$, if

(i) $Q^s \subseteq Q, \Sigma^s \subseteq \Sigma, Z^s \subseteq Z$, and

(ii) $\tilde{\delta}^s = \tilde{\delta}|(Q^s \times [0, 1]) \times \Sigma^s \times Q^s, \omega^s = \omega|Q^s$.

In the following, we give an example illustrating the capabilities achievable from these definitions

Example 3.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ and

$$\tilde{F}' = (Q', \Sigma', \tilde{R}', Z', \tilde{\delta}', \omega', F_1, F_2)$$

be general fuzzy automata as follows:

If we choose $F_1(\mu, \delta) = F_1'(\mu, \delta) = Min(\mu, \delta)$, $F_2 = F_2'' = Max$ and F_2' not applicable, then by definition of general fuzzy automata we have the active states and mv's at different times as follows:

time	t_0	t_1		t_2		t_3		t_4		t_5	
input	Λ	a		a		b		b		a	
$Q_{act}(t_i)$	q_0	q_2	q_3	q_1	q_4	q_2	q_3	q_2	q_1	q_1	q_4
mv	1	0.8	0.9	0.6	0.2	0.6	0.2	0.5	0.2	0.5	0.2

Table 1. The Active State and mv's at Different Times of $\tilde{F}$ in Example 3.3

For example:

$$\begin{aligned} \mu^{t_3}(q_2) &= F_2(F_1(\mu^{t_2}(q_1), \delta(q_1, b, q_2)), F_1(\mu^{t_2}(q_4), \delta(q_4, b, q_2)) \\ &= F_2([0.6 \wedge 0.6], [0.6 \wedge 0.1]) = 0.6 \vee 0.1 = 0.6 \end{aligned}$$

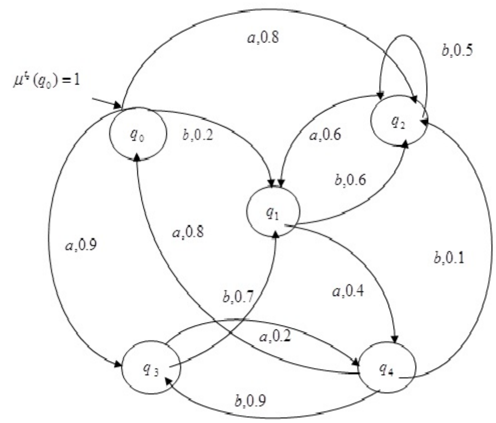

FIGURE 1. The $\tilde{F}$ in Example 3.3

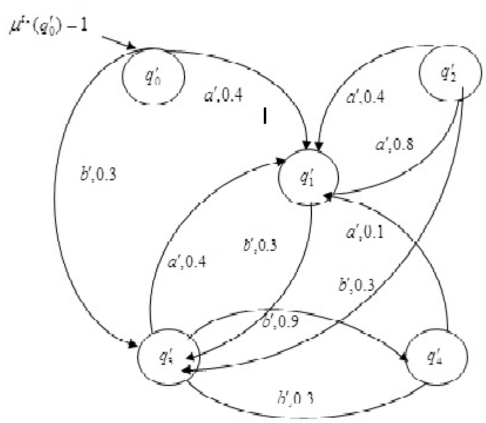

FIGURE 2. The $\tilde{F}'$ in Example 3.3

time	t_0	t_1	t_2	t_3	t_4	t_5
input	Λ	a'	a'	b'	b'	a'
$Q_{act}(t_i)$	q'_0	q'_1	q'_2	q'_3	$q'4$	q'_1
mv	1	0.4	0.4	0.3	0.3	0.1

TABLE 2. The Active State and mv's at Different Times of $\tilde{F}'$ in Example 3.3

Now by Definition 3.1 we have the general fuzzy automaton

$$\tilde{F} \times \tilde{F}' = (Q \times Q', \Sigma \times \Sigma', \tilde{R} \times \tilde{R}', Z \times Z', \tilde{\delta} \times \tilde{\delta}', \omega \times \omega', F_1'', F_2'').$$

For example:

$$\begin{aligned}
\mu^{t_{3+4+1}}(q_2,q_1') &= F_2''[\tilde{\delta}\times\tilde{\delta}'(((q_2,q_4'),\mu^{t_{3+4}}(q_2,q_4'))),(b,a'),(q_2,q_1')),\\
&\quad \tilde{\delta}\times\tilde{\delta}'(((q_3,q_4'),\mu^{t_{3+4}}(q_3,q_4'))),(b,a'),(q_2,q_1'))]\\
&= [\tilde{\delta}\times\tilde{\delta}'(((q_2,q_4'),\mu^{t_{3+4}}(q_2,q_4'))),(b,a'),(q_2,q_1'))]\\
&\vee\ [\tilde{\delta}\times\tilde{\delta}'(((q_3,q_4'),\mu^{t_{3+4}}(q_3,q_4'))),(b,a'),(q_2,q_1'))]\\
&= [\tilde{\delta}((q_2,\mu^{t_3}(q_2)),b,q_2)\wedge\tilde{\delta}'((q_4',\mu^{t_4}(q_4')),a',q_1')]\\
&\vee\ [\tilde{\delta}((q_3,\mu^{t_3}(q_3)),b,q_2)\wedge\tilde{\delta}'((q_4',\mu^{t_4}(q_4'))a',q_1')]\\
&= [F_1(\mu^{t_3}(q_2),\tilde{\delta}(q_2,b,q_2))\wedge F_1'(\mu^{t_4}(q_4'),\tilde{\delta}'(q_4',a',q_1'))]\\
&\vee\ [F_1(\mu^{t_3}(q_3),\tilde{\delta}(q_3,b,q_2))\wedge F_1'(\mu^{t_4}(q_4'),\tilde{\delta}'(q_4',a',q_1'))]\\
&= Min[F_1(0.6,0.5),F_1'(0.3,0.1)]\vee Min[F_1(0.2,0),F_1'(0.3,0.1)]\\
&= Min[0.5,0.1]\vee Min[0,0.1]=0.1\vee 0=0.1
\end{aligned}$$

The following definition is a straightforward generalization of a well-known claim about ordinary automata ([5]).

Definition 3.4. (Operations on (Fuzzy) Subsets)

Let $\tilde{F}^*=(Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2)$ be a max-min general fuzzy automaton,
Let $A\subseteq\Sigma^*$, $B\subseteq\Sigma^*$ and $a,b\in\Sigma^*$. Now we give the following notations:
N1. $AB^{-1}=\{x\in\Sigma^*|xb\in A,\ \text{for some } b\in B\}$.
N2. $A^{-1}B=\{x\in\Sigma^*|ax\in B,\ \text{for some } a\in A\}$.
N3. $Ab^{-1}=\{x\in\Sigma^*|xb\in A\}$.
N4. $a^{-1}B=\{x\in\Sigma^*|ax\in B\}$.
N5. $AB=\{xy|x\in A,y\in B\}$.
N6. $Ab=\{xb|x\in A\}$.
N7. $aB=\{ay|y\in B\}$.
Clearly, $AB^{-1}=\bigcup_{b\in B}Ab^{-1}, A^{-1}B=\bigcup_{a\in A}a^{-1}B, AB=\bigcup_{b\in B}Ab=\bigcup_{a\in A}aB$.

Definition 3.5. (see also [5])

Let $\eta:Q\to[0,1]$, $x\in\Sigma^*$, $p\in Q_{act}(t_i)$ and $A\subseteq\Sigma^*$. We define some operations on fuzzy subsets of Q as follows:

N8. $p*x:Q\to[0,1]$ is defined by $\forall q\in Q,(p*x)(q)=\tilde{\delta}^*((p,\mu^{t_i}(p)),x,q)$.

