

PRICING STOCKS BY USING FUZZY DIVIDEND DISCOUNT MODELS

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ABSTRACT. Although the classical dividend discount model (DDM) is a well-known and widely used model in evaluating the intrinsic price of common stock, the practical pattern of dividends, required rate of return or growth rate of dividend do not generally coincide with any of the model's assumptions. It is just the opportunity to develop a fuzzy logic system that takes these vague parameters into account. This paper extends the classical DDMs to more realistic fuzzy pricing models in which the inherent imprecise information will be fuzzified as triangular fuzzy numbers, and introduces a novel λ -signed distance method to defuzzify these fuzzy parameters without considering the membership functions. Through the conscientious mathematical derivation, the fuzzy dividend discount models (FDDMs) proposed in this paper can be regarded as one more explicit extension of the classical (crisp) DDMs, so that stockholders can use it to make a specific analysis and insight into the intrinsic value of stock.

1. Introduction

Stock represents an ownership interest in a corporation, but for the typical stockholders, the purpose of holding the stock can be simply characterized by two features: one is to expect to receive the continuous dividend payments by holding the stock; another is to earn a capital gain by selling the stock at a certain higher price. Generally speaking, common stocks are evaluated on the basis of either a corporation's expected profitability or its expected dividend distributions to stockholders. Since the analysis of common stocks can be based on the analysis of its dividend payments, if stockholders can estimate the future cash-flow streams of dividend payments by matching up an appropriate discount rate, then the theoretical value of the stock can be easily determined. The evaluating method is commonly known as the "dividend discount models (DDMs)" [1, 6, 15].

However, such the classical DDMs do not take the imprecision into account that may be inherent in these parameters used in it. These parameters include future cash-flow streams of dividend distributions, required rate of return, and growth rate of dividend which are always subjectively quantified and determined by the listed companies. Thus, previous study has applied such a model to derive the upper and lower bounds of fluctuation of stock price, but the empirical result showed that the real stock price obviously went beyond the scope [16]. In addition, most

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of the existing financial literature has incorporated uncertainty in the similar field of investing decision making based on intuitive method or probabilistic approach [1, 6, 11], and in their studies the uncertain information is usually estimated in advance by using educated guesses or other statistical technique to obtain them. Admittedly, although probability theory is still the one important tool that the majority of financial analysts rely on resolving market uncertainty, it requires the fulfillment of some assumptions for probabilistic distribution, especially needs more complete information. Therefore, in the applicability of classical DDMs, there exists the defect of depending too much on the intuition or requiring the fulfillment of some assumptions for probabilistic distribution. On the other hand, in practice, it is difficult for stockholder to precisely evaluate the intrinsic stock price by using classical (crisp) DDMs. In order to predict the future cash-flow streams of dividend payments, the growth rate of dividend and the required rate of return for some specific stocks, it is necessary to make several assumptions for the dividend policy and further to estimate these above-mentioned parameters in DDMs by linking the growth rate of dividend (g) with the other financial data such as ROE, P/E ratio or pay-out instead to directly estimate g [10]. Similarly, the required rate of return (k) is usually derived from using the CAPM framework by considering the risk factor, market expected rate of return, and the expectations about the risk-free rate [15]. Since these financial data are uncertain and imprecise, these magnitudes should be more suitable to be treated by fuzzy logic [18].

Some developments in fuzzy-financial mathematics have been applied to deal with financial issues. Buckley [3] studied the fuzzy extension of the mathematics of finance to concentrate on the compound interest law. Then, Li Calzi [13] investigated a possible general setting by considering both compact fuzzy intervals and invertible fuzzy intervals for the fuzzy financial mathematics. Kuchta [8] also generalized the fuzzy equivalents for evaluating investment projects. Especially in the complicated and uncertain stock market, stockholders usually try to find some credible ways in order to predict stock prices accurately, but have less than successful results. For the reason, several researchers have proposed a series of excellent studies about the fuzzy techniques in order to assess the stock market and to predict stock prices correctly. For example, Dourra and Siy [4] applied fuzzy information technologies to investments through technical analysis, and used it to examine various companies to achieve a substantial investment return. Kuo et al. [9] used genetic algorithm based on fuzzy neural network to measure the quantitative and qualitative effects on the stock market. Wang [19] proposed a fuzzy grey prediction system to analyze stock data and to predict stock price, and then he employed fuzzy rough set system to predict the stronger rules of stock price and achieved a higher accuracy [20]. In these previous fuzzy valuation models, a variety of defuzzification methods have been proposed to deal with defuzzification of fuzzy interval such as mean-of-maxima (MOM) method, center-of-area (COA) method, and fuzzy mean (FM) method, etc. [22], but none of these methods dominate over the others for the defuzzification of fuzzy intervals. In [22], they consider the fuzzification of fuzzy interval \tilde{Q} , and formulate the membership function of fuzzy interval \tilde{Q} . Afterward they use MOM method, COA method, and FM method

