BOUNDEDNESS AND CONTINUITY OF FUZZY LINEAR ORDER-HOMOMORPHISMS ON I-TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, a new definition of bounded fuzzy linear order homomorphism on I-topological vector spaces is introduced. This definition differs from the definition of Fang [The continuity of fuzzy linear order-homomorphism. J. Fuzzy Math. **5(4)**(1997), 829–838]. We show that the "boundedness" and " boundedness on each layer" of fuzzy linear order homomorphisms do not imply each other. On the basis, characterizations of continuity of fuzzy linear order-homomorphisms, and the relation between continuity and boundedness are studied.

1. Introduction

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981 [6]. According to the terminology of standardization in [4], it has now been renamed as I-topological vector space (briefly, I-tvs), where $I = [0, 1]$.

As it is well-known, boundedness of linear operators is an important concept in the theory of classical topological vector spaces. The notion of fuzzy linear order-homomorphism (for short, FLOH) is a reasonable generalization of that of linear operator (see [1]). So, it is natural and necessary to define rationally the boundedness of FLOH in the research of I-tvs.

In this paper, we propose a new definition of bounded FLOH, and rename "the bounded FLOH" defined in [2] as "the bounded FLOH on each layer". We investigate the relations between these two boundednesses of FLOH. We give two examples showing that the two kinds of boundedness of FLOH do not imply each other in general, and prove that under certain condition, the boundedness on each layer of FLOH implies the boundedness of FLOH. We also study characterizations of continuity of FLOH and discuss the relations between continuity and boundedness of FLOH.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, and $\mathbb{N}$ denotes the set of all natural numbers. Let $I = [0, 1]$ and I^X denote a family of all fuzzy

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subsets on X. A fuzzy subset which takes the constant value r on X $(0 \leq r \leq 1)$ is denoted by $\underline{r}_{\mathrm{x}}$, or briefly $\underline{r}$. A fuzzy subset of X is called a fuzzy point [8], denoted by x_λ $(0 < \lambda \leq 1)$, if it takes value 0 at $y \in X \setminus \{x\}$ and its value at x is λ. The set of all fuzzy points on X is denoted by $\mathrm{Pt}(I^X)$. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy subset U, denoted by $x_\lambda \widetilde{\in} U$, if $U(x) > 1 - \lambda$.

Definition 2.1. [14] Let (X, δ) be an I-topological space and $x \in X$. A fuzzy subset U of X is called a W-neighborhood of x iff there exists $G \in \delta$ such that $G \subseteq U$ and $G(x) = U(x) > 0$.

Definition 2.2. [7] Let (X, δ) be an I-topological space and $x \in X$. A family $\mathcal{B}_x$ of W-neighborhoods of x is called a W-neighborhood base of x, iff for every W-neighborhood A of x and every $\alpha \in [0, A(x))$ there exists $U \in \mathcal{B}_x$ such that $U \subseteq A$ and $U(x) > \alpha$.

Definition 2.3. [10] Let (X, δ) be an I-topological space and $x_\lambda \in \mathrm{Pt}(I^X)$. A fuzzy subset U of X is called a Q-neighborhood (neighborhood) of x_λ iff there exists $G \in \delta$ such that $x_\lambda \widetilde{\in} G \subseteq U$ $(x_\lambda \in G \subseteq U)$.

Definition 2.4. [10] Let (X, δ) be an I-topological space and $x_\lambda \in \mathrm{Pt}(I^X)$. A family $\mathcal{U}_{x_\lambda}$ of Q-neighborhoods of x_λ is called a Q-neighborhood base of x_λ iff for every Q-neighborhood W of x_λ, there exists $U \in \mathcal{U}_{x_\lambda}$ such that $U \subseteq W$.

Lemma 2.5. [3] *Let (X, δ) be an I-topological space, $x \in X$, $U \in I^X$ and $\mathcal{B} \subset I^X$. Then*

1) *U is a W-neighborhood of the point x iff U is a Q-neighborhood of the fuzzy point x_λ for each $\lambda \in (1 - U(x), 1]$.*

2) *$\mathcal{B}$ is a W-neighborhood base of the point x iff for each $A \in \mathcal{B}$, $A(x) > 0$ and for each $\lambda \in (0, 1]$, $\mathcal{U}_\lambda = \{A \in \mathcal{B} \mid A(x) > 1 - \lambda\}$ is a Q-neighborhood base of the fuzzy point x_λ.*

Lemma 2.6. [14] *Let (X, δ) be an I-topological space. Then $G \in \delta$ if and only if G is a W-neighborhood of x for each $x \in X$ with $G(x) > 0$.*

In the following, let X be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and θ denote the zero element of X.

Definition 2.7. [6] Let $A, B \in I^X$ and $k \in \mathbb{K}$. Then $A + B$ and kA are defined respectively by

$$(A + B)(x) = \bigvee \{A(s) \wedge B(t) \mid s + t = x\},$$

$$(kA)(x) = A(x/k), \quad \text{whenever } k \neq 0;$$

$$(0A)(x) = \begin{cases} \bigvee_{z \in X} A(z), & \text{if } x = \theta, \\ 0, & \text{if } x \neq \theta. \end{cases}$$

In particular, for $x_\lambda,\ y_\mu \in \mathrm{Pt}(I^X)$, we have

$$x_\lambda + y_\mu = (x + y)_{\lambda \wedge \mu}, \quad kx_\lambda = (kx)_\lambda.$$

Definition 2.8. [7] A stratified I-topology δ on X is said to be a I-vector topology, if the following two mappings are continuous:

$$f : X \times X \to X, \quad (x, y) \mapsto x + y \quad \text{and} \quad g : \mathbb{K} \times X \to X, \quad (k, x) \mapsto kx,$$

where $\mathbb{K}$ is equipped with the I-topology induced by the usual topology, $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product I-topologies.