N9. $p*A:Q\to[0,1]$ is defined by $p*A=\bigcup_{y\in A}p*y$, it means that:

$$\forall q\in Q, p*A(q)=\vee\{\tilde{\delta}^*((p,\mu^{t_i}(p)),y,q)|y\in A\}.$$

N10. $\eta*x:Q\to[0,1]$ is defined by,

$$\forall q\in Q,\eta*x(q)=\vee\{\eta(p)\wedge\tilde{\delta}^*((p,\mu^{t_i}(p)),x,q)|p\in Q_{act}(t_i)\}.$$

N11. $\eta*A:Q\to[0,1]$ is defined by $\eta*A=\bigcup_{x\in A}\eta*x$, it means that:

$$\forall q\in Q,\eta*A(q)=\vee\{\eta(p)\wedge\tilde{\delta}^*((p,\mu^{t_i}(p)),x,q)|p\in Q_{act}(t_i),x\in A\}.$$

N12. $\eta * x^{-1} : Q \to [0,1]$ is defined by,

$$\forall q \in Q_{act}(t_j), \eta * x^{-1}(q) = \vee\{\eta(p) \wedge \tilde{\delta}^*((q,\mu^{t_j}(q)),x,p)|p \in Q_{act}(t_i), t_j \leq t_i\}.$$

N13. $\eta * A^{-1} : Q \to [0,1]$ is defined by $\eta * A^{-1} = \bigcup_{x\in A} \eta * x^{-1}$, it means that:

$$\forall q \in Q_{act}(t_j), \eta * A^{-1}(q) = \vee\{\eta(p) \wedge \tilde{\delta}^*((q,\mu^{t_j}(q)),x,p)|p \in Q_{act}(t_i), t_j \leq t_i\}.$$

Example 3.6. Consider the max-min general fuzzy automaton

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

in Example 3. 3. Let $q_3 \in Q_{act}(t_1)$ and $q_1 \in Q_{act}(t_2)$. Then by Definition 3. 4 we have:

$$\begin{aligned}
q_3 * ab(q_1) &= \tilde{\delta}^*((q_3,\mu^{t_1}(q_3)),ab,q_1) = [\tilde{\delta}((q_3,\mu^{t_1}(q_3)),a,q_1) \wedge \tilde{\delta}((q_1,\mu^{t_2}(q_1)),b,q_1)] \\
&\vee [\tilde{\delta}((q_3,\mu^{t_1}(q_3)),a,q_4) \wedge \tilde{\delta}((q_4,\mu^{t_2}(q_4)),b,q_1)] \\
&= [F_1(\mu^{t_1}(q_3),\delta(q_3,a,q_1)) \wedge F_1(\mu^{t_2}(q_1),\delta(q_1,b,q_1))] \\
&\vee [F_1(\mu^{t_1}(q_3),\delta(q_3,a,q_4)) \wedge F_1(\mu^{t_2}(q_4),\delta(q_4,b,q_1))] \\
&= [F_1(0.9,0) \wedge F_1(0.6,0)] \vee [F_1(0.9,0.2) \wedge F_1(0.2,0)] \\
&= [0 \wedge 0] \vee [0.2 \wedge 0] = 0 \vee 0 = 0
\end{aligned}$$

and

$$\begin{aligned}
q_1 * bb(q_2) &= \tilde{\delta}^*((q_1,\mu^{t_2}(q_1)),bb,q_2) \\
&= [\tilde{\delta}((q_1,\mu^{t_2}(q_1)),b,q_2) \wedge \tilde{\delta}((q_2,\mu^{t_3}(q_2)),b,q_2)] \\
&\vee [\tilde{\delta}((q_1,\mu^{t_2}(q_1)),b,q_3) \wedge \tilde{\delta}((q_3,\mu^{t_3}(q_3)),b,q_2)] \\
&= [F_1(\mu^{t_2}(q_1),\delta(q_1,b,q_2)) \wedge F_1(\mu^{t_3}(q_2),\delta(q_2,b,q_2))] \\
&\vee [F_1(\mu^{t_2}(q_1),\delta(q_1,b,q_3)) \wedge F_1(\mu^{t_3}(q_3),\delta(q_3,b,q_2))] \\
&= [F_1(0.6,0.6) \wedge F_1(0.6,0.5)] \vee [F_1(0.6,0) \wedge F_1(0.2,0)] \\
&= [0.6 \wedge 0.5] \vee [0 \wedge 0] = 0.5 \vee 0 = 0.5
\end{aligned}$$

Theorem 3.7. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, $\eta : Q \to [0,1]$, $x, y \in \Sigma^*$, and $A, B \subseteq \Sigma^*$. Then*

(1) $(\eta * x) * y = \eta * (xy)$;

(2) $(\eta * A) * y = \eta * (Ay)$;

(3) $(\eta * x) * B = \eta * (xB)$;

(4) $(\eta * A) * B = \eta * (AB)$;

(5) $(\eta * x^{-1}) * y^{-1} = \eta * (yx)^{-1}$;

(6) $(\eta * A^{-1}) * y^{-1} = \eta * (yA)^{-1}$;

(7) $(\eta * x^{-1}) * B^{-1} = \eta * (Bx)^{-1}$;

(8) $(\eta * A^{-1}) * B^{-1} = \eta * (BA)^{-1}$.

Proof. We prove one of them and the other ones are the same as that one.

(5)

$$\begin{aligned}
(\eta * x^{-1}) * y^{-1}(q) &= \vee\{(\eta * x^{-1})(p) \wedge \tilde{\delta}^*((q,\mu^{t_j}(q)),y,p)|p \in Q_{act}(t_i), t_j \leq t_i\} \\
&= \vee\{\vee\{(\eta(e) \wedge \tilde{\delta}^*((p,\mu^{t_i}(p)),x,e)|e \in Q_{act}(t_k), t_i \leq t_k\} \\
&\quad \wedge \tilde{\delta}^*((q,\mu^{t_j}(q)),y,p)|p \in Q_{act}(t_i), t_j \leq t_i\} \\
&= \bigvee_{e\in Q_{act}(t_k), p\in Q_{act}(t_i)} [\eta(e) \wedge [\vee(\tilde{\delta}((q,\mu^{t_j}(q)),y,p) \\
&\quad \wedge \tilde{\delta}((p,\mu^{t_i}(p)),x,e))] \\
&= \bigvee_{e\in Q_{act}(t_k)} [\eta(e) \wedge \tilde{\delta}^*((q,\mu^{t_j}(q)),yx,e)] \\
&= \eta * (yx)^{-1}(q)
\end{aligned}$$

□

Theorem 3.8. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, η, η_1 and η_2 be fuzzy subsets of Q and let $x, y \in \Sigma^*$. Then the following properties hold:*
(1) $\eta * \Lambda = \eta$;
(2) $(\eta_1 \cup \eta_2) * x = (\eta_1 * x) \cup (\eta_2 * x)$;
(3) $(\eta_1 \cap \eta_2) * x = (\eta_1 * x) \cap (\eta_2 * x)$;
(4) $(\eta_1 \cup \eta_2) * x^{-1} = (\eta_1 * x^{-1}) \cup (\eta_2 * x^{-1})$;
(5) $(\eta_1 \cap \eta_2) * x^{-1} = (\eta_1 * x^{-1}) \cap (\eta_2 * x^{-1})$;
(6) $\chi_\phi * x = \chi_\phi$, *where* χ_ϕ *is the characteristic function of* ϕ *in* Q;
(7) $\chi_Q * x = \bigcup_{p \in Q} p * x$, *where* χ_Q *is the characteristic function of* Q.

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and $\eta : Q \to [0,1]$, $\gamma : Q \to [0,1]$ be fuzzy subsets of Q.

N14. Define $\eta * \gamma : \Sigma^* \to [0,1]$ *by*

$$\forall x \in \Sigma^*, \eta * \gamma(x) = \vee\{\eta(p) \wedge \gamma(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) | p \in Q_{act}(t_i), q \in Q\}.$$

N15. $\eta^{-1} o_{\tilde{\delta}} \gamma = \{x \in \Sigma^* | \eta * \gamma(x) > 0\}$.

When $\tilde{\delta}$ is understood, we sometimes write $\eta^{-1} o \gamma$ for $\eta^{-1} o_{\tilde{\delta}} \gamma$.