based on finding the membership function of fuzzy interval \tilde{Q} to defuzzify the fuzzy interval \tilde{Q} , respectively. However, Zhao and Govind's [22] methods (e.g., MOM, COA and FM) are not available to defuzzify the fuzzy sets when the membership functions of fuzzy models are really difficult to be found out.

To sum up, the fuzzy reasoning seems to be very effective in an imprecise environment and it is necessary for stockholders to use a proper method to evaluate the stock prices. Therefore, the purpose of this paper is to extend the classical (crisp) stock pricing model that can be fed with a fuzzy system to evaluate a more realistic stock price. Meanwhile, a novel defuzzication method, λ -signed distance method, without considering membership function is also introduced to improve the classical DDMs.

2. Preliminaries

Before presenting the fuzzy DDMs (FDDMs) based on the λ -signed distance method, some definitions are provided in advance with some relevant operations as follows.

Definition 2.1. By [7], a fuzzy set $[a, b; \alpha]$, $a < b$ defined on $\mathfrak{R} = (-\infty, \infty)$, which has the following membership function, is called a level α fuzzy interval

$$\mu_{[a,b;\alpha]}(x) = \begin{cases} \alpha, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. By [14], fuzzy point \tilde{D} is a fuzzy set defined on \mathfrak{R} with the following membership function:

$$\mu_{\tilde{D}}(x) = \begin{cases} 1, & x = D, \\ 0, & x \neq D. \end{cases}$$

Definition 2.3. By [12, 17], the triangular fuzzy number \tilde{D} is defined on \mathfrak{R} with a membership function as follows, and denoted by $\tilde{D} = (D - \Delta_1, D, D + \Delta_2)$, where $0 < \Delta_1 < D$, $0 < \Delta_2$

$$\mu_{\tilde{D}}(x) = \begin{cases} \frac{x - D - \Delta_1}{\Delta_1}, & D - \Delta_1 \leq x \leq D, \\ \frac{D + \Delta_2 - x}{\Delta_2}, & D \leq x \leq D + \Delta_2, \\ 0, & \text{otherwise.} \end{cases}$$

Additionally, let the family of all triangular fuzzy numbers be denoted by $F_N = \{(a, b, c) \mid \forall a < b < c, a, b, c \in \mathfrak{R}\}$. In particular, the fuzzy point $\tilde{D} = (D, D, D)$ could be regarded as the degenerated case of fuzzy number $\tilde{D} = (D - \Delta_1, D, D + \Delta_2)$ if $\Delta_1, \Delta_2 \rightarrow 0$.

Let F_s be the family of fuzzy sets defined on \mathfrak{R} , for each fuzzy set $\tilde{D} \in F_s$, the α -cut of \tilde{D} is denoted by $D(\alpha) = \{x \mid \mu_{\tilde{D}}(x) \geq \alpha\} = [\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$, where

$\alpha \in [0, 1]$,

$$\begin{aligned}\tilde{D}_L(\alpha) &= D - (1 - \alpha)\Delta_1 = (D - \Delta_1) + \alpha\Delta_1 > 0; \\ \tilde{D}_U(\alpha) &= D + (1 - \alpha)\Delta_2 > 0,\end{aligned}\tag{1}$$

and both $\tilde{D}_L(0)$ and $\tilde{D}(0)$ are finite values. For each $\alpha \in [0, 1]$, the real numbers $\tilde{D}_L(\alpha)$, $\tilde{D}_U(\alpha)$ separately represent the left and right end points of $D(\alpha)$ and satisfy the conditions that both of $\tilde{D}_L(\alpha)$, $\tilde{D}_U(\alpha)$ exist in $\alpha \in [0, 1]$ and are continuous over $[0, 1]$. consider a fuzz set $\tilde{D} \in F_s$, by decomposition theory, we have