A vector space X with a I-vector topology δ, denoted by (X, δ), is called a I-topological vector space (for short, an I-tvs).

Let L be a *Hutton algebra* [11], i.e., it is a complete and completely distributive lattice equipped with an order-reversing involution $'$. 0 and 1 are its bottom and top elements, respectively.

Definition 2.9. [13] A mapping $f : L \to L$ is called an order-homomorphism, if the conditions hold:

(H-1) $f(0) = 0$;

(H-2) f is union-preserving;

(H-3) $f^\vee$ is complement-preserving, i.e., $f^\vee(b') = [f^\vee(b)]'$ for each $b \in L_2$, where $f^\vee$ is the right adjoint of f (See [11]), i.e., $f^\vee(b) = \bigvee\{a \in L : f(a) \le b\}$, $b \in L$.

Definition 2.10. [1] Let X and Y be two vector spaces. A mapping $\widetilde{T} : L^X \to L^Y$ is a fuzzy linear order-homomorphism (FLOH), it is an order-homomorphism satisfying

$$\widetilde{T}(\alpha A + \beta B) = \alpha\widetilde{T}(A) + \beta\widetilde{T}(B), \quad \text{for all } A, B \in L^X,\ \alpha, \beta \in \mathbb{K}.$$

Lemma 2.11. [1] *Let X and Y be two vector spaces. A mapping $\widetilde{T} : L^X \to L^Y$ is a FLOH iff there exist an ordinary linear operator $T : X \to Y$ and a finitely meet-preserving order-homomorphism $\phi : L \to L$ such that $\widetilde{T} = (T, \phi)^\to$, where $(T, \phi)^\to$ is defined by*

$$(T, \phi)^\to(A)(y) = \bigvee_{Tx=y} \phi(A(x)) \quad \text{for } A \in L^X,\ y \in Y,$$

it is called a bi-induced mapping of T and ϕ.

Below, we only deal with the fuzzy linear order-homomorphism under the case of $L = I$. Obviously, a mapping $\widetilde{T} : I^X \to I^Y$ is a FLOH iff there exist an ordinary linear operator $T : X \to Y$ and an order-homomorphism $\phi : I \to I$ such that $\widetilde{T} = (T, \phi)^\to$. In addition, for convenience we usually use $(T, \phi)^\leftarrow$ instead of $(\widetilde{T})^\vee$.

Lemma 2.12. *Let $\phi : I \to I$ be an order-homomorphism and $\lambda, \mu, r, s \in I$. Then*

1) $\mu > 1 - \lambda$ *implies that* $\phi(\mu) > 1 - \phi(\lambda)$;

2) $\phi(\mu) > 1 - \lambda$ *iff* $\phi^\vee(\lambda) > 1 - \mu$;

3) $\phi^\vee(s) > r$ *implies that* $\phi(r) < s$.

Proof. By Lemma 3.3 in [2], it is easy to know that conclusions 1 and 2 hold.

3) If $\phi^\vee(s) > r$, then we can choose a $\delta_0 > 0$ such that $r < \phi^\vee(s) - \delta_0 < \phi^\vee(s)$. By the conclusion 2, we have $\phi((\phi^\vee(s) - \delta_0)') > s'$.

Taking $0 < \delta < \phi((\phi^{\vee}(s) - \delta_0)') - s'$, we have $\phi((\phi^{\vee}(s) - \delta_0)') > 1 - (s - \delta)$. Again by the conclusion 2, we obtain $\phi^{\vee}(s - \delta) > \phi^{\vee}(s) - \delta_0 > r$, and so $\phi(r) \leq \phi\phi^{\vee}(s - \delta) \leq s - \delta < s$. □

Let $L_1 = L_2 = I$. From Lemma 3.1 and Lemma 3.2 of [2], we obtain the following

Lemma 2.13. *Let* $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ *be a FLOH,* $x_\lambda \in P_t(I^X)$, $A \in I^X$, $B \in l^Y$ *and* $r \in (0, 1]$. *Then*

1) $(T, \phi)^{\rightarrow}(x_\lambda) = (Tx)_{\phi(\lambda)}$;
2) $(T, \phi)^{\leftarrow}(B)(x) = \phi^{\vee}(B(Tx))$;
3) $(T, \phi)^{\rightarrow}(A \cap \underline{r}) = (T, \phi)^{\rightarrow}(A) \cap \underline{\phi(r)}$;
4) $(T, \phi)^{\rightarrow}(A \cap \underline{\phi^{\vee}(r)}) \subset (T, \phi)^{\rightarrow}(A) \cap \underline{r}$.

Let's recall two definitions of "bounded fuzzy sets" given respectively by Katsaras [8] and Wu and Fang [15] and some relevant results.

Definition 2.14. [8] Let (X, δ) be an I-tvs. A fuzzy set B in (X, δ) is said to be *bounded*, if it is absorbed by every W-neighborhood of θ, that is, for every W-neighborhood U of θ and every $0 < r < U(\theta)$, there exists $t > 0$ such that $B \cap \underline{r} \subseteq tU$.

Definition 2.15. [15] Let (X, δ) be an I-tvs. A fuzzy set B in (X, δ) is said to be λ-*bounded* (λ is a given number in $(0, 1]$), if for each Q-neighborhood U of θ_λ in X, there exists $t > 0$ and $r \in (1 - \lambda, 1]$ such that $B \cap \underline{r} \subseteq tU$. B is said to be *bounded*, if it is λ-bounded for each $\lambda \in (0, 1]$.