Theorem 3.9. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, $\eta : Q \to [0,1]$ be a fuzzy subset of Q and $x \in \Sigma^*$. Then $\eta * \chi_Q > 0$ if and only if $\exists q \in Q$ such that $\eta * x(q) > 0$.*

Proof.

$$\begin{aligned} \eta * \chi_Q(x) > 0 \quad &\Leftrightarrow \quad \vee\{\eta(p) \wedge \chi_Q(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) | p \in Q_{act}(t_i), q \in Q\} > 0 \\ &\Leftrightarrow \quad \vee\{\eta(p) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) | p \in Q_{act}(t_i), q \in Q\} > 0 \\ &\Leftrightarrow \quad \vee\{\eta(p) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) | p \in Q_{act}(t_i)\} > 0 \\ &\Leftrightarrow \quad \eta * x(q) > 0. \end{aligned}$$

for some $q \in Q$. □

Theorem 3.10. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, η and γ be fuzzy subsets of Q, $q \in Q_{act}(t_j)$, $A, B \subseteq \Sigma^*$, and $x, y \in \Sigma^*$. Then the following properties hold:*
(1) $\eta^{-1} o(\gamma * A^{-1}) = (\eta^{-1} o \gamma) A^{-1}$;
(2) $\eta^{-1} o(\gamma * x^{-1}) = (\eta^{-1} o \gamma) x^{-1}$;
(3) $(\eta * A)^{-1} o \gamma = A^{-1}(\eta^{-1} o \gamma)$;
(4) $(\eta * x)^{-1} o \gamma = x^{-1}(\eta^{-1} o \gamma)$;
(5) $(q * A)^{-1} o \gamma = A^{-1}(q^{-1} o \gamma)$;
(6) $(q * x)^{-1} o \gamma = x^{-1}(q^{-1} o \gamma)$;

Proof. We prove (1), (3), and (5). (1)

$$\begin{aligned} z \in \eta^{-1} o(\gamma * A^{-1}) \quad &\Leftrightarrow \quad \eta * (\gamma * A^{-1})(z) > 0 \\ &\Leftrightarrow \quad \eta(p_0) \wedge (\gamma * A^{-1})(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), z, q_0) > 0 \end{aligned}$$

for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q_{act}(t_j)$, $t_i \leq t_j$,

$$\Leftrightarrow \eta(p_0) \wedge \gamma(t_0) \wedge \tilde{\delta}^*((q_0, \mu^{t_j}(q_0)), x_0, t_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), z, q_0) > 0$$

for some $p_0 \in Q_{act}(t_i)$, $q_0 \in Q_{act}(t_j)$, $t_i \leq t_j$, $t_0 \in Q$ and $x_0 \in A$,

$$\Leftrightarrow \eta(p_0) \wedge \gamma(t_0) \wedge [\tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), z, q_0) \wedge \tilde{\delta}^*((q_0, \mu^{t_j}(q_0)), x_0, t_0)] > 0$$

for some $p_0 \in Q_{act}(t_i)$, $q_0 \in Q_{act}(t_j)$, $t_0 \in Q$, $t_i \leq t_j$, and $x_0 \in A$,

$$\Leftrightarrow \eta(p_0) \wedge \gamma(t_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), zx_0, t_0) > 0$$

for some $p_0 \in Q_{act}(t_i)$, $t_0 \in Q$ and $x_0 \in A$, $\Leftrightarrow zx_0 \in \eta^{-1} o\gamma$ for some $x_0 \in A \Leftrightarrow z \in (\eta^{-1} o\gamma)A^{-1}$.

(3)

$$\begin{aligned} z \in (\eta * A)^{-1} o\gamma \quad &\Leftrightarrow \quad ((\eta * A) * \gamma)(z) > 0 \\ &\Leftrightarrow \quad (\eta * A)(p_0) \wedge \gamma(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), z, q_0) > 0 \end{aligned}$$

for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q$,

$$\Leftrightarrow \eta(t_0) \wedge \tilde{\delta}^*((t_0, \mu^{t_j}(t_0)), x_0, p_0) \wedge \gamma(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), z, q_0) > 0$$

for some $p_0 \in Q_{act}(t_i)$, $q_0 \in Q$, $t_0 \in Q_{act}(t_j)$, $t_j \leq t_i$, and $x_0 \in A$,

$$\Leftrightarrow \eta(t_0) \wedge \gamma(q_0) \wedge \tilde{\delta}^*((t_0, \mu^{t_j}(t_0)), x_0 z, q_0) > 0$$

for some $q_0 \in Q$, $t_0 \in Q_{act}(t_j)$, $t_j \leq t_i$, and $x_0 \in A$, $\Leftrightarrow x_0 z \in \eta^{-1} o\gamma$ for some $x_0 \in A \Leftrightarrow z \in A^{-1}(\eta^{-1} o\gamma)$.

(5)

$$\begin{aligned} y \in (q * A)^{-1} o\gamma \quad &\Leftrightarrow \quad ((q * A) * \gamma)(y) > 0 \\ &\Leftrightarrow \quad (q * A)(p_0) \wedge \gamma(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), y, q_0) > 0 \end{aligned}$$

for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q$,

$$\Leftrightarrow \tilde{\delta}^*((q, \mu^{t_j}(q)), x, p_0) \wedge \gamma(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), y, q_0) > 0$$

for some $p_0 \in Q_{act}(t_i)$, $q_0 \in Q$ and $y \in A$,

$$\Leftrightarrow \gamma(q_0) \wedge [\tilde{\delta}^*((q, \mu^{t_j}(q)), x, p_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), y, q_0)] > 0$$

for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q$, and $y \in A$, $\Leftrightarrow \gamma(q_0) \wedge [\tilde{\delta}^*((q, \mu^{t_j}(q)), xy, q_0) > 0$. for some $q_0 \in Q$ and $y \in A$, $\Leftrightarrow y \in A^{-1}(q^{-1} o\gamma)$. □

N16. Let $\eta_1 : Q \to [0,1]$ and $\eta_2 : Q' \to [0,1]$. Then define $\eta_1 \cap \eta_2 : Q \times Q' \to [0,1]$ by, $\forall (q, q') \in Q \times Q', (\eta_1 \cap \eta_2)(q, q') = \eta_1(q) \wedge \eta_2(q')$.

N17. Let $\eta_1 : Q \to [0,1]$ and $\eta_2 : Q' \to [0,1]$ be such that $Q \cap Q' = \emptyset$. Then define $\eta_1 \cup \eta_2 : Q \cup Q' \to [0,1]$ by $\forall q \in Q \cup Q'$,

$$\eta_1 \cup \eta_2(q) = \begin{cases} \eta_1(q) & if q \in Q \\ \eta_2(q) & if q \in Q'. \end{cases}$$

Theorem 3.11. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *and*

$$\tilde{F}'^* = (Q', \Sigma, \tilde{R}', Z, \tilde{\delta}^*, \omega', F_1', F_2')$$

be max-min general fuzzy automata. Let η_1, γ_1 *be fuzzy subsets of* Q *and* η_2, γ_2 *be fuzzy subsets of* Q'. *Then the following properties hold:*
(1) $(\eta_1 \cup \eta_2)^{-1} o_{\tilde{\delta} \cup \tilde{\delta}'} (\gamma_1 \cup \gamma_2) = (\eta_1^{-1} o_{\tilde{\delta}} \gamma_1) \cup (\eta_2^{-1} o_{\tilde{\delta}'} \gamma_2)$;
(2) $(\eta_1 \cap \eta_2)^{-1} o_{\tilde{\delta} \cap \tilde{\delta}'} (\gamma_1 \cap \gamma_2) = (\eta_1^{-1} o_{\tilde{\delta}} \gamma_1) \cap (\eta_2^{-1} o_{\tilde{\delta}'} \gamma_2)$.