$$\tilde{D} = \bigcup_{0 \leq \alpha \leq 1} \alpha D(\alpha), \quad \text{or} \quad \mu_{\tilde{D}}(x) = \bigvee_{0 \leq \alpha \leq 1} (\alpha \wedge I_{D(\alpha)}(x)),$$

where $I_{D(\alpha)}$ is the characteristic function of $D(\alpha)$. By Definition 2.1, if $x \in D(\alpha)$, then $\alpha D(\alpha)(x) = \alpha = \mu_{[D_L(\alpha), D_U(\alpha); \alpha]}(x)$, and if $x \notin D(\alpha)$, then $\alpha D(\alpha)(x) = 0 = \mu_{[D_L(\alpha), D_U(\alpha); \alpha]}(x)$. Hence, we have

$$\tilde{D} = \bigcup_{0 \leq \alpha \leq 1} \alpha D(\alpha) = \bigcup_{0 \leq \alpha \leq 1} [\tilde{D}_L(\alpha), \tilde{D}_U(\alpha); \alpha].\tag{2}$$

By extending the definition of signed distance initiated by [21], we have the Definition 2.4.

Definition 2.4. (a) For each fuzzy set and each $\tilde{D} \in F_s$ and each $\lambda \in (0, 1)$, by Definition 2.2, the λ -signed distance from \tilde{D} to $\tilde{0}$ is defined by $d(\tilde{D}, \tilde{0}; \lambda) = \int_0^1 [\lambda \tilde{D}_L(\alpha) + (1 - \lambda) \tilde{D}_U(\alpha)] d\alpha$, where $\tilde{0}$ is a fuzzy point.

(b) When \tilde{D} is a fuzzy point, for all $\alpha \in [0, 1]$, $\tilde{D}_L(\alpha) = \tilde{D}_U(\alpha) = D$, then by Definition 2.4 (a) yields: $d(\tilde{D}, \tilde{0}; \lambda) = D$, for all $\lambda \in (0, 1)$.

Definition 2.5. Let $\tilde{A}, \tilde{B}, \tilde{C} \in F_s$, and for each $\lambda \in (0, 1)$, define the metric ρ_λ by $\rho_\lambda(\tilde{A}, \tilde{B}) = |d(\tilde{A}, \tilde{0}; \lambda) - d(\tilde{B}, \tilde{0}; \lambda)|$, where ρ_λ satisfies the following three metric axioms: (a) $\rho_\lambda(\tilde{A}, \tilde{B}) = 0$; iff $\tilde{A} \approx \tilde{B}$; (b) $\rho_\lambda(\tilde{A}, \tilde{B}) \geq 0$; (c) $\rho_\lambda(\tilde{A}, \tilde{B}) + \rho_\lambda(\tilde{B}, \tilde{C}) \geq \rho_\lambda(\tilde{A}, \tilde{C})$.

Definition 2.6. BY THEOREM OF DINI. [2], let $\tilde{A}_n (n = 1, 2, 3, \dots)$, $\tilde{A} \in F_s$ for any fixed $\lambda \in (0, 1)$, for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$,

(a) when $n > N(\varepsilon)$, such that $\rho_\lambda(\tilde{A}_n, \tilde{A}) < \varepsilon$, then the sequence $\{\tilde{A}_n; n = 1, 2, \dots\}$ of the fuzzy sets is said to converge to the fuzzy set \tilde{A} with respect to ρ_λ , which is denoted by $\lim_{n \rightarrow \infty} \tilde{A}_n = \tilde{A}$ with respect to ρ_λ .

(b) when $n > N(\varepsilon)$, such that $\rho_\lambda(\tilde{A}_n, \tilde{A}) < \varepsilon$, for all $\lambda \in (0, 1)$, which is denoted by $\lim_{n \rightarrow \infty} \tilde{A}_n = \tilde{A}$.

