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MULTI-OBJECTIVE OPTIMIZATION WITH PREEMPTIVE PRIORITY SUBJECT TO FUZZY RELATION EQUATION CONSTRAINTS

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Abstract. This paper studies a new multi-objective fuzzy optimization problem. The objective function of this study has different levels. Therefore, a suitable optimized solution for this problem would be an optimized solution with preemptive priority. Since, the feasible domain is non-convex; the traditional methods cannot be applied. We study this problem and determine some special structures related to the feasible domain, and using them some methods are proposed to reduce the size of the problem. Therefore, the problem is being transferred to a similar 0-1 integer programming and it may be solved by a branch and bound algorithm. After this step the problem changes to solve some consecutive optimized problem with linear objective function on discrete region. Finally, we give some examples to clarify the subject.

1. Introduction

The theory of fuzzy relational equations (FRE) forms a generalized version of Boolean relation equations. There are a large number of papers that have dealt with fuzzy relation equation[75, 76, 61, 30, 28, 56, 41, 16, 17, 64, 12, 26, 40]. The paper of Sanchez [60] started a development of the theory and applications of FRE treated as a formalized model for non – precise concepts. Generally, fuzzy set theory has a number of properties that make it suitable for formalizing the uncertain information upon which many applied concepts [17] such as medical diagnosis and treatment are usually based. For this reason the theory of FRE was originally applied to problems of the medical diagnosis [60]. Later, the concept of FRE was used in many problems such as system analysis [47], decision making [1], fuzzy controller [12], fuzzy modeling [69], fuzzy analysis and especially fuzzy arithmetic can be implemented using FRE. Pedrycz [48] categorized and extended two ways of the generalizations of FRE in terms of sets under discussion and various operations which are taken into account. Since then, many theoretical improvements have been investigated and many applications have been presented. For instance, we can refer to [5, 7, 9, 16, 18, 21, 20, 13, 53, 65, 25, 36, 45, 50, 51, 49, 55, 68, 72, 74, 23, 62, 63].

Klement et al. [27, 29]presented the basic analytical and algebraic properties of triangular norms and important classes of fuzzy operators' generalization such as Archimedean, strict and nilpotent t - norms. Pedrycz and Vasilakos [52]converted a

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highly dimensional relational equation into a series of single input FREs. In [54] the author shows how problems of interpolation and approximation of fuzzy functions are connected with solvability of systems of FRE. In [48], FRE has been extended to the setting of interval - valued FRE with a max - t - norm composition and three types of solution sets have been proposed. Markovskii showed that solving max - product FRE is closely related to the covering problem which is an NP - hard problem [39]. In [3] Chen and Wang designed an algorithm for obtaining the logarithm to find all minimal solutions of FRE with max - min composition may not exist. An interesting application of the fuzzy relations theory is in the field of image processing [11]and [8]. Also, the various types of images are considered [24, 46, 37, 6]. In [6] the image was divided in to blocks and then the FRE with a t - norm were utilized for compression each block where results were obtained using the Lukasiewicz t - norm. In [44] monochromatic images are interpreted as a fuzzy relation such that the entries may be the normalized values of the pixels. This method is based essentially on the fact that the reconstructed values of the pixels. This method is based essentially on the fact that the reconstructed images are obtained as the greatest or the smallest solution of a system of fuzzy equations. Also, authors showed processes for coding / decoding color images in the RGB and YUV spaces by using FRE of max - t type where t is the Yager t - norm. In [43] the authors used particular FRE of compression / decompression of color images in the RGB and YUV spaces. Nobuhara et al. [42] formulated and solved a problem of image reconstruction using eigen fuzzy sets. They proposed two algorithms of generating eigen fuzzy sets used in the reconstruction process. Di Nola and Russo [10] focused on the algebraic structures of such problems by considering the Lukasiewicz transform as a resituated map. Theoretically, the well -known consequence has been showed that the solutions set of FRE with max - min operator can be completely determined by a unique solution and finite number of minimal solutions [22, 4]. The same results have been proved by Di Nola et al. [11] if continuous t - norm replaces max - min operator. This fact, especially, plays an important role in linear optimization problem subject to FRE formulated as follows:

$$\min c^t x$$

$$x^t \circ A = b, 0 \leq x \leq 1 \tag{1}$$

That "$\circ$" is a fuzzy operation.

Fang and Li [14]converted the linear optimization problem subjected to FRE constraints with max - min operation into an integer programming problem and solved that problem by branch and bound method using jump - tracking technique. Fang and Li's method was improved by Wu et al. [71] who presented a procedure decreasing the search domain. Also, Wu and Guu studied this problem and simplified its optimization process by giving three rules resulted from a necessary condition [73]. The topic of the linear optimization problem with max - product was investigated by Leotamonphong and Fang [35]. They defined two sub - problems by separating negative and non - negative coefficients in the objective function and then obtained the optimal solution by combining these two sub problems. The sub problem with negative coefficient is converted into a $0-1$ integer programming problem and is solved by branch and bound method. Guu and Wu [19] provided a necessary condition for an optimal solution in terms of the maximum solution derived from fuzzy relational equations. They employed this condition to derive an efficient procedure for solving that linear optimization problem.