Proof. (1) Let $x \in (\eta_1 \cup \eta_2)^{-1} o(\gamma_1 \cup \gamma_2)$. Then $(\eta_1 \cup \eta_2)^*(\gamma_1 \cup \gamma_2)(x) > 0$. Thus $(\eta_1 \cup \eta_2)(p) \wedge (\gamma_1 \cup \gamma_2)(q) \wedge (\tilde{\delta} \cup \tilde{\delta}')^*((p, \mu^{t_i}(p)), x, q) > 0$ for some $p, q \in Q \cup Q'$. It follows that both p, q belong to either Q or Q'. For, if $p \in Q$ and $q \in Q'$ and vice versa, then by definition $(\tilde{\delta} \cup \tilde{\delta}')^*((p, \mu^{t_i}(p)), x, q) = 0$ and consequently,

$$(\eta_1 \cup \eta_2)(p) \wedge (\gamma_1 \cup \gamma_2)(q) \wedge (\tilde{\delta} \cup \tilde{\delta}')^*((p, \mu^{t_i}(p)), x, q) = 0,$$

which is a contradiction. If $p, q \in Q$, then clearly

$$\begin{aligned} \eta_1(p) \wedge \gamma_1(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) \quad &= \quad (\eta_1 \cup \eta_2)(p) \wedge (\gamma_1 \cup \gamma_2)(q) \\ & \quad \wedge (\tilde{\delta} \cup \tilde{\delta}')^*((p, \mu^{t_i}(p)), x, q) > 0. \end{aligned}$$

Therefore, $\eta_1 * \gamma_1(x) > 0$ and hence $x \in \eta_1^{-1} o \gamma_1$. Similarly, if $p, q \in Q'$, then $x \in \eta_2^{-1} o \gamma_2$.

On the other hand, let $x \in \eta_1^{-1} o \gamma_1$. Then $\eta_1(p) \wedge \gamma_1(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) > 0$ for some $p, q \in Q$, i. e. $(\eta_1 \cup \eta_2)(p) \wedge (\gamma_1 \cup \gamma_2)(q) \wedge (\tilde{\delta} \cup \tilde{\delta}')^*((p, \mu^{t_i}(p)), x, q) > 0$. Note that a similar situation holds if $x \in \eta_2^{-1} o \gamma_2$.

(2) $x \in (\eta_1 \cap \eta_2)^{-1} o(\gamma_1 \cap \gamma_2) \Leftrightarrow (\eta_1 \cap \eta_2) * (\gamma_1 \cap \gamma_2)(x) > 0 \Leftrightarrow (\eta_1 \cap \eta_2)(p_1, p_2) \wedge (\gamma_1 \cap \gamma_2)(q_1, q_2) \wedge (\tilde{\delta} \cup \tilde{\delta}')^*(((p_1, p_2), \mu^{t_{i+j}}(p_1, p_2)), x, (q_1, q_2)) > 0$
and

$$[\eta_1(p_1) \wedge \gamma_1(q_1) \wedge \tilde{\delta}'^*((p_1, \mu^{t_i}(p_1)), x, q_1)] \wedge [\eta_2(p_2) \wedge \gamma_2(q_2) \wedge \tilde{\delta}'^*((p_2, \mu^{t_j}(p_2)), x, q_2)] > 0$$

for some $(p_1, p_2), (q_1, q_2) \in Q \times Q'$.
$\Leftrightarrow [\eta_1 * \gamma_1(x)] \wedge [\eta_2 * \gamma_2(x)] > 0$ for some $p_1, q_1 \in Q$ and $p_2, q_2 \in Q'$,
$\Leftrightarrow [\eta_1 * \gamma_1(x)] > 0$ and $[\eta_2 * \gamma_2(x)] > 0$
$\Leftrightarrow x \in \eta_1^{-1} o \gamma_1$ and $x \in \eta_2^{-1} o \gamma_2 \Leftrightarrow x \in (\eta_1^{-1} o \gamma_1) \cap (\eta_2^{-1} o \gamma_2)$. □

Theorem 3.12. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *and*

$$\tilde{F}'^* = (Q', \Sigma, \tilde{R}', Z, \tilde{\delta}^*, \omega', F_1', F_2')$$

be max-min general fuzzy automata. Let η_1, γ_1 *be fuzzy subsets of* Q *and* η_2, γ_2 *be fuzzy subsets of* Q'. *Then*

$$(\eta_1 \cap \eta_2)^{-1} o_{\tilde{\delta} \times \tilde{\delta}'} (\gamma_1 \cap \gamma_2) = (\eta_1^{-1} o_{\tilde{\delta}} \gamma_1) \times (\eta_2^{-1} o_{\tilde{\delta}'} \gamma_2)$$

Proof. $(x, y) \in (\eta_1 \cap \eta_2)^{-1} o_{\tilde{\delta} \times \tilde{\delta}'} (\gamma_1 \cap \gamma_2) \Leftrightarrow (\eta_1 \cap \eta_2) * (\gamma_1 \cap \gamma_2)(x, y) > 0$
$\Leftrightarrow (\eta_1 \cap \eta_2)(p_1, p_2) \wedge (\gamma_1 \cap \gamma_2)(q_1, q_2) \wedge (\tilde{\delta} \times \tilde{\delta}')^*(((p_1, p_2), \mu^{t_{i+j}}(p_1, p_2)), (x, y), (q_1, q_2)) > 0$
and

$$[\eta_1(p_1) \wedge \gamma_1(q_1) \wedge \tilde{\delta}'^*((p_1, \mu^{t_i}(p_1)), x, q_1)] \wedge [\eta_2(p_2) \wedge \gamma_2(q_2) \wedge \tilde{\delta}'^*((p_2, \mu^{t_j}(p_2)), y, q_2)] > 0$$

for some $(p_1, p_2), (q_1, q_2) \in Q \times Q'$.
$\Leftrightarrow [\eta_1 * \gamma_1(x)] \wedge [\eta_2 * \gamma_2(y)] > 0$ for some $p_1, q_1 \in Q$ and $p_2, q_2 \in Q'$,
$\Leftrightarrow [\eta_1 * \gamma_1(x)] > 0$ and $[\eta_2 * \gamma_2(y)] > 0$
$\Leftrightarrow x \in \eta_1^{-1} o\gamma_1$ and $y \in \eta_2^{-1} o\gamma_2 \Leftrightarrow (x, y) \in (\eta_1^{-1} o\gamma_1) \cap (\eta_2^{-1} o\gamma_2)$. □

The proof of the following theorem is similar to the proof of Theorem 3. 11.

Theorem 3.13. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *and*

$$\tilde{F}'^* = (Q', \Sigma, \tilde{R}', Z, \tilde{\delta}^*, \omega', F_1', F_2')$$

be max-min general fuzzy automata, $\eta_1 : Q \to [0, 1]$*,* $\eta_2 : Q' \to [0, 1]$ *and* $x \in \Sigma^*$*. Then*
(1) $(\eta_1 \cap \eta_2) * x = (\eta_1 * x) \cap (\eta_2 * x)$*;*
(2) $(\eta_1 \cap \eta_2) * x^{-1} = (\eta_1 * x^{-1}) \cap (\eta_2 * x^{-1})$.

4. Construction of General Fuzzy Recognizers and Recognizable Sets

In this section we construct the general fuzzy recognizers and recognizable sets and by considering the notions of section 3 we define direct product, restricted direct product and join of two Σ-recognizers. In particular, we define the notion of the language of a given general fuzzy recognizer and show that the language of direct product of two Σ-recognizers is equal to direct product of their languages.

Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ denote a general fuzzy automaton and $f : (\Sigma')^* \to \Sigma^*$ be a monoid homomorphism, where Σ' is a nonempty set. If $f(\Sigma') \subseteq \Sigma \cup \{\Lambda\}$, then f is called a fine homomorphism.

Definition 4.1. A general fuzzy Σ-recognizer of a general fuzzy automaton $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is an $M = (\tilde{F}, l, \tau)$, where τ and $l \in \tilde{R}$ are fuzzy subsets of Q. l is called a fuzzy set of initial states and τ is called a fuzzy set of final states.

A general fuzzy Σ-recognizer will be called a general recognizer when the set of input symbols is understood.