The following proposition shows that the above two definitions, i.e., Definition 2.9 and Definition 2.10 are equivalent.

Proposition 2.16. [5] *A fuzzy set* B *in* I*-tvs* (X, δ) *is bounded in the sense of Katsaras if and only if* B *is bounded in the sense of Wu and Fang.*

Proposition 2.17. [2] *In* I*-tvs* (X, δ)*, the following statements are hold:*

1) *The union of finitely many bounded fuzzy sets is bounded.*
2) *Every fuzzy point* x_α *is bounded;*
3) *Every finite fuzzy set* A *(i.e.,* supp$A$ *is a finite subset of* X*) is bounded.*

3. Boundedness of Fuzzy Linear Order Homomorphisms

Definition 3.1. [2] Let (X, δ_1) and (Y, δ_2) be two I-tvs and $\lambda \in (0, 1]$. A FLOH $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ is said to be λ-bounded, if $(T, \phi)^{\rightarrow}$ maps every λ-bounded fuzzy set in X into a $\phi(\lambda)$-bounded fuzzy set in Y. $(T, \phi)^{\rightarrow}$ is said to be bounded, if it is λ-bounded for every $\lambda \in (0, 1]$.

Remark 3.2. Definition 3.1 is a special case of Definition 4.2 in [2] with $L = [0, 1]$.

Now, we give a new definition of the bounded FLOH, which is a natural generalization of the definition of a bounded linear operator.

Definition 3.3. Let (X, δ_1) and (Y, δ_2) be two I-tvs. A FLOH $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ is said to be bounded if it maps every bounded fuzzy set in (X, δ_1) into a bounded fuzzy set in (Y, δ_2).

Remark 3.4. From now on, for distinction, we rename bounded FLOH in the sense of Definition 3.1 as bounded FLOH on every layer.

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a normed space satisfying $\sup_{x\in X}\|x\| = \infty$. Put $U_\varepsilon = \{x \in X : \ \|x\| < \varepsilon\}$ $(\varepsilon > 0)$. For given $c \in (0,1)$ and each $\lambda \in (0,1]$, we define $\mathcal{U}_\lambda$ as follows:*

$$\mathcal{U}_\lambda = \begin{cases} \{\underline{r} \mid r \in (1-\lambda, 1]\}, & \text{if } \lambda \in (0, c], \\ \{U_\varepsilon \cap \underline{r} \mid \varepsilon > 0,\ r \in (1-\lambda, 1]\}, & \text{if } \lambda \in (c, 1]. \end{cases}$$

Then

1) *There exists a unique fuzzy topology on X, denoted by $N_c(\tau)$, such that $(X, N_c(\tau))$ is an I-tvs and $\mathcal{U}_\lambda$ is a Q-neighborhood base of θ_λ for each $\lambda \in (0,1]$.*

2) *B is a bounded fuzzy set in $(X, N_c(\tau))$ if and only if the set $\operatorname{supp} B$ is bounded in $(X, \|\cdot\|)$, where $\operatorname{supp} B = \{x \in X \mid B(x) > 0\}$*

Proof. 1) It is easy to verify that $\{\mathcal{U}_\lambda \mid \lambda \in (0,1]\}$ satisfies condition (1)−(5) of Theorem 4.1 in [3]. Hence the conclusion 1 holds.

2) Suppose that B is bounded in $(X, N_c(\tau))$. Then B is 1-bounded in $(X, N_c(\tau))$. Note that $\mathcal{U}_1 = \{U_\varepsilon \cap \underline{s} \mid \varepsilon > 0,\ s \in (0,1]\}$ is a Q-neighborhood base of θ_1. Hence, for each $\varepsilon > 0$ and $s \in (0,1]$, there exist $t > 0$ and $r \in (0,1]$ such that $B \cap \underline{r} \subseteq tU_\varepsilon \cap \underline{s}$, which implies that $\operatorname{supp} B \subseteq tU_\varepsilon$. Note that $\{U_\varepsilon \mid \varepsilon > 0\}$ is a neighborhood of θ in $(X, \|\cdot\|)$. Therefore, we conclude that $\operatorname{supp} B$ is bounded in $(X, \|\cdot\|)$.

Conversely, suppose that $\operatorname{supp} B$ is bounded in $(X, \|\cdot\|)$. Then for each $\varepsilon > 0$ there exists $t > 0$ such that $\operatorname{supp} B \subset tU_\varepsilon$. So, for each $\lambda \in (c, 1]$ and $r \in (1-\lambda, 1]$, we have $B \cap \underline{r} \subseteq (\operatorname{supp} B) \cap \underline{r} \subseteq tU_\varepsilon \cap \underline{r}$. In addition, for each $\lambda \in (0, c]$ and $r \in (1-\lambda, 1]$, we have $B \cap \underline{r} \subseteq \underline{r}$. This shows that for each $\lambda \in (0,1]$, B is λ-bounded in $(X, N_c(\tau))$. Therefore B is bounded in $(X, N_c(\tau))$. □

The following two examples show that "boundedness" and "boundedness on each layer" of FLOH do not imply each other.

Example 3.6. Let $(X, N_{3/4}(\tau))$ and $(X, N_{1/4}(\tau))$ be defined as in Lemma 3.5. We denote by T_0 and ϕ_0 the identity mappings on X and I, respectively. Then mapping $(T_0, \phi_0)^{\rightarrow} : (X, N_{3/4}(\tau)) \to (X, N_{1/4}(\tau))$ is a bounded FLOH, but it is not bounded on each layer.