Another interesting generalization of such optimization problems is related to objective function. If the objective function is $Z(x) = \max_{i\in I}\{\min(c_i\ ,\ x_i)\}$ with $c_i \in [0\ ,\ 1]$ the model is called the latticized linear problem [67]. Also, Wang [50] studied optimization problem subject to FRE with multiple linear objective functions. Loetamonphong et al. [34] have investigated non - linear multi - objective optimization problem with a FRE constraints and proposed a genetic algorithm to find the Pareto optimal solutions (For further study see [58,66]). On the other hand, Lu and Fang solved a nonlinear optimization problem with fuzzy relation equation constraints and max-min composition [38]. Additionally, we refer the readers to [33] in which an interesting application of optimization linear objective with max-min composition has been employed for the streaming media provider seeking a minimum cost while fulfilling the requirements assumed by a three-tier framework.

In this paper, we aim to find the optimal solution of the following problem

$$\begin{aligned} &min\ (c_i x\ ,\ 1 \le i \le p) \\ &s.t\ x^t oA = b, 0 \le x \le 1 \end{aligned} \tag{2}$$

In the problem (2), p is the existing objective function which should be simultaneously optimized on the feasible domain of this problem. The values of the objectives are not equal and have been set to an order according to their priority. That means the objective function $f_i(x) = c_i x$ is the prior of the other objective functions $f_j(x) = c_j x$ for$j = i+1, \ldots, p$. Suppose X is the feasible domain of the system (2). Now we may have the following definition.

Definition 1.1. The point $\overline{x} \in X$ is called an effective solution or Pareto optimal solution if and only if there is no $x \in X$ such that $c_k x \le c_k \overline{x}$ where $k = 1, 2, \ldots, p$ whereas $c_k x < c_k \overline{x}$ for some k. Otherwise $\overline{x}$ is called a *non*-effective point. The set of all Pareto optimal solutions is called Pareto optimal set.

The Pareto optimal set is being obtained in the majority of papers that the solution of (2) has been discussed so far. The Pareto optimal set is a suitable choice when there is no utility function or the objective function at hand. If the decision maker has an utility function or the objective function has a priority level in the decision maker's point of view (conditions we are faced to in this paper), then it will not be suitable to find the Pareto optimal set. Because, apart from the way the Pareto optimal set is obtained, it would cost a lot. When the objective function has a priority level, an optimal solution for this problem is an optimal solution with preemptive priority. Although such a solution for this problem is an optimal solution as well [57, 31]and we may find it by the Pareto optimal set, it is costly and one should find the optimal solution with preemptive priority using a direct method. The optimal solution with preemptive priority is defined as follows.

Definition 1.2. Suppose X is a feasible domain of problem (2) and c_i are coefficients of the objective function. Optimization with preemptive priority means finding the minimum $c_1^t x$ on$X$, and then finding the minimum $c_2^t x$ on alternative optimal solutions of the first problem and in the next step finding the minimum $c_3^t x$ on alternative optimal solutions of the second problem and so on. The solution is found using this procedure is called an optimal solution with preemptive priority.

In the problem (2) the objective functions are linear and the restrictions are fuzzy relation equations [77, 32], therefore the feasible domain is not convex. Therefore the classic programming methods such as simplex cannot be used. In this paper, the problem is solved as well as some special structures related to its feasible region is found. We also provide some reduction methods.

The concept of fuzzy relation equations first was considered by Sanchez [59]. He showed that the Max - min fuzzy relation equations have one maximum solution and several minimal solutions. Bourke and Fisher [2] have extended the study on fuzzy relation system of equations using max - product composition. The result shows that the complete set of solutions may be determined by the maximum solution and some of the minimal solutions.

To solve the problem, its parts must be completely determined. Therefore, at first we introduce its parts .

Suppose $A = [a_{ij}]$ where $0 \leq a_{ij} \leq 1$ is an $(n \times m)$-matrix, $b = (b_1, b_2, \ldots, b_m)$ where $0 \leq b_j \leq 1$ is an m-vector and $C^t = (c_1, c_2, \ldots, c_p)$, where $c_k = (c_{k1}, c_{k2}, \ldots, c_{kn})$. In fact $C = [c_{ki}]$ is a $(p \times n)$ matrix, where $c_{ki} \in Z$ and $I = \{1, 2, \ldots, m\}, J = \{1, 2, \ldots, m\}$and $k = 1, 2, \ldots, p$.

A system of fuzzy relation equation is considered which is defined by A and b as follows:

$$x^t oA = b \tag{3}$$

Where "o"is defined as follows:

$$max_{1 \leq i \leq n}(x_i + a_{ij} - x_i a_{ij}) = b_j, for\ j = 1, 2, \ldots, m \tag{4}$$

and $x^t = (x_1, x_2, \ldots, x_n) is\ an\ n - vector\ with\ 0 \leq x_i \leq 1.$

In fact the problem we aim to solve is as follows:

$$min\{c_1x, c_2x, \ldots, c_px\}|s.t\ x^t oA = b,\ and\ 0 \leq\ x_i \leq 1 \tag{5}$$

One should note that the solution sets obtained by max - min operator [2, 22, 4] and max - product operator are similar, that is, the non - empty set of solutions may be determined completely by a unique max solution and a finite set of minimal solutions. Since the solution set is not convex, the classic linear programming such as simplex algorithm and interior point cannot be used [15]. In this paper we consider the solution set of (5) and solve it.