Definition 4.2. Let $M = (\tilde{F}, l, \tau)$ be a general fuzzy Σ-recognizer of a general fuzzy automaton $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ and $M' = (\tilde{F}', l', \tau')$ be a general fuzzy Σ'-recognizer of a general fuzzy automaton

$$\tilde{F}' = (Q', \Sigma', \tilde{R}', Z', \tilde{\delta}', \omega', F_1', F_2').$$

Then the general fuzzy $\Sigma \times \Sigma'$-recognizer $M \times M' = (\tilde{F} \times \tilde{F}', l \times l', \tau \times \tau')$ is called the direct product of M and M', where $\tilde{F} \times \tilde{F}' = (Q \times Q', \Sigma \times \Sigma', \tilde{R} \times \tilde{R}', Z \times Z', \tilde{\delta} \times \tilde{\delta}', \omega \times \omega', F_1'', F_2'')$ is called the direct product of $\tilde{F}$ and $\tilde{F}'$ (see Definition 3. 1), $l \times l' : Q \times Q' \to [0, 1]$ is defined by, $\forall (q, q') \in Q \times Q', (l \times l')(q, q') = l(q) \wedge l'(q')$ and $\tau \times \tau' : Q \times Q' \to [0, 1]$ is defined the same as $l \times l'$.

Let $M = (\tilde{F}, l, \tau)$ and $M' = (\tilde{F}', l', \tau')$ be general fuzzy Σ-recognizers of $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ and $\tilde{F}' = (Q', \Sigma', \tilde{R}', Z, \tilde{\delta}', \omega', F_1', F_2')$, respectively. Then the general fuzzy Σ-recognizer $M \cap M' = (\tilde{F} \cap \tilde{F}', l \cap l', \tau \cap \tau')$ is called the restricted direct product of M and M', where $\tilde{F} \cap \tilde{F}' = (Q \times Q', \Sigma, \tilde{R} \cap \tilde{R}', Z, \tilde{\delta} \cap \tilde{\delta}', \omega \cap$

$\omega', F_1'', F_2'')$ is called the restricted direct product of $\tilde{F}$ and $\tilde{F}'$ (see Definition 3.1), $l \cap l' : Q \times Q' \to [0,1]$ is defined by, $\forall (q,q') \in Q \times Q', (l \cap l')(q,q') = l(q) \wedge l'(q')$ and $\tau \cap \tau' : Q \times Q' \to [0,1]$ is defined the same as $l \cap l'$.

Definition 4.3. Let $M = (\tilde{F}, l, \tau)$ and $M' = (\tilde{F}', l', \tau')$ be general fuzzy Σ-recognizers of $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ and $\tilde{F}' = (Q', \Sigma', \tilde{R}', Z', \tilde{\delta}', \omega', F_1', F_2')$, respectively, where $Q \cap Q' = \emptyset$. Then the general fuzzy recognizer $M \cup M' = (\tilde{F} \cup \tilde{F}', l \cup l', \tau \cup \tau')$ is called the join of M and M', where $\tilde{F} \cup \tilde{F}' = (Q \cup Q', \Sigma, \tilde{R} \cup \tilde{R}', Z, \tilde{\delta} \cup \tilde{\delta}', \omega \cup \omega', F_1'', F_2'')$ is the join of $\tilde{F}$ and $\tilde{F}'$ (see Definition 3. 2), $l \cup l' : Q \cup Q' \to [0,1]$ is defined by,

$$(l \cup l')(q) = \begin{cases} l(q) & if \quad q \in Q \\ l'(q) & if \quad q \in Q' \end{cases}$$

and $\tau \cup \tau'$ is defined the same as $l \cup l'$.

Definition 4.4. A mapping $\rho : \Sigma^* \to \Sigma^*$ is called a reversal of Σ^* if the following conditions hold:
(1) $\rho(\Lambda) = \Lambda$,
(2) $\rho(a) = a$,
(3) $\rho(xa) = \rho(a)\rho(x)$, and
(4) $\rho(\rho(x)) = x$,
$\forall a \in X$ and $x \in \Sigma^*$.

It follows by induction that $\rho(xy) = \rho(y)\rho(x) : \forall x, y \in \Sigma^*$.

Let ρ be a reversal of Σ^*. Define the augmented transition function

$$\tilde{\delta}^\rho : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$$

by

$$\tilde{\delta}^\rho((p, \mu^{t_i}(p)), a, q) = \tilde{\delta}((q, \mu^{t_j}(q)), \rho(a), p) \forall a \in \Sigma, \forall p \in Q_{act}(t_i), q \in Q_{act}(t_j), t_i \leq t_j.$$

Definition 4.5. If $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is a general fuzzy automaton, then the general fuzzy automaton $\tilde{F}^\rho = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^\rho, \omega, F_1, F_2)$ is called the reversal of $\tilde{F}$.

Example 4.6. Consider the general fuzzy automaton $\tilde{F}$ in Example 3.3. By Definition 4.5 we have the reversal of $\tilde{F}$ as $\tilde{F}^\rho = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^\rho, \omega, F_1, F_2)$.

For example:

$$\begin{aligned} \tilde{\delta}^\rho((q_4, \mu^{t_2}(q_4)), a, q_1) &= \tilde{\delta}((q_1, \mu^{t_2}(q_1)), \rho(a), q_4) \\ &= \tilde{\delta}((q_1, \mu^{t_2}(q_1)), a, q_4) \\ &= F_1(\mu^{t_2}(q_1), \delta(q_1, a, q_4)) \\ &= F_1(0.6, 0.4) = 0.4. \end{aligned}$$

Theorem 4.7. *Let $\tilde{F}^\rho = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^\rho, \omega, F_1, F_2)$ be the reversal of*

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2).$$

Then $(\tilde{\delta}^\rho)^*((p, \mu^{t_i}(p)), xy, q) = \tilde{\delta}^*((q, \mu^{t_j}(q)), \rho(y)\rho(x), p) \forall x, y \in \Sigma^*$.

Let $A \subseteq \Sigma^*$ and $\rho : \Sigma^* \to \Sigma^*$ be the reversal mapping. Let $A^\rho = \{\rho(x) | x \in A\}$.

Theorem 4.8. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and η, γ be fuzzy subset of Q. Then $(\eta^{-1} o\gamma)^\rho = \gamma^{-1} o\eta$.*

Proof. $\rho(x) \in (\eta^{-1} o\gamma)^\rho \Leftrightarrow x \in \eta^{-1} o\gamma \Leftrightarrow \eta * \gamma(x) > 0 \Leftrightarrow \eta(p) \wedge \gamma(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) > 0$ for some $p \in Q_{act}(t_i)$ and $q \in Q_{act}(t_j)$,

$\Leftrightarrow \eta(p) \wedge \gamma(q) \wedge (\tilde{\delta}^\rho)^*((q, \mu^{t_j}(q)), \rho(x), p) > 0$ for some $p \in Q_{act}(t_i)$ and $q \in Q_{act}(t_j)$

$\Leftrightarrow \gamma(q) \wedge \eta(p) \wedge (\tilde{\delta}^\rho)^*((q, \mu^{t_j}(q)), \rho(x), p) > 0$ for some $p \in Q_{act}(t_i)$ and $q \in Q_{act}(t_j)$

$\Leftrightarrow \gamma * \eta(\rho(x)) > 0 \Leftrightarrow \rho(x) \in \gamma^{-1} o\eta$. □

Definition 4.9. Let $M = (\tilde{F}, l, \tau)$ be a general fuzzy Σ-recognizer and let $\rho : \Sigma^* \to \Sigma^*$ be the reversal mapping. Then the general fuzzy Σ-recognizer $M^\rho = (\tilde{F}^\rho, \tau, l)$ of $\tilde{F}^\rho$ is called the reversal of $\tilde{F}$.

Definition 4.10. Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy Σ-recognizer of a max-min general fuzzy automaton $\tilde{F}^*$. Then $x \in \Sigma^*$ is said to be recognized by M^* if $l * \tau(x) > 0$.

Let $B(M^*)$ denote the set of all words that are recognized by M^*. $B(M^*)$ is called the language of M^*.