Definition 2.7. By [7], the triangular fuzzy numbers \tilde{A} and \tilde{B} are defined on \mathfrak{R} , for each $\alpha \in [0, 1]$, the operations of the level α fuzzy intervals are:

$$\begin{aligned}\text{(a)} \quad A(\alpha)(+)B(\alpha) &= [A_L(\alpha), A_U(\alpha)](+) [B_L(\alpha), B_U(\alpha)] \\ &= [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)];\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad A(\alpha)(-)B(\alpha) &= [A_L(\alpha), A_U(\alpha)](-)[B_L(\alpha), B_U(\alpha)] \\
&= [A_L(\alpha) - B_U(\alpha), A_U(\alpha) - B_L(\alpha)]; \\
\text{(c)} \quad A(\alpha)(\times)B(\alpha) &= [A_L(\alpha), A_U(\alpha)](\times)[B_L(\alpha), B_U(\alpha)] \\
&= [A_L(\alpha) \times B_L(\alpha), A_U(\alpha) \times B_U(\alpha)]; \\
\text{(d)} \quad A(\alpha)(\div)B(\alpha) &= [A_L(\alpha), A_U(\alpha)](\div)[B_L(\alpha), B_U(\alpha)] \\
&= [A_L(\alpha)/B_U(\alpha), A_U(\alpha)/B_L(\alpha)], \quad B_L(\alpha), B_U(\alpha) > 0.
\end{aligned}$$

3. Crisp DDMs

Crisp DDMs are analyzed in [1, 15]. The generalized DDM given in (3) describes that the present value of stock can be treated as a stream of discount dividends for stockholders.

$$P_t^* = \frac{D_{t+1}}{(1+k_{t+1})} + \frac{D_{t+2}}{(1+k_{t+1})(1+k_{t+2})} + \cdots + \frac{D_{t+T}}{(1+k_{t+1})(1+k_{t+2}) \cdots (1+k_{t+T})} + \cdots, \quad (3)$$

or shorter

$$P_t^* = \sum_{i=1}^{\infty} \left[\prod_{j=1}^i \frac{1}{1+k_{t+j}} \right] D_{t+i}, \quad t = 0, 1, 2, \dots, \quad (4)$$

where

P_t^* : the intrinsic, expected present value at time t of an infinite stream of dividends that is based on the stockholder's estimate for some specific stock.

D_{t+i} : dividends, the stockholder expects to receive at the end of time $t+i$ depending on the stock policy of an issuing company. All future dividends are expected values, so the estimation of D_t may differ among stockholders.

k_{t+j} : required rate of return at time $t+j$, the stockholder considers the returns available on other investments.

Equation (4) is a generalized stock pricing model in the sense that the time pattern of D_{t+i} should be a non-negative number, and D_{t+i} may be rising, falling, or constant, or it may be fluctuating randomly. Generally speaking, the projected stream of dividends follows a systematic pattern and can be developed the following two simplified versions.

Case 1: Stock values with zero dividend growth

In case of the stockholder's required rate of return in each period is equalized as $k = k_1 = k_2 = \cdots = k_T = \cdots$, and the dividends remain constant, namely $D = D_1 = D_2 = \cdots = D_T = \cdots$. Then (3) can be simplified as a zero dividend growth model as (5). This model is usually used to evaluate the intrinsic value of *preferred stock*.

$$P_0^* = \frac{D}{(1+k)^1} + \frac{D}{(1+k)^2} + \cdots = \lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D}{(1+k)^t} = \frac{D}{k}. \quad (5)$$

Case 2: Stock values with constant dividend growth

In case of a constant growth company has paid the last dividend D_0 (i.e., the dividend at time 0 for the following sequence of dividends), then its follow-up cash flows of dividend payments at time t can be analogized as $D_t = D_0(1+g)^t$, where $t = 1, 2, 3, \dots$ and g is the constant expected growth rate in each period.

In the same way, supposing the stockholder's required rate of return in each period is equalized, then equation (3) may be rewritten as a constant dividend growth model that is so-called the Gordon model [5] given by

$$\begin{aligned} P_0^* &= \frac{D_1}{(1+k)^1} + \frac{D_1(1+g)}{(1+k)^2} + \cdots + \frac{D_1(1+g)^{T-1}}{(1+k)^T} + \cdots \\ &= \lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D_0(1+g)^t}{(1+k)^t} = \frac{D_0(1+g)}{k-g}, \end{aligned} \quad (6)$$

where k should be a positive real number, g may be positive, zero, or negative. Note that the necessary condition for (6) is $k > g$. In addition, if $g = 0$, then (6) can be simplified as (5) (i.e., Case 1 may be regarded as the special case of Case 2), and if $g \leq -1$, then the company will pay no dividend.