In fact, by $(T_0, \phi_0)^{\rightarrow}(B) = B$ and Lemma 3.5, it is easy to see that $(T_0, \phi_0)^{\rightarrow}$ is a bounded FLOH. On the other hand, note that $(T_0, \phi_0)^{\rightarrow}(\underline{1}) = \underline{1}$, $\phi_0(\frac{1}{2}) = \frac{1}{2}$ and $\underline{1}$ is $\frac{1}{2}$-bounded in $(X, N_{3/4}(\tau))$, but $\underline{1}$ is not $\frac{1}{2}$-bounded in $(X, N_{1/4}(\tau))$. Therefore, $(T_0, \phi_0)^{\rightarrow}$ is not bounded on each layer.

Example 3.7. Let X be a linear space and $\delta_0 = \{\underline{r} \mid r \in I\}$. Then it is easy to see that (X, δ_0) is an I-tvs and $\mathcal{U}_\lambda = \{\underline{r} \mid r \in (1-\lambda, 1]\}$ is a Q-neighborhood base of θ_λ for each $\lambda \in (0,1]$. Also, let $(X, N_{3/4}(\tau))$ be an I-tvs defined as in Lemma 3.5 and T_0 an identity mapping on X. Define the mapping $\phi : I \to I$ as follows:

$$\phi(x) = \begin{cases} 0, & x = 0, \\ \frac{1}{3}(x+1), & x \in (0,1]. \end{cases}$$

Note that

$$\phi^{\vee}(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{3}, \\ 3x - 1, & \frac{1}{3} < x \le \frac{2}{3}, \\ 1, & \frac{2}{3} < x \le 1. \end{cases}$$

It is easy to see that $(T_0, \phi)^{\rightarrow} : (X, \delta_0) \to (X, N_{3/4}(\tau))$ is a FLOH. We can prove that $(T_0, \phi)^{\rightarrow}$ is bounded on each layer, but it is not bounded.

In fact, it is easy to see that for each $\lambda \in (0, 1]$, $B \in I^X$ is λ-bounded in (X, δ_0). Since $\phi(\lambda) < 3/4$ for each $\lambda \in (0, 1]$, $\mathcal{U}_{\phi(\lambda)} = \{\underline{r} \mid r \in (1 - \phi(\lambda), 1]\}$ is a Q-neighborhood base of $\theta_{\phi(\lambda)}$ in $(X, N_{3/4}(\tau))$, and so $(T_0, \phi)^{\rightarrow}(B)$ is $\phi(\lambda)$-bounded in $(X, N_{3/4}(\tau))$. This shows that $(T_0, \phi)^{\rightarrow}$ is bounded on each layer. On the other hand, $\underline{1}$ is bounded in (X, δ_0). Note that supp $(T_0, \phi)^{\rightarrow}(\underline{1}) = X$ and X is not bounded in $(X, \| \cdot \|)$. Hence, by Lemma 3.5, we know that $(T_0, \phi)^{\rightarrow}(\underline{1})$ is not bounded in $(X, N_{3/4}(\tau))$. This shows that $(T_0, \phi)^{\rightarrow}$ is not bounded.

Lemma 3.8. *Let $\phi : I \to I$ be an order-homomorphism. Then $\phi^{\vee}$ is injective if and only if ϕ is continuous.*

Proof. Necessity: Suppose that $\phi^{\vee}$ is injective, but ϕ is not continuous. Without loss of generality, we may assume that $t_0 \in (0, 1)$ is a discontinuous point of ϕ (if $t_0 = 0$ or $t_0 = 1$, the proof is similar). Since ϕ is nondecreasing, we have $\phi(t_0 - \circ) < \phi(t_0 + \circ)$. Choose $b_1, b_2 \in (\phi(t_0 - \circ), \phi(t_0 + \circ))$ with $b_1 < b_2$. Since $\phi^{\vee}$ is nondecreasing and injective, we have $\phi^{\vee}(b_1) < \phi^{\vee}(b_2)$. Note that for any $t \in (0, t_0)$, we have $\phi(t) \le \phi(t_0 - \circ) < b_1$, and so $\phi^{\vee}(b_1) \ge t$. By the arbitrariness of $t \in (0, t_0)$, we obtain $\phi^{\vee}(b_1) \ge t_0$. On the other hand, for any $t \in (t_0, 1]$, we have $b_2 < \phi(t_0 + \circ) \le \phi(t)$, and so $\phi^{\vee}(b_2) < t$. By the arbitrariness of $t \in (t_0, 1]$, we obtain $\phi^{\vee}(b_2) \le t_0$. Hence, $\phi^{\vee}(b_1) \ge \phi^{\vee}(b_2) > \phi^{\vee}(b_1)$, which is a contradiction. Therefore ϕ is continuous.

Sufficiency: Suppose that $\phi^{\vee}$ is not injective. Then there exist $b_1, b_2 \in I$ with $b_1 < b_2$ such that $\phi^{\vee}(b_1) = \phi^{\vee}(b_2)$. Putting $t_0 = \phi^{\vee}(b_1)$, we shall prove that t_0 is a discontinuous point of ϕ. In fact, since $\phi(t) \le \phi(t_0) = \phi(\phi^{\vee}(b_1)) \le b_1$ for all $t < t_0$, we have $\phi(t_0 - \circ) \le b_1$. On the other hand, we can prove that $\phi(t_0 + \circ) \ge b_2$. If there exists $t_1 > t_0$ such that $\phi(t_1) < b_2$, then we have $\phi^{\vee}(b_2) \ge t_1 > t_0 = \phi^{\vee}(b_1)$, which contradicts $\phi^{\vee}(b_1) = \phi^{\vee}(b_2)$. This shows that $\phi(t_0 + \circ) \ge b_2 > b_1 \ge \phi(t_0 - \circ)$. Therefore, t_0 is a discontinuous point of ϕ. □

Lemma 3.9. *Let $\phi : I \to I$ be a surjective order-homomorphism. Then ϕ is continuous.*

Proof. Suppose that ϕ is not continuous. Without loss of generality, we may assume that $t_0 \in (0, 1)$ is a discontinuous point of ϕ (if $t_0 = 0$ or $t_0 = 1$, the proof is similar). Since ϕ is nondecreasing, we can choose $b \in (\phi(t_0 - \circ), \phi(t_0 + \circ))$ such that $\phi(t_0) \ne b$. Thus, it is not difficult to see that $\phi(t) \ne b$ for all $t \in I$, i.e., ϕ is not surjective. □

The following theorem shows that "boundedness on each layer" of FLOH implies "boundedness" under certain condition.