2. Characterization of Feasible Domain

The solution set of (2) is as follows:

$$X(A, b) = \{x^t = (x_1, x_2, \ldots, x_n) \mid x_i \in [0, 1],\ x^t oA = b\} \tag{6}$$

For x^1 , $x^2 \in X$ it is said $x^1 \leq x^2$ if and only if $x_i^2 \leq x_i^2$ for all $i \in I$. therefor , " $\leq$ " is a partial order relation on X.

Definition 2.1. The solution $\hat{x} \in X(A,b)$ is the maximum solution if for all $x \in X(A,b)$, we have $x \leq \hat{x}$. Similarly, $\check{x} \in X(A,b)$ is a minimal solution if for all $x \in X(A,b)$, $x \leq \check{x}$ implies $x = \check{x}$

It can easily be found out from (3) that $x \in [0,1]^n$ is a feasible solution if and only if the following conditions hold,

$$x_i(1-a_{ij}) \leq b_j - a_{ij} \ , \ \forall j \in J \ , \ \forall \ i \in I$$
$$x_i(1-a_{ij}) \leq b_j - a_{ij} \ , \ \forall j \in J \ , \ \exists \ i \in I \tag{7}$$

It is clear that $b_j - a_{ij} \geq 0$ is a necessary condition for the feasible space to be nonempty. If the problem (5) is nonempty, $\hat{x}$ is obtained as follows [2].

$$\hat{x_i} = min_{j=1,2,\ldots,n}(a_{ij} \otimes b_j) \ for \ all \ i \in I \tag{8}$$

Where

$$a_{ij} \otimes b_j = \begin{cases} 1 & \text{if} \quad b_j = 1 \\ \frac{b_j - a_{ij}}{1-a_{ij}} & otherwise \end{cases} \tag{9}$$

Definition 2.2. Suppose $x \in X(A,b)$. The variable x_i is called binding if for some $j_0 \in J$, we have $x_i(1-a_{ij_0}) + a_{ij_0} = b_{ij_0}$.

Convention . From now on the set $\check{X}$(A,b) is the finite set of all minimal elements of X(A,b) and it is briefly shown by $\check{X}$

Theorem 2.3.

$$X(A,b) \subseteq \bigcup_{\check{x} \in \check{X}} \left\{ x | \check{x} \leq x \leq \hat{x} \right\}$$

Proof. See [2, 22, 4]. □

3. The Impact of the Cost Vector

As we have mentioned above, in order to find the optimal solutions of (5) with preemptive priority, one should find the optimal solution of the following problem

$$min \ c_1 x, \quad s.t \quad x^t \circ A = b, \ 0 \leq x_i \leq 1 \tag{10}$$

Before solving this problem, we should mention that there have been several papers dealing with the solving problems like (10), but the point is that the main goal of those papers is to find the optimal solution of the problem and not finding the set of all optimal solutions (alternative optimal solutions). Our goal is to find alternative optimal solutions; hence we should use an algorithm to provide this capability. The applied method here is similar to that of Fang used in [38], but there are also some minor differences. One of those we should mention is changing the problem to a $0-1$ programming problem by fang. Finding the optimal solutions of the second problem helps us to find the optimal solutions of the main problem, but here the optimal solutions of the problem are found directly. This easy to understand this algoritm, also the expenses of using this algorithm is far less. Now the algorithm is being introduced.

Due to the fact that the coefficients of the objective function (10) can be positive, negative or zero, the index of the variable by the nature of these coefficients is partitioned.

$$I^+ = \{i \in I | c_{1i} > 0\} \ , \ I^- = \{i \in I | c_{1i} < 0\} \ , \ I^- = \{i \in I | c_{1i} = 0\} \qquad (11)$$

Lemma 3.1. *In any optimal solution x^* of the problem (10) we have $x_i^* = \hat{x}_i$, where $i \in I^-$*

Proof. Suppose the optimal solution x^* is given. Suppose there exists $i' \in I^-$ such that $x_{i'}^* \neq \hat{x}_{i'}$. Vector $x'$is defined as follows:

$$x_i' = \begin{cases} x_i^* & i \in I - \{i'\} \\ \hat{x}_{i'} & i = i' \end{cases} \qquad (12)$$

since $x^* \in X$, $x^* \leq \hat{x}$. Therefore, $x' \leq \hat{x}$. By definition of x', we have$x^* \leq x'$. Therefore by Theorem 2.3 we get $x' \in X$. On the other hand, due to the fact that $\hat{x}_{i'} = x'_{i'} > x^*_{i'}$ we have

$$c_1 x^* = \sum_{i \in I - \{i'\}} c_{1i} x_i^* + c_{1i'} x_{i'}^* > \sum_{i \in I - \{i'\}} c_{1i} x_i' + c_{1i'} x_{i'}' = c_1 x'$$

and this contradict the fact that x^* is an optimal solution.