Example 4.11. Consider the general fuzzy automaton $\tilde{F}$ in Example 3.3. Let $F = \{q_1, q_4\}$ be the final states set. We define $\tau : Q \to [0, 1]$ by $\tau(q_1) = 0.6$ and $\tau(q_4) = 0.2$ and consider $M^* = (\tilde{F}^*, l, \tau)$. By Definition 4.10 we have $a^2 b \notin B(M^*)$ since

$$\begin{aligned} l * \tau(a^2 b) &= \vee\{l(p) \wedge \tau(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), a^2 b, q) | p \in Q_{act}(t_0), q \in Q\} \\ &\geq [l(q_0) \wedge \tau(q_1) \wedge \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^2 b, q_1)] \\ &= [1 \wedge 0.6 \wedge 0] = 0 \end{aligned}$$

and $a^2 \in B(M^*)$ because

$$\begin{aligned} l * \tau(a^2 b) &= \vee\{l(p) \wedge \tau(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), a^2 b, q) | p \in Q_{act}(t_0), q \in Q\} \\ &\geq [l(q_0) \wedge \tau(q_4) \wedge \tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a^2, q_4)] \\ &= [1 \wedge 0.2 \wedge 0.2] = 0.2 > 0. \end{aligned}$$

Theorem 4.12. *Let M^* be a recognizer of a max-min general fuzzy automaton $\tilde{F}^*$ and $x \in \Sigma^*$. Then the following condition are equivalent.*
(1) $x \in B(M^*)$;
(2) $x \in l^{-1} o\tau$;
(3) $l * \tau(x) > 0$;
(4) $\exists p_0 \in Q_{act}(t_i)$ such that $l \cap (\tau * x^{-1})(p_0) > 0$;
(5) $\exists p_0 \in Q_{act}(t_i)$ such that $(l * x) \cap \tau(p_0) > 0$.

Proof. Clearly (1), (2), and (3) are equivalent.
(3)$\Longleftrightarrow$(4) $l * \tau(x) > 0$
$\Longleftrightarrow \vee\{l(p) \wedge \tau(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), x, q) | p \in Q_{act}(t_i), q \in Q\} > 0$
$\Longleftrightarrow \exists p_0 \in Q_{act}(t_i), q_0 \in Q$ s.t $l(p_0) \wedge \tau(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x, q_0) > 0$

$\Longleftrightarrow \exists p_0 \in Q_{act}(t_i), q_0 \in Q$ such that $l(p_0) > 0$ and $\tau(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x, q_0) > 0 \Longleftrightarrow \exists p_0 \in Q_{act}(t_i)$ such that $l(p_0) > 0$ and $\vee\{\tau(q) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x, q) | q \in Q\} > 0 \Longleftrightarrow \exists p_0 \in Q_{act}(t_i)$ such that $l(p_0) > 0$ and $\tau * x^{-1}(p_0) > 0 \Longleftrightarrow \exists p_0 \in Q_{act}(t_i)$ such that $l \cap (\tau * x^{-1})(p_0) > 0$.
(3)$\Longleftrightarrow$(5) this can be proved by interchanging the roles of p_0 and q_0 and using the definition of $l * x$. □

Definition 4.13. A subset A of Σ^* is called Σ-recognizable, if there exists a general fuzzy Σ-recognizer M^* such that $B(M^*) = A$.

Theorem 4.14. *Let Σ be a nonempty finite set. Then the following sets are Σ-recognizable:*
(1) $\{a\}$, where $a \in \Sigma$.
(2) Λ.
(3) A^*, where $A \subseteq \Sigma$.
(4) Σ.

Theorems 3.10, 4.8, and 4.12 immediately lead to the following theorem.

Theorem 4.15. *Let M^* and M'^* be general fuzzy recognizers of max-min general fuzzy automata $\tilde{F}^*$ and $\tilde{F}'^*$, respectively. Then the following assertions hold.*
(1) $B(M^* \cup M'^*) = B(M^*) \cup B(M'^*)$.
(2) $B(M^* \cap M'^*) = B(M^*) \cap B(M'^*)$.
(3) $B(M^\rho) = B(M)^\rho$.

Proof. We have $B(M)^\rho = (l^{-1} o \tau)^\rho = \tau^{-1} o l = B(M^\rho)$. □

The following corollary is a direct consequence of Theorem 4. 15.

Corollary 4.16. *Let A and B be recognizable of Σ^*. Then the following sets are recognizable.*
(1) $A \cup B$.
(2) $A \cap B$.
(3) A^ρ.

Corollary 4.17. *A subset A of Σ^* is recognizable if and only if A^ρ is recognizable.*

Proof. The proof follows from $(A^\rho)^\rho = A$. □

According to Theorems 3.11 and 4.12, we obtain the following theorem.

Theorem 4.18. *Let M^* be a general fuzzy Σ-recognizer of a max-min general fuzzy automaton $\tilde{F}^*$ and M'^* be a general fuzzy Σ'-recognizer of a max-min general fuzzy automaton $\tilde{F}'^*$. Then $B(M^* \times M'^*) = B(M^*) \times B(M'^*)$.*

Corollary 4.19. *If A is Σ-recognizable and B is Σ'-recognizable, then $A \times B$ is $\Sigma \times \Sigma'$-recognizable.*

Theorem 4.20. *Let A be Σ-recognizable and B be any subset of Σ^*. Then the following sets are Σ-recognizable:*
(1) AB^{-1}.

(2) Ax^{-1}.
(3) $B^{-1}A$.
(4) $x^{-1}A$.

Proof. We prove (1) and (3) and the other ones are the same.

Since A is Σ-recognizable, there exists a general fuzzy recognizer $M^* = (\tilde{F}^*, l, \tau)$ of a max-min general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that $B(M^*) = A$.
(1) Consider a general fuzzy recognizer $M'^* = (\tilde{F}^*, l, \tau * B^{-1})$. Then by Theorem 3.9(1), $l^{-1}o(\tau * B^{-1}) = (l^{-1}o\tau)B^{-1} = B(M^*)B^{-1} = AB^{-1}$. Thus AB^{-1} is recognizable.
(3) Consider the general fuzzy recognizer $M'^* = (\tilde{F}^*, l, B^{-1} * \tau)$. Then by Theorem 3.9(4), $(l * B)^{-1}o\tau = B^{-1}(l^{-1}o\tau) = B^{-1}B(M^*) = B^{-1}A$. Hence $B^{-1}A$ is recognizable. □

Let Σ and Σ' be sets and let $f : (\Sigma')^* \to \Sigma^*$ be a fine homomorphism. If $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ is a max-min general fuzzy automaton, then define

$$\tilde{\delta}' : (Q \times [0,1]) \times \Sigma' \times Q \to [0,1]$$

by $\tilde{\delta}'((p, \mu^{t_i}(p)), a', q) = \tilde{\delta}^*((p, \mu^{t_i}(p)), f(a'), q), p \in Q_{act}(t_i), q \in Q, a' \in \Sigma'$.

This defines a general fuzzy automaton $\tilde{F}' = (Q, \Sigma', \tilde{R}, Z, \tilde{\delta}', \omega, F_1, F_2)$ and a general fuzz Σ'-recognizer $M' = (\tilde{F}', l, \tau)$.

Definition 4.21. The general fuzzy Σ'-recognizer M' defined immediately above is called the inverse image of M^* and is denoted by $f^{-1}(M^*)$, where f is a fine homomorphism.

Theorem 4.22. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, and $\eta : Q \to [0,1]$ and $\gamma : Q \to [0,1]$. Then $\eta^{-1}o_{\tilde{\delta}'}\gamma = f^{-1}(\eta^{-1}o_{\tilde{\delta}}\gamma)$.*

Proof. $x \in \eta^{-1}o_{\tilde{\delta}'}\gamma \Leftrightarrow \eta * \gamma(x) > 0 \Leftrightarrow \eta(p) \wedge \gamma(q) \wedge \tilde{\delta}'((p, \mu^{t_i}(p)), x, q) > 0$ for some $p \in Q_{act}(t_i), q \in Q \Leftrightarrow \eta(p) \wedge \gamma(q) \wedge \tilde{\delta}^*((p, \mu^{t_i}(p)), f(x), q) > 0$ for some $p \in Q_{act}(t_i), q \in Q \Leftrightarrow f(x) \in \eta^{-1}o_{\tilde{\delta}}\gamma$. □

The following corollaries are easily shown to hold.