Theorem 3.10. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and let $(T, \phi)^{\rightarrow} : I^X \to I^Y$ be a FLOH and ϕ be continuous (or $\phi^{\vee}$ be injective). If $(T, \phi)^{\rightarrow}$ is bounded on each layer, then $(T, \phi)^{\rightarrow}$ is bounded.*

Proof. By Lemma 3.8, we only need to consider the case of when $\phi^\vee$ is injective. Let A be an arbitrary bounded fuzzy set in (X, δ_1) and V be an arbitrary W-neighborhood of θ in (Y, δ_2). We need to prove that for each $\alpha \in (0, V(\theta))$, there exists $t > 0$ such that $(T, \phi)^{\rightarrow}(A) \cap \underline{\alpha} \subset tV$. Since $\phi^\vee$ is injective, $\alpha \in (0, V(\theta))$ implies that $\phi^\vee(V(\theta)') < \phi^\vee(\alpha')$. Take $\lambda \in (\phi^\vee(V(\theta)'), \phi^\vee(\alpha')]$. It follows from $\lambda \leq \phi^\vee(\alpha')$ that $\phi(\lambda) \leq \alpha' = 1-\alpha$, i.e., $\alpha \leq 1-\phi(\lambda)$. It follows from $\lambda > \phi^\vee(V(\theta)')$ that $\phi^\vee(V(\theta)) > \lambda' = 1 - \lambda$, and so by the conclusion 2 of Lemma 2.12, we get $\phi(\lambda) > 1 - V(\theta)$, i.e., $V(\theta) > 1 - \phi(\lambda)$. Hence, by Lemma 2.5 we know that V is a Q-neighborhood of $\theta_{\phi(\lambda)}$ in (Y, δ_2).

Note that A is λ-bounded for each $\lambda \in (0, 1]$ and $(T, \phi)^{\rightarrow}$ is bounded on each layer. Hence $(T, \phi)^{\rightarrow}(A)$ is $\phi(\lambda)$-bounded. So, there exist $t > 0$ and $r > 1 - \phi(\lambda)$ such that

$$(T, \phi)^{\rightarrow}(A) \cap \underline{r} \subset tV. \tag{1}$$

Since $r > 1 - \phi(\lambda) \geq \alpha$, (1) implies that $(T, \phi)^{\rightarrow}(A) \cap \underline{\alpha} \subset tV$, which shows that $(T, \phi)^{\rightarrow}(A)$ is a bounded fuzzy set in (Y, δ_2). Therefore $(T, \phi)^{\rightarrow}$ is bounded. □

By Theorem 3.10 and Lemma 3.9, we have the following corollaries:

Corollary 3.11. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and let $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ be a FLOH and ϕ be surjective. If $(T, \phi)^{\rightarrow}$ is bounded on each layer, then $(T, \phi)^{\rightarrow}$ is bounded.*

4. Continuity of Fuzzy Linear Order Homomorphisms

Definition 4.1. Let (X, δ_1) and (Y, δ_2) be two I-tvs, $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ be a FLOH.

1) $(T, \phi)^{\rightarrow}$ is said to be W-continuous at a point x if for every W-neighborhood V of Tx in (Y, δ_2), $(T, \phi)^{\leftarrow}(V)$ is a W-neighborhood of x in (X, δ_1).

2) $(T, \phi)^{\rightarrow}$ is said to be continuous at a fuzzy point x_λ if for every Q-neighborhood U of $(Tx)_{\phi(\lambda)}$ in (Y, δ_2), $(T, \phi)^{\leftarrow}(U)$ is a Q-neighborhood V of x_λ in (X, δ_1).

Lemma 4.2. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and let $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ be a FLOH and $x_\lambda \in \mathrm{Pt}(I^X)$. If $(T, \phi)^{\rightarrow}$ is continuous at x_λ, then $\phi^\vee(t) > 1 - \lambda$ for each $t \in (1 - \lambda, 1]$.*

Proof. Let $t \in (1-\lambda, 1]$. It is obvious that $\underline{t}$ is a Q-neighborhood of $(Tx)_\lambda$ in (Y, δ_2). By continuity of $(T, \phi)^{\rightarrow}$ at $x_\lambda \in X$, we know that $(T, \phi)^{\leftarrow}(\underline{t})$ is a Q-neighborhood of x_λ, and so $(T, \phi)^{\leftarrow}(\underline{t})(x) > 1 - \lambda$. Note that $(T, \phi)^{\leftarrow}(\underline{t})(x) = \phi^\vee(\underline{t}(x)) = \phi^\vee(t)$. Hence $\phi^\vee(t) > 1 - \lambda$. □

Theorem 4.3. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and let $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ be a FLOH and $x \in X$. Then $(T, \phi)^{\rightarrow}$ is W-continuous at x iff for each $\lambda \in (0, 1]$, $(T, \phi)^{\rightarrow}$ is continuous at the fuzzy point x_λ.*