4. Pricing Stocks with FDDMs

In general, although the listed companies may declare their dividend policy in advance and anticipate to continuously provide dividends D_t in the future t ($t = 1, 2, 3, \dots$), the real distributions of dividends in each year may vary with the shifting economy. Thus, the real dividends received by the stockholders would not be exactly equal to the former expectations. Therefore, for estimating the uncertain dividends, it will more fit in with real situation to predict the future dividend payout by estimating a possible dividend payment interval such as $[D - \Delta_1, D + \Delta_2]$ instead of point estimation, where $0 < \Delta_1 < D$, $0 < \Delta_2$. Since the dividend interval $[D - \Delta_1, D + \Delta_2]$ is not a certain value, the stockholder must estimate a value from the interval. If a stockholder takes an estimation of dividend by \hat{D} as the same as the former expected D , then the estimation error will be "0". Based on this, we can connect the statistical concept of confidence level with membership grade of fuzzy theory and hereby set the maximum confidence level as "1". Thus, if the estimation of dividend determined during the interval $[D - \Delta_1, D]$ or $(D, D + \Delta_2]$ is more far away from prior expectation of D , then the confidence level would be smaller. Namely, the right and left end points ($D - \Delta_1$ and $D + \Delta_2$) have the same minimum confidence levels set as "0".

According to the fuzzy theory, if the membership grade of \tilde{D} at D is closer to "1", the larger distance from the right or left end point ($D - \Delta_1$ and $D + \Delta_2$) to D would be. Contrarily, if the membership grade is closer to "0", the smaller distance from the end points would be. Based on this, there are similar characters between membership grade and confidence level. Thus it is reasonable to substitute membership grade for confidence level. Namely, the triangular fuzzy number \tilde{D} should be corresponding to the closed interval $[D - \Delta_1, D + \Delta_2]$.

Two FDDMs derived from the crisp cases are shown in the following subsections.

4.1. Fuzzification of Dividend and Required Rate of Return in Crisp Case

3.1. Corresponding to the dividend interval $[D - \Delta_1, D + \Delta_2]$, the fuzzy dividend can be given as the triangular fuzzy number, $\tilde{D} = (D - \Delta_1, D, D + \Delta_2)$, where the variations Δ_1, Δ_2 may be appropriately determined by the subjective judgment of

stockholders satisfying $0 < \Delta_1 < D$, $0 < \Delta_2$. According to Definitions 2.3 and 2.4, we have the estimate value of dividend in the fuzzy sense based on λ -signed distance as

$$D_\lambda^* \equiv d(\tilde{D}, \tilde{0}; \lambda) = D + \frac{1}{2}((1 - \lambda)\Delta_2 - \lambda\Delta_1) (> 0), \quad (7)$$

where $D_\lambda^* > 0$ and $D_\lambda^* \in [D - \Delta_1, D + \Delta_2]$.

Similarly, it is also difficult for the typical stockholders to precisely determine the required rate of return for some stock by taking a certain value k , but relatively easy to determine that by taking a possible interval as $[k - \Delta_3, k + \Delta_4]$. In such a closed interval, $0 < \Delta_3 < k$, $0 < \Delta_4$, and Δ_3, Δ_4 can be appropriately determined by stockholders. Corresponding to the interval $[k - \Delta_3, k + \Delta_4]$, the fuzzy required rate of return can be given as the triangular fuzzy number, $\tilde{k} = (k - \Delta_3, k, k + \Delta_4)$, where $0 < \Delta_3 < k$, $0 < \Delta_4$. According to Definitions 2.3 and 2.4, we have the estimate value of required rate of return in the fuzzy sense based on λ -signed distance as

$$k_\lambda^* \equiv d(\tilde{k}, \tilde{0}; \lambda) = k + \frac{1}{2}((1 - \lambda)\Delta_4 - \lambda\Delta_3) (> 0), \quad (8)$$

where $k_\lambda^* > 0$ and $k_\lambda^* \in [k - \Delta_3, k + \Delta_4]$.