Corollary 4.23. *$B(f^{-1}(M^*)) = f^{-1}(B(M^*))$, where f is a fine homomorphism.*

Corollary 4.24. *If A is Σ-recognizable, then $f^{-1}(A)$ is Σ'-recognizable.*

Let $M = (\tilde{F}, l, \tau)$ be a general fuzzy recognizer of a general fuzzy automaton $\tilde{F}$ and let $a \in \Sigma$. Consider $Q' = Q \cup \{e\}$, where $e \notin Q$. Define $\tilde{\delta}_a : (Q' \times [0,1]) \times \Sigma \times Q' \to [0,1]$ by:

$$\tilde{\delta}_a((p, \mu^{t_i}(p)), b, q) = \begin{cases} \tilde{\delta}((p, \mu^{t_i}(p)), b, q), & if p, q \in Q \\ 1, & if p \in Q, \tau(p) > 0, b = a and q = e \\ 0, & otherwise \end{cases}$$

Clearly, $\tilde{F}_a = (Q', \Sigma, \tilde{R}_a, Z, \tilde{\delta}_a, \omega, F_1, F_2)$ is a general fuzzy automaton.

Define $l_a : Q' \to [0,1]$ as follow:

$$l_a(p) = \begin{cases} l(p), & if p \in Q \\ 0, & otherwise \end{cases}$$

Let τ_a be the characteristic function of $\{e\}$ and $\omega' = \omega|Q'$. Then $M_a = (\tilde{F}_a, l_a, \tau_a)$ is the general fuzzy recognizer of $\tilde{F}_a$.

Let Σ be a nonempty finite set, $A \subseteq \Sigma^*$, and $u \in \Sigma$. Recall that $Au = \{xu | x \in A\}$.

Theorem 4.25. *Let $M_a = (\tilde{F}_a, l_a, \tau_a)$ be a general recognizer of $\tilde{F}_a$. Then $l^{-1} o_{\tilde{\delta}_a} \tau_a = (l^{-1} o_{\tilde{\delta}} \tau)a$.*

Proof. Let $x \in l_a^{-1} o_{\tilde{\delta}_a} \tau_a$. Then $l_a * \tau_a(x) > 0$.

Thus $l_a(p_0') \wedge \tau_a(q_0') \wedge \tilde{\delta}_a^*((p_0', \mu^{t_i}(p_0')), x, q_0') > 0$ for some $p_0' \in Q'_{act}(t_i)$, $q_0' \in Q'$. Hence $q_0' = e$ and $p_0' \in Q_{act}(t_i)$. Now $\tilde{\delta}_a^*((p_0', \mu^{t_i}(p_0')), x, q_0') > 0$ is possible only if either $\tau(p_0') > 0$ and $x = a$ or there exists $y \in \Sigma^*$ such that $\tilde{\delta}_a^*((p_0', \mu^{t_i}(p_0')), ya, e) > 0$. We consider two cases.

Supposes $\tau(p_0') > 0$ and $x = a$.

Now $l_a(p_0') > 0$, $\tau(p_0') > 0$ and $\tilde{\delta}^*((p_0', \mu^{t_i}(p_0')), \Lambda, p_0') > 0$.

Hence $l(p_0') \wedge \tau(p_0') \wedge \tilde{\delta}^*((p_0', \mu^{t_i}(p_0')), \Lambda, p_0') > 0$, i. e. $\Lambda \in l^{-1} o_{\tilde{\delta}} \tau$.

Therefore, $x = \Lambda a \in (l^{-1} o_{\tilde{\delta}} \tau)a$.

Suppose now that $y \in \Sigma^*$ such that $\tilde{\delta}_a^*((p_0', \mu^{t_i}(p_0')), ya, e) > 0$. Then there exists $t_0' \in Q'_{act}(t)$ such that $\tilde{\delta}_a^*((p_0', \mu^{t_i}(p_0')), y, t_0') \wedge \tilde{\delta}_a((t_0', \mu^t(t_0')), a, e) > 0$. This is true only when $\tau(t_0') > 0$ and $t_0' \in Q_{act}(t)$. Thus $l(p_0') \wedge \tau(t_0') \wedge \tilde{\delta}^*((p_0', \mu^{t_i}(p_0')), y, t_0') > 0$,. Therefore, $l * \tau(y) > 0$. Hence $x = ya \in (l^{-1} o_{\tilde{\delta}} \tau)$. Thus $l_a^{-1} o_{\tilde{\delta}_a} \tau_a \subseteq (l^{-1} o_{\tilde{\delta}} \tau)a$.

We now show inclusion in the other direction. Let $x = ya \in (l^{-1} o_{\tilde{\delta}} \tau)a$. Then $y \in (l^{-1} o_{\tilde{\delta}} \tau)$.

Therefore, $l(p_0) \wedge \tau(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), y, q_0) > 0$ for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q$.

Thus $l(p_0) > 0$ and $\tau(q_0) > 0$.

Now $\tau(q_0) > 0 \implies \tilde{\delta}_a((q_0, \mu^{t_j}(q_0)), a, e) = 1$, and $\tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), y, q_0) = \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), y, q_0) > 0$.

Hence

$$\begin{aligned} \tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), ya, e) &= \vee\{\tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), y, r') \wedge \tilde{\delta}_a((r', \mu^{t_p}(r')), a, e) | \\ &\quad r' \in Q'_{act}(t_p), t_i \le t_p\} \\ &> \tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), y, q_0) \wedge \tilde{\delta}_a((q_0, \mu^{t_j}(q_0)), a, e) > 0. \end{aligned}$$

Therefore, $l(p_0) \wedge \tau_a(e) \wedge \tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), ya, e) > 0$ and hence $l(p_0) \wedge \tau_a(e) \wedge \tilde{\delta}_a^*((p_0, \mu^{t_i}(p_0)), ya, e) > 0$ since $p_0 \in Q_{act}(t_i)$, i. e. $x = ya \in l_a^{-1} o_{\tilde{\delta}_a} \tau_a$. Thus $(l^{-1} o_{\tilde{\delta}} \tau)a \subseteq l_a^{-1} o_{\tilde{\delta}_a} \tau_a$. Hence $l_a^{-1} o_{\tilde{\delta}_a} \tau_a = (l^{-1} o_{\tilde{\delta}} \tau)a$. □

Corollary 4.26. $B(M_a) = B(M)_a$.

Corollary 4.27. *If a subset A of Σ^* is recognizable and $u \in \Sigma$, then Au is recognizable.*

5. Accessible and Coaccessible General Fuzzy Recognizer

In this section by considering the definition of general fuzzy recognizer we define accessible and coaccessible general Fuzzy recognizer and prove: if a general fuzzy recognizer M^* is an accessible (coaccessible, trim) then $M^a = M^*$ ($M^b = M^*$, $M^t = M^*$ respectively), where M^a, M^b and M^t denote the accessible, coaccessible and trim parts of M^* respectively.

Definition 5.1. Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy Σ-recognizer of a max-min general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ and $K = \{q \in Q | l{*}\Sigma^*(q) > 0\}$. If $Q = K$, then M^* is called accessible.

Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy recognizer of a general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$. Let $K = \{q \in Q | l * \Sigma^*(q) > 0\}$.

Consider $\tilde{\delta}^a = \tilde{\delta}^* | (K \times [0,1]) \times \Sigma^* \times K$, $l^a = l|K$, $\tau^a = \tau|K$ and $\omega^a = \omega|K$.