Proof. *Necessity:* Let $\lambda \in (0, 1]$ and U be an arbitrary Q-neighborhood of $(Tx)_{\phi(\lambda)}$ in (Y, δ_2), and let $\mathcal{B}_{Tx}$ be a W-neighborhood base of Tx. Then by Lemma 2.5 there exists $B \in \mathcal{B}_{Tx}$ with $B(Tx) > 1 - \phi(\lambda)$ such that $B \subseteq U$. Since $(T, \phi)^{\rightarrow}$ is W-continuous at x, $(T, \phi)^{\leftarrow}(B)$ is a W-neighborhood of x in (X, δ_1). Note that

$\phi(\lambda) > 1 - B(Tx)$. It follows from Lemmas 2.12 and 2.13 that $(T,\phi)^{\leftarrow}(B)(x) = \phi^{\vee}(B(Tx)) > 1-\lambda$. So, by Lemma 2.5, $(T,\phi)^{\leftarrow}(B)$ is a Q-neighborhood of x_λ. Note that $B \subseteq U$ implies that $(T,\phi)^{\leftarrow}(B) \subseteq (T,\phi)^{\leftarrow}(U)$. Hence $(T,\phi)^{\leftarrow}(U)$ is also a Q-neighborhood of x_λ in (X,δ_1). Therefore, $(T,\phi)^{\rightarrow}$ is continuous at x_λ.

Sufficiency: Suppose that P is an arbitrary W-neighborhood of Tx in (Y,δ_2). Clearly, $P(Tx) > 0$. Take $\lambda \in (1-P(Tx),1]$. Note that $(T,\phi)^{\rightarrow}$ is continuous at x_λ. It follows from Lemma 4.2 that $\phi^{\vee}(P(Tx)) > 1-\lambda$, and so by Lemma 2.12 we have $\phi(\lambda) > 1-P(Tx)$, i.e., $P(Tx) > 1-\phi(\lambda)$, which shows that P is a Q-neighborhood of $(Tx)_{\phi(\lambda)}$. Since $(T,\phi)^{\rightarrow}$ is continuous at x_λ, $(T,\phi)^{\leftarrow}(P)$ is a Q-neighborhood of x_λ in (X,δ_1). So, there exists $G_\lambda \in \delta_1$ such that $x_\lambda \widetilde{\in} G_\lambda \subseteq (T,\phi)^{\leftarrow}(P)$. Put $\mu = 1 - P(Tx)$ and $G = \bigcup_{\lambda\in(\mu,1]} G_\lambda$. It is evident that $G \in \delta_1$, $G \subseteq (T,\phi)^{\leftarrow}(P)$ and

$$G(x) = \sup_{\lambda\in(\mu,1]} G_\lambda(x) \geq \sup_{\lambda\in(\mu,1]} (1-\lambda) = 1-\mu = P(Tx) > 0.$$

Hence, $(T,\phi)^{\leftarrow}(P)$ is a W-neighborhood of x. This shows that $(T,\phi)^{\rightarrow}$ is W-continuous at x. □

Theorem 4.4. *Let (X,δ_1) and (Y,δ_2) be two I-tvs, and $(T,\phi)^{\rightarrow} : I^X \to I^Y$ be a FLOH. Then the following statements are equivalent:*

(a) $(T,\phi)^{\rightarrow}$ *is continuous, i.e.,* $(T,\phi)^{\leftarrow}(G) \in \delta_1$ *for each* $G \in \delta_2$*;*
(b) $(T,\phi)^{\rightarrow}$ *is continuous at each* $x_\lambda \in \mathrm{Pt}(I^X)$*;*
(c) $(T,\phi)^{\rightarrow}$ *is continuous at* θ_λ *for each* $\lambda \in (0,1]$.

Proof. (a) $\Rightarrow$ (*b*). Suppose that $x_\lambda \in \mathrm{Pt}(I^X)$ and U is an arbitrary Q-neighborhood of $(Tx)_{\phi(\lambda)}$ in (Y,δ_2). Then there exists $G \in \delta_2$ such that $G \subseteq U$ and $G(Tx) > 1-\phi(\lambda)$, and so by Lemma 2.12 we have $(T,\phi)^{\leftarrow}(G) = \phi^{\vee}(G(Tx)) > 1-\lambda$. By (a), we know that $(T,\phi)^{\leftarrow}(G) \in \delta_1$. Note that $(T,\phi)^{\leftarrow}(G) \subseteq (T,\phi)^{\leftarrow}(U)$. Hence $(T,\phi)^{\leftarrow}(U)$ is a Q-neighborhood of x_λ in (X,δ_1). This shows that $(T,\phi)^{\rightarrow}$ is continuous at the point x_λ.

(*b*) $\Rightarrow$ (*c*). It is evident.