It should be noted that, if $\exists i_0, j_0$ such that $a_{i_0 j_0} = 1$, then $\forall x \in X(A, b)$

$$max_{1 \leq i \leq n}(x_i + a_{ij_0} - x_i a_{ij_0}) = 1$$

Now, let $b_{j_0} < 1$, then the feasible domain will be empty. Also, let $b_{j_0} = 1$, then the$j_0 th$ constraint is redundant and so it can be eliminated. Therefore,
$\forall i, j : a_{ij} < 1$. □

Lemma 3.2. *Suppose $x \in X$ is a solution. If $x_i < \hat{x}_i$ then x_i cannot be a binding variable.*

Proof. Let $x \in X$ be a solution, and $x_i < \hat{x}_i$ for some i and be a binding variable of $j\ th$ constraint. Therefore, we have $x_i(1 - a_{ij}) + a_{ij} = b_{ij}$. Now, since $\hat{x}_i < \hat{x}_i$, $\hat{x}_i(1 - a_{ij}) + a_{ij} > b_{ij}$ and this is a contradiction. □

Lemma 3.3. *In any optimal solution x^* of the problem (10) the value of x^* for $i \in I^+$ is neither zero nor equal to $\hat{x}_i$.*

Proof. Suppose in an optimal solution x^* the value of $x_{i'}^*$ for $i' \in I^+$ is neither zero nor it is equal to $\hat{x}_i$. This means $0 < x_{i'}^* < \hat{x}_{i'}$. Then, by Lemma 3.2, $x_{i'}^*$ is not a binding variable. Now we define the vector x' as follows:

$$x_i' = \begin{cases} x_i^* & i \in I - \{i'\} \\ 0 & i = i' \end{cases} \qquad (13)$$

By the definition of x', it is clear that the components of x' and x^* are identical except that of index i', for which we have $x'_{i'} = 0$. The only difference is that the i' th element is now zero. Thus, we have the following

$$\forall i, j \quad x'_i(1 - a_{ij}) + a_{ij} \leq b_j$$

Let us consider an arbitrary constraint $x'_i(1-a_{ij})+a_{ij} = b_j$ for$j \in J$. Since,$x^* \in X$, then $\exists i_0 \in I$ such that $x^*_{i_0}$ is the binding variable of the jth constraint. Hence by the definition of x', x'_{i_0} is the binding variable of the jth constraint. So, we have the following

$$\forall j \ \ \exists i \ \ s.t. \ \ x'_i(1 - a_{ij}) + a_{ij} = b_j$$

Therefore, the condition on (7) is satisfied. So that $x' \in X$.

On the other hand, we have

$$c_1x^* = \sum_{i \in I-\{i'\}} c_{1i}x^*_i + c_{1i'}x^*_{i'} = \sum_{i \in I-\{i'\}} c_{1i}x'_i + c_{1i'}x^*_{i'} > \sum_{i \in I-\{i'\}} c_{1i}x'_i + 0 = c_1x'$$

Thus, $c_1x^* > c_1x'$, this contradict the fact that x^* is an optimal solution and so the proof is completed. □

By Lemma 3.1 and Lemma 3.3, if the objective function of the problem (10) has no zero cost coefficients, then it has a finite number of solutions. The reason is that by Lemma 3.1 in any optimal solution, the value of x^*_i for $i \in I^-$ is a constant and it is equal to $\hat{x}_i$ and the components corresponding to I^+ are either $\hat{x}_i$ or zero. The total number of such solutions is finite. It is clear that in case the objective function has no zero cost vectors, one should look for the optimal solutions of the problem (10) among these solutions. Therefore, we consider two cases in order to find the optimal solution with preemptive priority.

(1) The vector c_1 has no zero cost coefficients.
(2) The vector c_1 has some zero cost coefficients.

4. Looking for the Optimal Solution of the Problem (5) in Case 1

For the current situation we should find all optimal solutions of the problem (10). By Lemma 3.1 and Lemma 3.3, for any optimal solution x^* we may see that for $i \in I^-$, we have $x_i = \hat{x}_i$, which is a constant and for $i \in I^+$, we have $x^*_i = 0$ or $x^*_i = \hat{x}_i$.

It is clear that in order to find the optimal solutions of the problem (10) we should look for such solutions among the above solutions. In other words, we should find the solutions of the following optimal problem.

$$\begin{aligned} & min \ \ c_1^t x \\ & s.t \ \ \ x^t \circ A = b \\ & x_i = \hat{x}_i, \ \ i \in I^- \\ & x_i = 0 \ \ or \ \ \hat{x}_i, i \in I^+ \end{aligned} \tag{14}$$

The problem (14) is similar to a $0-1$ programming. Therefore, the Branch and Bound algorithm [70] may be used to solve it. We can start from any constraint and the branch to the next level. Branching to the next level corresponds to adding another constraint which in turn tightens the feasible domain. The worst case scenario of the $B\&B$ method is that all possible solutions are enumerated. However, after evaluating some nods, some branches can be generally eliminated. For example, if solution of the partial branch which we are evaluating is larger than the best solution obtained from a complete branch, then evaluating the current branch can be stopped. The problem can be reduced before any attempt to solve problem (14), so the problem would be solved much easier.

5. Reducing the Problem

Consider the following set of indices

$$j^- = \{j \in J | \hat{x}_i(1 - a_{ij}) + a_{ij} = b_j,\ i \in I^-\} \tag{15}$$

The set J^- is the set corresponding to the constraints where $\hat{x}_i$ for $i \in I^-$ are the binding variables for them. By the fact that the value of $i \in I^-$ in any optimal solution is constant and satisfies $x_i = \hat{x}_i$, we may omit these variables. Hence, we may modify the parts of the problem as follows:

$$I' = I - I^-, J' = J - J^- \tag{16}$$

We omit ith row for $i \in I^-$ and jth column for $j \in J^-$ of matrix A. The obtained matrix by this procedure is called A'. We also omit the jth component for $j \in J^-$ of the vector b to get vector b'. Now, the index set is defined as follows:

$$I_j = \{i \in I | \hat{x}_i(1 - a_{ij}) + a_{ij} = b_j\} \tag{17}$$

Here, I_j is the index set corresponding to $\hat{x}_i$s which are binding in the jth restriction.