Then $\tilde{F}^a = (K, \Sigma, \tilde{R}^a, Z, \tilde{\delta}^a, \omega, F_1, F_2)$ is a max-min general fuzzy automaton and $M^a = (\tilde{F}^a, l^a, \tau^a)$ is a general fuzzy recognizer of $\tilde{F}^a$, where $\tilde{R}^a \subseteq \tilde{P}(K)$ such that:

$$\tilde{R}^a = \{l^a | l^a : K \to [0,1]\}.$$

Definition 5.2. Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy Σ-recognizer of $\tilde{F}^*$. Then $M^a = (\tilde{F}^a, l^a, \tau^a)$ is called accessible part of M^*.

Definition 5.3. Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy Σ-recognizer of a general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$. Let $S = \{p \in Q | \tau * (\Sigma^*)^{-1}(p) > 0\}$. If $Q = S$, then $\tilde{F}^*$ is called coaccessible.

Consider $\tilde{\delta}^b = \tilde{\delta}^* | (S \times [0,1]) \times \Sigma^* \times S$, $l^b = l|S$, $\tau^b = \tau|S$ and $\omega^b = \omega|S$.

Then $\tilde{F}^b = (S, \Sigma, \tilde{R}^b, Z, \tilde{\delta}^b, \omega^b, F_1, F_2)$ is a general fuzzy automaton and $M^b = (\tilde{F}^b, l^b, \tau^b)$ is a general fuzzy recognizer of $\tilde{F}^b$, where $\tilde{R}^b \subseteq \tilde{P}(S)$ such that:

$$\tilde{R}^b = \{l^b | l^b : S \to [0,1]\}.$$

Definition 5.4. Let $M^* = (\tilde{F}^*, l, \tau)$ be a general fuzzy Σ-recognizer of $\tilde{F}^*$.

Then $\tilde{F}^b = (S, \Sigma, \tilde{R}^b, Z, \tilde{\delta}^b, \omega^b, F_1, F_2)$ is called the coaccessible part of $\tilde{F}^*$.

If $l(q) > 0$, then $q \in K$. Thus whenever l is crisp, so is l^a and $l^a = l$ as sets. If $\tau(q) > 0$, then $q \in S$. Hence whenever τ is crisp, so is τ^a and $\tau^a = \tau$ as sets.

Theorem 5.5. *Let M^* be a general fuzzy recognizer. Then the following properties hold:*
(1) $l^{-1} o \tau = (l^a)^{-1} o \tau^a$;
(2) $l^{-1} o \tau = (l^b)^{-1} o \tau^b$,

Proof. (1) Let $x \in l^{-1} o \tau$. Then $l * \tau(x) > 0$. Therefore,

$$l(p_0) \wedge \tau(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x, q_0) > 0$$

for some $p_0 \in Q_{act}(t_i)$ and $q_0 \in Q$.

Clearly, $q_0 \in K$ since $l(p_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x, q_0) > 0$.

Also, $l(p_0) > 0$ and $\tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), \Lambda, p_0) = 1 > 0$ and so $p_0 \in K$. Hence

$$\tau^a(q_0) \wedge l^a(p_0) \wedge \tilde{\delta}^{a^*}((p_0, \mu^{t_i}(p_0)), x, q_0) > 0.$$

But then $l^a * \tau^a(x) > 0$ and thus $x \in (l^a)^{-1} o \tau^a$. Therefore, $l^{-1} o \tau \subseteq (l^a)^{-1} o \tau^a$.

We now show inclusion in the other direction. Let $x \in (l^a)^{-1} o \tau^a$. Then $l^a * \tau^a > 0$. i. e. $l^a(p_0^a) \wedge \tau^a(q_0^a) \wedge \tilde{\delta}^{a^*}((p_0, \mu^{t_i}(p_0)), x, q_0^a) > 0$ for some $p_0^a, q_0^a \in K$.

Therefore, $l^a(p_0^a) > 0$, $\tau^a(q_0^a) > 0$, and $\tilde{\delta}^{a^*}((p_0^a, \mu^{t_i}(p_0^a)), x, q_0^a) > 0$.

Thus $l(p_0^a) = l^a(p_0^a), \tau(q_0^a) = \tau^a(q_0^a)$ and

$$\tilde{\delta}^*((p_0^a, \mu^{t_i}(p_0^a)), x, q_0^a) = \tilde{\delta}^{a^*}((p_0^a, \mu^{t_i}(p_0^a)), x, q_0^a).$$

Therefore, $l(p_0^a) \wedge \tau(q_0^a) \wedge \tilde{\delta}^*((p_0^a, \mu^{t_i}(p_0^a)), x, q_0^a) > 0$. Hence $l * \tau(x) > 0$ and so $x \in l^{-1} o \tau$. Thus $(l^a)^{-1} o \tau^a \subseteq l^{-1} o \tau$. Consequently, $l^{-1} o \tau = (l^a)^{-1} o \tau^a$. □

Corollary 5.6. *Let M^* be a general fuzzy recognizer. Then*
(1) $B(M^*) = B(M^a)$;
(2) $B(M^*) = B(M^b)$.

Theorem 5.7. *Let M^* be a general fuzzy Σ-recognizer. Then M^* is coaccessible if and only if M^ρ is accessible.*

Proof. M is coaccessible general fuzzy recognizer $\Leftrightarrow Q = \{p \in Q | \tau * (\Sigma^*)^{-1}(p) > 0\} \Leftrightarrow \forall p_0 \in Q_{act}(t_i), \exists x_0 \in \Sigma^*$, and $q_0 \in Q$ such that $\tau(q_0) \wedge \tilde{\delta}^*((p_0, \mu^{t_i}(p_0)), x_0, q_0) > 0 \Leftrightarrow \forall p_0 \in Q_{act}(t_i), \exists x_0^\rho$ namely $\rho(x_0) \in \Sigma^*$ and $q_0 \in Q$ such that

$$\tau(q_0) \wedge \tilde{\delta}^{\rho^*}((p_0, \mu^{t_i}(p_0)), \rho(x_0), q_0) > 0 \Leftrightarrow Q = \{p \in Q | \tau * \Sigma^*(p) > 0\}$$

$\Leftrightarrow M^\rho$ is an accessible general fuzzy recognizer. □

Definition 5.8. A general fuzzy Σ-recognizer M^* is called trim if it is both an accessible and coaccessible general fuzzy Σ-recognizer.

Theorem 5.9. *Let $M^* = (\tilde{F}^*, l, \tau)$ be the general fuzzy Σ-recognizer of the max-min general fuzzy automaton*

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

and let $W = \{p \in Q | l * \Sigma^* \cap \tau * (\Sigma^*)^{-1}(p) > 0\}$. Then M^* is trim if and only if $Q = W$.

Corollary 5.10. *Consider $\tilde{\delta}^t = \tilde{\delta}^* | (W \times [0,1]) \times \Sigma^* \times W$, $l^t = l|W$, $\omega^t = \omega|W$ and $\tau^t = \tau|W$.*

Then $\tilde{F}^t = (W, \Sigma, \tilde{R}, Z, \tilde{\delta}^t, \omega^t, F_1, F_2)$ is a max-min general fuzzy automaton and $M^t = (\tilde{F}^t, l^t, \tau^t)$ is a general fuzzy Σ-recognizer.

Definition 5.11. Let M^* be a general fuzzy Σ-recognizer. Then $M^t = (\tilde{F}^t, \tilde{R}^t, \tau^t)$ is called the trim part of $\tilde{F}^*$.

The following theorem is immediate from Corollary 5.6.

Theorem 5.12. *If $M^* = (\tilde{F}^*, l, \tau)$ is a general fuzzy recognizer, then $B(M^*) = B(M^t)$.*

Corollary 5.13. *If M^* is an accessible (coaccessible, trim) general fuzzy Σ-recognizer, then $M^a = M^*$ ($M^b = M^*$, $M^t = M^*$, respectively).*

6. Concluding Remarks

In this paper, we generalize some concepts and results of crisp automata to general fuzzy automata. Specifically we show that the language of direct product of two Σ-recognizer is equal to direct product of their languages. Now there exist two important questions:
1) can obtain a relation between language and related grammars?
2) can examine regular languages, adjunctive languages, and dense language and present their algebraic properties?

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