(*c*) $\Rightarrow$ (a). Suppose that $G \in \delta_2$. To prove that $(T,\phi)^{\leftarrow}(G) \in \delta_1$, by Lemma 2.6, we only need to prove that for every $x \in X$ with $(T,\phi)^{\leftarrow}(G)(x) > 0$, $(T,\phi)^{\leftarrow}(G)$ is a W-neighborhood of x in (X,δ_1). Obviously, $(T,\phi)^{\leftarrow}(G)(x) = \phi^{\vee}(G(Tx)) > 0$ implies that $G(Tx) > 0$. Note that $-Tx+G \in \delta_2$ and $(-Tx+G)(\theta) = G(Tx) > 0$. Hence, by Lemma 2.6, $-Tx+G$ is a W-neighborhood of θ in (Y,δ_2). From (c) and Theorem 4.3, it is easy to see that $(T,\phi)^{\leftarrow}(-Tx+G)$ is a W-neighborhood of θ in (X,δ_1). In addition, $(T,\phi)^{\leftarrow}(-Tx+G) = -x + (T,\phi)^{\leftarrow}(G)$. Hence, $(T,\phi)^{\leftarrow}(G)$ is a W-neighborhood of x in (X,δ_1). This shows that $(T,\phi)^{\rightarrow}$ is continuous. □

By Theorem 4.3 and Theorem 4.4, we have the following

Theorem 4.5. *Let (X,δ_1) and (Y,δ_2) be two I-tvs, and $(T,\phi)^{\rightarrow} : I^X \to I^Y$ be a FLOH. Then the following statements are equivalent:*

(a) $(T,\phi)^{\rightarrow}$ *is continuous, i.e.,* $(T,\phi)^{\leftarrow}(G) \in \delta_1$ *for each* $G \in \delta_2$*;*
(b) $(T,\phi)^{\rightarrow}$ *is W-continuous at x for each* $x \in X$*;*
(c) $(T,\phi)^{\rightarrow}$ *is W-continuous at* θ.

The following theorem is a special case of Theorem 4.4 in [2] with $L_1 = L_2 = I$.

Theorem 4.6. [2] *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and let $(T, \phi)^{\rightarrow} : I^X \to I^Y$ be a continuous FLOH. Then $(T, \phi)^{\rightarrow}$ is bounded on each layer.*

From Theorem 4.6 and Theorem 3.10, we obtain the following theorem.

Theorem 4.7. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and $(T, \phi)^{\rightarrow} : I^X \to I^Y$ be a FLOH. If $(T, \phi)^{\rightarrow}$ is continuous and ϕ is continuous (or $\phi^{\vee}$ is injective), then $(T, \phi)^{\rightarrow}$ is bounded.*

Definition 4.8. An I-tvs (X, δ) is said to satisfy the first axiom of countability if there exists a countable family of neighborhoods $\{V_n\}$ of θ_1 such that the family $\{V_n \cap \underline{r} \mid r > 0,\ n \in \mathbb{N}\}$ is a W-neighborhood base of θ.

Lemma 4.9. *Let (X, δ) be an I-tvs satisfying the first axiom of countability. Then there exists a countable family of balanced neighborhoods $\{V_n\}_{n\in\mathbb{N}}$ of θ_1 satisfying $V_{n+1} \subset V_n$ $(n = 1, 2, \cdots)$ such that $\{V_n \cap \underline{r} \mid r > 0,\ n \in \mathbb{N}\}$ is a W-neighborhood base of θ.*

Proof. Since (X, δ) is an I-tvs satisfying the first axiom of countability, there exists a countable family of neighborhoods $\{U_n\}$ of θ_1 such that $\{U_n \cap \underline{r} \mid r > 0,\ n \in \mathbb{N}\}$ is a W-neighborhood base of θ. By Theorem 3 in [16], we can assume that $\mathcal{B}$ is a balanced neighborhood base of θ_1. So, for U_1, there exists $V_1 \in \mathcal{B}$ such that $V_1 \subseteq U_1$. Note that $U_2 \cap V_1$ is also a neighborhood of θ_1. Hence there exists $V_2 \in \mathcal{B}$ such that $V_2 \subseteq U_2 \cap V_1$. Proceeding with this way, we obtain a sequence $\{V_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that $V_n \subseteq U_n \cap V_{n-1}$, $n = 2, 3, \cdots$. It is evident that $\{V_n\}_{n\in\mathbb{N}}$ is needed in conclusion of the lemma. □

Lemma 4.10. *Let $\phi : I \to I$ be an order-homomorphism. Then ϕ is continuous from right at $t = 0$ if and only if $\phi^{\vee}(t) > 0$ for each $t > 0$.*

Proof. Suppose that ϕ is continuous from right at $t = 0$. Note that $\phi(0) = 0$. Hence for each $t > 0$, there exists $0 < s < t$ such that $\phi(s) = \phi(s) - \phi(0) < t$, and so $\phi^{\vee}(t) \geq s > 0$.

Conversely, if $\phi^{\vee}(t) > 0$ for each $t > 0$, then there exists $\delta > 0$ such that $\phi(\delta) \leq t$. Note that ϕ is non-decreasing. Hence, $|\phi(s) - \phi(0)| \leq \phi(\delta) \leq t$ whenever $0 < s < \delta$. Therefore ϕ is continuous from right at $t = 0$. □

Theorem 4.11. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, (X, δ_1) satisfy the first axiom of countability, and let $(T, \phi)^{\rightarrow} : I^X \to I^Y$ be a bounded FLOH and ϕ be continuous from right at $t = 0$ (or, equivalently, $\phi^{\vee}(t) > 0$ for all $t > 0$). Then $(T, \phi)^{\rightarrow}$ is continuous.*

Proof. Suppose that $(T, \phi)^{\rightarrow}$ is bounded, but is not continuous. Then by Theorem 4.5, we know that $(T, \phi)^{\rightarrow}$ is not W-continuous at θ. Hence there exists a W-neighborhood U_0 of θ in (Y, δ_2) such that $(T, \phi)^{\leftarrow}(U_0)$ is not a W-neighborhood of θ in (X, δ_1).