Lemma 5.1. *if* $|I_j| = 1$, *then for* $i \in I_j, \forall \hat{x} \in \check{X}$ *we have* $\hat{x}_i = \check{x}_i = \frac{b_j - a_{ij}}{1 - a_{ij}}$

Proof. See [14] for details. □

Consider the jth constraint where$j \in J^-$. If $j \in I_j$ has only one element, then the only one of $\hat{x}_i$s can change the jth constraint to equality, and by Lemma 5.1 for each $x \in X, x_j$ is constant and it is equal to x_i.

Consider the following sets of indices

$$\bar{J} = \{j \in J' \big| \ |I_j| = 1\}\ ,\ \bar{I} = \{i \in I_j \big| j \in \bar{J}\} \tag{18}$$

Once more we omit ith row for $i \in \bar{I}$ and jth column for $j \in \bar{J}$ of matrix A' to get the matrixA''. We also omit jth elements for $j \in \bar{J}$ of b' to get b''. We set

$$I'' = I' - \bar{I}, J'' = j' - \bar{J} \tag{19}$$

Convention: Suppose $|I''| = k$ is the cardinal of I'' and

$$I'' = \{i_t | t = 1, 2, \ldots, k\} \tag{20}$$

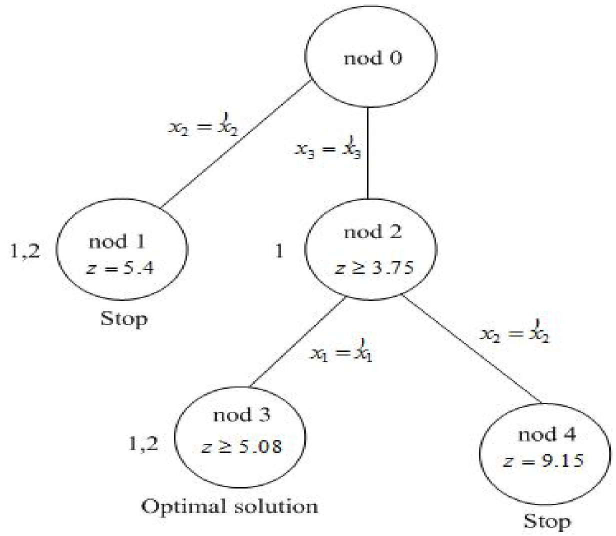

FIGURE 1. Diagram of Example 1

From now on i_t means one of the indices in I''.

Finally, after using the reduction method, the problem (10) changes to

$$\begin{array}{ll} min & c_1''x'' \\ s.t & x''^t \circ A'' = b'' \\ & x_{i_t} = 0 \ or \ \hat{x}_{i_t} \ for \ t = 1, 2, \ldots, k \end{array} \tag{21}$$

where $c_1'' = (c_{1i_1}, c_{1i_2}, \ldots, c_{1i_k})$, $x''^t = (x_{1i_1}, x_{1i_2}, \ldots, x_{1i_k})$

Example 5.2. Solve the following problem

$$\begin{array}{ll} min & c^t x \\ s.t & x^t \circ A = b \\ & 0 \leq x_i \leq 1 \end{array} \tag{22}$$

where $,max_{1 \leq i \leq n}(x_i + a_{ij} - x_i a_{ij}) = b_j$ and

$$A = \begin{pmatrix} 0.6 & 0.7 \\ 0.75 & 0.5 \\ 0.6 & 0.15 \end{pmatrix} \quad , \quad b = (0.9, 0.8) \quad , \quad c = (4, 9, 5)$$

Solution. We have $\hat{x}^t = (\frac{1}{3}, \frac{3}{5}, \frac{3}{4})$ Hence $I_1 = \{2, 3\}$, $I_2 = \{1, 2\}$

Starting from the first constraint, the algorithm is being started from the nod 0. The set I_1 has two elements, so the nod 0 is branched into two branches. Set $x_2 = \hat{x}_2$ and $x_3 = \hat{x}_3$ on the first and second branches respectively. As soon as a new nod is received, we write the number of restrictions changed already to equalities, just next to equalities, just next to the nod. Therefore we write 1, 2 next to the first nod. The process stops here because both constraints of the problem changed to equality. We write 1 next to the second nod. We branch again on the nod 2. On the branch ending nod 3 we write $x_1 = \hat{x}_1$ and on the branch ending nod 4 we write $x_2 = \hat{x}_2$. As both of the constraints of problem changed to equality the process stops on both of these nods. Since the nod 3 is smaller than both nods 1 and 4, the direction towards 3 is the optimal solution of the problem. Therefore, we have $x^* = (\hat{x}_1, 0, \hat{x}_3)$

Remark 1. In general if there are several nods to carry on, it is usually better to use the nod which has the smallest value of z.