Since (X, δ_1) satisfy the first axiom of countability, by Lemma 4.9 there exists a countable family of balanced neighborhoods $\{V_n\}$ of θ_1 satisfying condition $V_{n+1} \subseteq$

V_n $(n = 1, 2, \cdots)$ such that $\{V_n \cap \underline{r} \mid r > 0,\ n \in \mathbb{N}\}$ is a W-neighborhood base of θ. Obviously, $\{(1/n)V_n \cap \underline{r} \mid r > 0,\ n \in \mathbb{N}\}$ is also a W-neighborhood base of θ in (X, δ_1). Since ϕ is continuous from right at $t = 0$, by Lemma 4.10 we have $[(T, \phi)^{\leftarrow}(U_0)](\theta) = \phi^{\vee}(U_0(\theta)) > 0$. Note that $(T, \phi)^{\leftarrow}(U_0)$ is not a W-neighborhood of θ in (X, δ_1). Hence there exists $\alpha \in (0, [(T, \phi)^{\leftarrow}(U_0)](\theta))$ such that

$$\frac{1}{n}V_n \cap \underline{r} \nsubseteq (T, \phi)^{\leftarrow}(U_0) \quad \text{for all } n \in \mathbb{N} \text{ and } r > \alpha.$$

In particular, we take $r_0 \in (\alpha, [(T, \phi)^{\leftarrow}(U_0)](\theta))$, then $\frac{1}{n}V_n \cap \underline{r_0} \nsubseteq (T, \phi)^{\leftarrow}(U_0)$. And then, by Lemma 2.13, we have

$$(T, \phi)^{\rightarrow}(V_n) \cap \underline{\phi(r_0)} \nsubseteq nU_0 \quad \text{for all } n \in \mathbb{N}.$$

Hence there exists $x^{(n)} \in \operatorname{supp} V_n$ such that

$$(T, \phi)^{\rightarrow}\Big((x^{(n)})_{\lambda_n}\Big) \cap \underline{\phi(r_0)} \notin nU_0 \quad \text{for all } n \in \mathbb{N}, \tag{2}$$

where $\lambda_n = V_n(x^{(n)})$. Put $A = \bigcup\Big\{(x^{(n)})_{\lambda_n} : n \in \mathbb{N}\Big\}$. We can prove that A is a bounded fuzzy set in (X, δ_1).

In fact, if P is an arbitrary W-neighborhood of θ in (X, δ_1), then for each $\alpha \in (0, P(\theta))$, there exist $n_0 \in \mathbb{N}$ and $r > \alpha$ such that $V_{n_0} \cap \underline{r} \subseteq P$. Put

$$A_0 = \bigcup_{i=1}^{n_0 - 1} (x^{(n)})_{\lambda_n} \quad \text{and} \quad A_1 = \bigcup_{n \geq n_0} (x^{(n)})_{\lambda_n}.$$

By Proposition 2.17, we know that A_0 is bounded. Note that $V_n \subset V_{n_0}$ for all $n \geq n_0$. We have

$$A_1 \cap \underline{\alpha} \subseteq V_{n_0} \cap \underline{r}. \tag{3}$$

Note that A_0 is bounded, $V_{n_0} \cap \underline{r}$ is a W-neighborhood of θ and $\alpha \in (0, V_{n_0}(\theta) \wedge r)$. Hence there exists $t_0 > 0$ such that

$$A_0 \cap \underline{\alpha} \subseteq t_0(V_{n_0} \cap \underline{r}). \tag{4}$$

It follows from (3) and (4) that

$$\begin{aligned} A \cap \underline{\alpha} &= (A_0 \cap \underline{\alpha}) \cup (A_1 \cap \underline{\alpha}) \\ &\subseteq (t_0 V_{n_0} \cap \underline{r}) \cup (V_{n_0} \cap \underline{r}) \subseteq (t_0 + 1)(V_{n_0} \cap \underline{r}), \end{aligned}$$

which shows that A is bounded. On the other hand, it is not difficult to prove that $(T, \phi)^{\rightarrow}(A)$ is not bounded in (Y, δ_2).

In fact, by the definition of A and (2), we have

$$(T, \phi)^{\rightarrow}(A) \cap \underline{\phi(r_0)} \nsubseteq nU_0 \text{ for all } n \in \mathbb{N}. \tag{5}$$

Note that $r_0 < [(T, \phi)^{\leftarrow}(U_0)](\theta) = \phi^{\vee}(U_0(T\theta)) = \phi^{\vee}(U_0(\theta))$. By Lemma 2.12, we have $0 < \phi(r_0) < U_0(\theta)$. Thus, by Definition 2.14 and (5), we conclude that $(T, \phi)^{\rightarrow}(A)$ is not bounded in (Y, δ_2). This contradicts that $(T, \phi)^{\rightarrow}$ is bounded. Therefore, $(T, \phi)^{\rightarrow}$ is continuous. □

By Theorem 4.7 and Theorem 4.11, we have the following Theorem:

Theorem 4.12. *Let (X, δ_1) and (Y, δ_2) be two I-tvs, and (X, δ_1) satisfy the first axiom of countability. Let $(T, \phi)^{\rightarrow} : I^X \rightarrow I^Y$ be a FLOH and ϕ be continuous (or $\phi^{\vee}$ be injective). Then $(T, \phi)^{\rightarrow}$ is continuous if and only if $(T, \phi)^{\rightarrow}$ is bounded.*

In particular, if ϕ in Theorem 4.12 is the identity mapping on I, i.e., $\phi(x) = x$ for all $x \in I$, at this time, we use $T^{\rightarrow}$ to replace $(T, \phi)^{\rightarrow}$, then we obtain the following corollary.

Corollary 4.13. *Let (X, δ_1) and (Y, δ_2) be two I-tvs. Suppose that (X, δ_1) satisfy the first axiom of countability and $T : X \rightarrow Y$ be a linear operator. Then $T^{\rightarrow}$ is continuous if and only if $T^{\rightarrow}$ is bounded.*

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