We are now ready to find the optimal solution with preemptive priority of the problem (4). For this purpose, the optimal solutions of problem (10) are found and let X_1 be the set of these optimal solutions. Now, c_2x of the set X_1 should be optimized. This is very simple. For each $x \in X_1$ the value of c_2x is calculated and x with the smallest value for its objective functions is chosen and it as an alternative optimal solution is selected and the set X_2 is assumed to be the set of these optimal solutions. By the same manner we proceed until to reach the far end.

Case 2: In this case, some components of c_1 have zero cost coefficients. This case is in fact a generalization of the case 1.

Lemma 5.3. *Let the optimal solution x^* of the problem (10) be given. Suppose some of its components corresponding to I^0 be zero. Define the vector x' as follows:*

$$x'_i = \begin{cases} \lambda_i & i \in I^0 \ , \ x^*_i = 0 \\ x^*_i & otherwise \end{cases}$$

Then, for each $\lambda_i \in [0, \hat{x}_i]$, x' is the optimal solution of the problem (10).

Proof. Since, $x^* \in X$, $x^* \leq \hat{x}$. By the definition of x' it is clear that $x^* \leq x' \leq \hat{x}$.

Therefore, by Theorem 2.3, $x' \in X$. We have also

$$c_1x^* = \sum_{i \in I \setminus I^0} c_{1i}x^*_i + 0 = \sum_{i \in I \setminus I^0} c_{1i}x'_i + 0 = c_1x'$$

Which means x' is the optimal solution of (10). □

Lemma 5.4. *Suppose an optimal solution x^* is given. $i \; E \; I^0$ be such that $x^*_i < \hat{x}_i$. Therefore, the vector x' which is defined as follows*

$$x'_i = \begin{cases} 0 & i \in I^0 \ , \ x^*_i < \hat{x}_i \\ x^*_i & otherwise \end{cases}$$

is the optimal solution of the problem (10).

Proof. Since $x^* \in X$, $x^* < \hat{x}$ and by the definition of x', we have$x' < x^*$. Therefore, the following is obtained

$$x'_i(1 - a_{ij}) + a_{ij} \leq b_j \quad , \quad \forall i, j$$

Choose the jth constraint arbitrarily. Since $x^* \in X$, at least one of the components of $x^*(e.g. i_0)$ changes the jth constraint to equality. By Lemma 3.2, and by definition of x' we have $x'_{i_0} = x^*_{i_0}$. That is x'_{i_0} is the binding variable of jth constraint. By the fact that the jth constraint has been chosen arbitrarily, the following relation holds:

$$\forall j \ , \ \exists i : \ x_i'(1 - a_{ij}) + a_{ij} = b_i$$

Thus, the condition (7) is satisfied and $x' \in X$. Also,

$$c_1 x^* = \sum_{i \in I \setminus I^0} c_{1i} x_i^* + \sum_{i \in I^0} c_{1i} x_i' = \sum_{i \in I \setminus I^0} c_{1i} x_i' + 0$$

$$= \sum_{i \in I \setminus I^0} c_{1i} x_i' + 0 = \sum_{i \in I^0} c_{1i} x_i' + \sum_{i \in I \setminus I^0} c_{1i} x_i' = c_1 x'$$

Hence, x' is the optimal solution of the problem (10). Now, using lemmas we have already proved the optimal solution of the problem (10) in case 2 can be found. But before doing so the reduction method is used. □

6. An Algorithm for Finding the Optimal Solutions of the Problem (5) with Preemptive Priority

•First, we solve the problem (21) by branch and bound algorithm.

•As soon as we have all the optimal solutions of the problem (21), we check if any component of any of these solutions which corresponds to I^0 is equal zero. If this is the case, then we proceed as follows:

Suppose $x^{(k)}$ is the optimal solution whit this property. Set

$$x^{(k)\prime} = \begin{cases} \lambda_i & i \in I'' \cap I^0 \ , \ x_i^{(k)} = 0 \\ x^{(k)} & otherwise \end{cases}$$

It can easily be seen that for all $\lambda_i \in [0, x_i], x^{(k)}$ is an optimal solution of the problem (21). This can be seen using Lemma 5.3. Now replace $x^{(k)}$ by $x^{(k)\prime}$. We have two sets of optimal solutions at the end. These two sets are parametric optimal solutions (λ_i is the parameter) and nonparametric optimal solutions. Let X_1 be the set of all optimal solutions of the problem (21).

•Now, what we need to do is minimizing the objective function $c_2 x$ on the set X_1 and finding out the optimal solutions.

$$\min_{x \in X_1} c_2 x \tag{23}$$

Therefore, the following steps are summarized:

(1). To Know that the set X_1 has two types of solution the set of parametric solutions is denoted by $S$and the set of nonparametric solutions byS'.

(2). For all $x^* \in S$ the values of λ_i is set in a way that the objective function $c_2 x$ is minimized. For this purpose, if the coefficients corresponding to λ_i in the objective function are greater than zero, $\lambda_i = 0$ is set, and if the coefficients corresponding to λ_i in the objective function is less than zero, then $\lambda_i = \hat{x}_i$ is set. In the case, that the coefficients corresponding to λ_i in the objective function are zero no there is value to assigned λ_i and it remains parametric. Replacing x^* by this new solution

the set S is now modified.
(3). Now, looking for solutions $X_1 = S \cup S'$ such that they minimize the objective function $c_2 x$ and X_2 is the set of optimal solutions of the problem below

$$\min_{x \in X_1} c_2 x \tag{24}$$

(4). Now, the minimum value of $c_3 x$ on $X - 2$ should be found and in fact the following problem might be solved

$$\min_{x \in X_2} c_3 x \tag{25}$$

The method is quite similar to that we have already used to solve the problem (23). Using this method until solving the following problem is carried on:

$$\min_{x \in X_{p-1}} c_p x \tag{26}$$

The optimal solution of the problem (26) is the optimal solution with preemptive priority of that problem.

Example 6.1. Find the optimal solutions with preemptive priority of the following problem:

$$\begin{array}{ll} min & c^t x \\ s.t. & x^t \circ A = b \\ & 0 \leq x_i \leq 1 \end{array}$$

Where, $max_{1 \leq i \leq n}(x_i + a_{ij} - x_i a_{ij}) = b_j$, $b = (\frac{9}{10}, \frac{4}{5}, \frac{4}{5}, \frac{7}{8}, \frac{7}{9})$

$c_1 = (-3, 0, -5, 4, 9, 0, 5), c_2 = (4, 0, 0, 2, 5, 0, -6), c_3 = (0, 7, 4, 9, -2, 5, 0)$

$$A = \begin{bmatrix} \frac{3}{5} & \frac{1}{4} & 0 & \frac{3}{4} & \frac{5}{9} \\ \frac{5}{6} & \frac{2}{5} & \frac{3}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{3} & \frac{5}{6} & \frac{1}{2} \\ \frac{3}{5} & \frac{7}{10} & \frac{7}{10} & \frac{1}{5} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & \frac{2}{9} \\ \frac{3}{10} & \frac{1}{2} & \frac{13}{20} & 0 & \frac{5}{9} \\ \frac{3}{5} & \frac{1}{10} & \frac{1}{5} & \frac{2}{5} & \frac{1}{10} \end{bmatrix}$$

Solution. The feasible domain of this problem is nonempty, so we have

$$\hat{x} = (\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{3}, \frac{3}{5}, \frac{3}{7}, \frac{3}{4})$$

$$I_1 = \{2, 5, 7\}, I_2 = \{4, 5\}, I_3 = \{4, 6, 7\}, I_4 = \{1, 3\}, I_5 = \{1\}$$

First, the reduction method is used to reduce the problem. Therefore, the fourth and the fifth columns and the first and the third rows must be omitted. The same

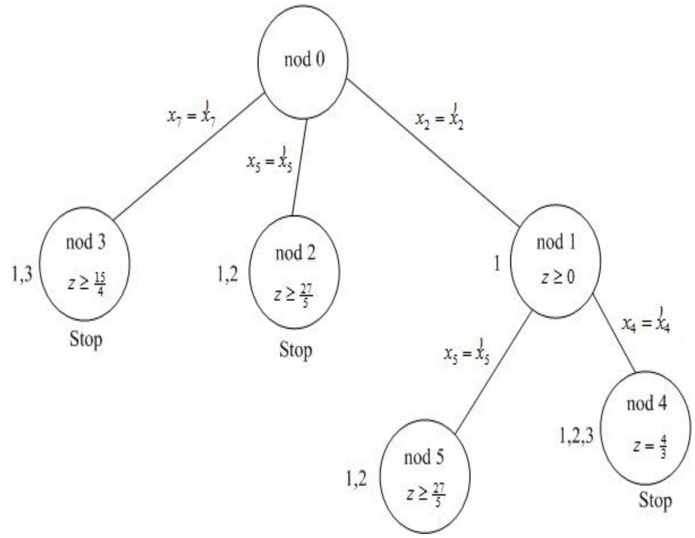

Figure 2. Diagram of Example 2

story happens to the fourth and the fifth components of the vector b. So, we have

$$A = \begin{bmatrix} \frac{5}{6} & \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{7}{10} & \frac{7}{10} \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{3} \\ \frac{3}{10} & \frac{1}{2} & \frac{13}{20} \\ \frac{3}{5} & \frac{1}{10} & \frac{1}{5} \end{bmatrix}$$

Now, the following problem is used:

$$\begin{aligned} min \quad & c^t x \\ s.t. \quad & x'^t \circ A' = b' \\ & x'_i = 0 \ or \ \hat{x}_i \end{aligned} \tag{27}$$

Branch and Bound algorithm is used to solve this problem.

Therefore, x'^* is the optimal solution of problem (27). In other words, $\hat{x} = (\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{3}, 0, 0, 0)$ is the optimal solution of the following problem:

$$min \ c_1 x \quad s.t. \quad x \circ A = b, \ x_i = 0 \ or \ \hat{x}_i \tag{28}$$

Because, the sixth component of x^* is zero, for each $\lambda_6 \in [0, \frac{3}{7}]$ we have

$$\hat{x}(\lambda_6) = (\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{3}, 0, \lambda_6, 0)$$

This is also an optimal solution of problem (28). Hence, we have

$$X_1 = \{x^*(\lambda_6) | \lambda_6 \in [0, \frac{3}{7}]\}$$

Now, the optimal solution of the following problem should be found.

$$\min_{x \in X_1} \quad c_2 x$$

According to the algorithm defined already, the set X_1 is being partitioned into two separated sets, S and S' where

$$S = \{x^*(\lambda_6)\} \ , \ S' = \phi$$

The suitable parameter λ_6 to minimize the objective function c_2 is chosen. Since the corresponding component to λ_6, of c_2 is zero, we heave λ_6 as it is. Therefore, $X_2 = \{x^*(\lambda_6)\}$. So, this set is being partitioned into two sets as we did for X_1 in the previous step, that is

$$S = \{x^*(\lambda_6)\} \ , \ S' = \phi$$

The parameter λ_6 has to be chosen in a way that minimize the objective function $c_3 x$. Since, the component of c_3 corresponding to λ_6 is negative, we set $\lambda_6 = \hat{x}_6$. This means that $\lambda_6 = \frac{3}{7}$. Now $x^*(\lambda_6)$ is being replaced by this new solution. Hence, we have

$$S = \big\{(\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{3}, 0, \frac{3}{7}, 0)\big\} \quad , \quad S' = \phi$$

As a result, vector $x^{*t} = (\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{3}, 0, \frac{3}{7}, 0)$ is the optimal solution with the preemptive priority of the problem of under consideration.

By inspiration of the Example 2 from [17]an applicable example is provided as follows:

An Applicable Example:

A doctor has diagnosed three tumors after visiting a patient. The doctor is going to use five different medicines. These medicines have to be used one after the other in a specific period and order. Based on prior experience there is a rate of efficiency for each medicine which are shown in the matrix A. In fact the efficiency rate of the ith medicine on the jth tumor is shown by a_{ij}. If each medicine is prescribed in usual dosage it might affect the tumor less than doctor's expectation. Therefore, the doctor is permitted to prescribe each medicine up to two times of usual dosage. Any extra dosage of the ith medicine is shown by x_i which is in fact the percentage of extra usage of the ith medicine. On one hand, using extra medicine may affect the tumor and make it smaller and on the other hand it might weaken the patient. So, some other tumors may be created. If $\mu_j(i)$ is the efficiency rate of the ith medicine on the jth tumor after the extra dosage prescription, we may have

$$\mu_j(i) = (a_{ij} + x_j) - a_{ij} x_j.$$

According to prior experience, the doctor guesses that the effect of these medicines on the jth tumor should be less than or equal b_j. Otherwise, it may cause some serious problems for the other important organs such as heart, liver and kidney. On the other hand, the effect of one of these medicines should be at least b_j for achieving a complete cure. Hence, we have

$$max_{i=1,2,\dots,5} \mu_j(i) = b_j \quad , \quad j = 1, 2, 3$$

Or equivalently

$$max_{i=1,2,\ldots,5}\{(a_{ij}+x_j)-a_{ij}x_j\}=b_j \quad , \quad j=1,2,3 \tag{29}$$

The values for x_js have to satisfy the following three conditions :
1. The side effect of the extra usage of medicines has to be minimized.
2. The price has to be minimized even in the case we use extra medicine.
3. Using extra medicine may increase the curing period. This also has to be minimized.

Hence, we have three objective functions which have to be minimized on the feasible domain of (29). These functions have different priorities. The first function has the first priority among the others. The second function has the second priority. The matrices A and b the coefficients of the objective functions are shown below.

$$b=(\frac{9}{10},\frac{4}{5},\frac{4}{5})$$

$$c_1=(0,4,9,0,5), c_2=(1,2,5,1,5), c_3=(7,9,0,5,0)$$

$$A=\begin{bmatrix} \frac{5}{6} & \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{7}{10} & \frac{7}{10} \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{3} \\ \frac{3}{10} & \frac{1}{2} & \frac{13}{20} \\ \frac{3}{5} & \frac{1}{10} & \frac{1}{5} \end{bmatrix}$$

Therefore, we have the following problem

$$\begin{gathered} min(c_1x, c_2x, c_3x) \\ x^t \circ A = b \\ 0 \leq x \leq 1 \end{gathered}$$

where, $max_{1\leq i\leq n}(a_{ij}+x_i-x_ia_{ij})=b_j$.

The feasible domain of the above problem is the same as the feasible domain in the previous example. It is an optimization problem with preemptive priority. These kinds of problems can be solved using the method provided in this paper. In fact solving these kinds of problems is the main motivation for writing this paper.

7. **Conclusion**

The algorithm presented in this paper may be used to find the optimal solution with preemptive priority of a fuzzy optimization problem. solving a multi- objective optimization problem in this method, part of the solution is finding the alternative optimal solutions of the problem (10). The problem will be divided into two cases by the nature of the coefficients of the first objective function ($i.e. c_1 x$). The first case is when the objective function of the problem (10) has no zero coefficients and the second case is when the objective function of the problem (10) has some zero coefficients. In the first case any optimal solution may be obtained by the branch and bound algorithm, but in the second case the problem has some alternative optimal solutions which may be determined parametrically. In fact in the second case the problem (10) has two sets of solutions. One of them is parametric and the other one is nonparametric. We include these two sets in a set called X_1 and then we optimize the objective function c_2x on this set. As soon as this is done we include the optimized solutions in a set called X_2 and optimize the objective function on this set and so on. Before solveing the problem we use a reduction method to reduce the problem. Therefore, the problem may be solved much easier. Finally, two examples were provided to clarify the situation.

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