By Definition 2.7, let $\sum_{t=1}^n \tilde{A}_t$ be represented as $\tilde{A}_1(+)\tilde{A}_2(+)\cdots(+)\tilde{A}_n$, using triangular fuzzy numbers \tilde{D} and \tilde{k} to fuzzify $\sum_{t=1}^n \frac{D}{(1+k)^t}$ in (5), we have

$$\sum_{t=1}^n (\tilde{D}(\div)\tilde{M}^t), \quad (9)$$

where $\tilde{M} = \tilde{1}(+)\tilde{k}$, $\tilde{M}^t = \tilde{M}(\times)\tilde{M}(\times)\cdots(\times)\tilde{M}$ (t times), $\tilde{1}$ is a fuzzy point at 1, then the left and the right end points of α -cut of \tilde{M}^t are

$$\begin{aligned} \tilde{M}_L^t(\alpha) &= [1 + k - (1 - \alpha)\Delta_3]^t (> 0); \\ \tilde{M}_U^t(\alpha) &= [1 + k + (1 - \alpha)\Delta_4]^t (> 0). \end{aligned} \quad (10)$$

By (10), we can derive the right and the left end points of α -cut of $\tilde{D}(\div)\tilde{M}^t$ respectively shown as below.

$$(\tilde{D}(\div)\tilde{M}^t)_L(\alpha) = \frac{D - (1 - \alpha)\Delta_1}{(1 + k + (1 - \alpha)\Delta_4)^t} (> 0), \quad t = 1, 2, 3, \dots; \quad (11)$$

$$(\tilde{D}(\div)\tilde{M}^t)_U(\alpha) = \frac{D + (1 - \alpha)\Delta_1}{(1 + k - (1 - \alpha)\Delta_3)^t} (> 0), \quad t = 1, 2, 3, \dots \quad (12)$$

Substituting (11) and (12) into (9), and using decomposition theory to yield

$$\sum_{t=1}^n (\tilde{D}(\div)\tilde{M}^t) = \bigcup_{0 \leq \alpha \leq 1} \left[\sum_{t=1}^n \frac{D - (1 - \alpha)\Delta_1}{(1 + k + (1 - \alpha)\Delta_4)^t}, \sum_{t=1}^n \frac{D + (1 - \alpha)\Delta_1}{(1 + k - (1 - \alpha)\Delta_3)^t}; \alpha \right]. \quad (13)$$

Theorem 4.1. *By fuzzifying D and k in (5), the fuzzy stock value \tilde{P}_0^* with zero growth dividend corresponding to (5) can be represented as*

$$\tilde{P}_0^* = \lim_{n \rightarrow \infty} \sum_{t=1}^n (\tilde{D}(\div)\tilde{M}^t) = \tilde{D}(\div)\tilde{k}.$$

Proof. Let $\widetilde{M} = \widetilde{1}(+) \widetilde{k} = (1+k-\Delta_3, 1+k, 1+k+\Delta_4)$, we have

$$0 < \frac{1}{\widetilde{M}_L(\alpha)} = \frac{1}{1+k-(1-\alpha)\Delta_3} < 1,$$

and

$$0 < \frac{1}{\widetilde{M}_U(\alpha)} = \frac{1}{1+k+(1-\alpha)\Delta_4} < 1, \quad \forall \alpha \in [0, 1].$$

Incorporating (13) and employing the crisp formula of infinite geometric progression, for each $\alpha \in [0, 1]$, we have

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{[1+k+(1-\alpha)\Delta_4]^t} = \frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4} = (\widetilde{D}(\div) \widetilde{k})_L(\alpha); \quad (14)$$

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{[1+k-(1-\alpha)\Delta_3]^t} = \frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3} = (\widetilde{D}(\div) \widetilde{k})_U(\alpha). \quad (15)$$

Because $D-(1-\alpha)\Delta_1 > 0$, $D+(1-\alpha)\Delta_2 > 0$ and $1+k-(1-\alpha)\Delta_3 > 0$, $1+k+(1-\alpha)\Delta_4 > 0$, both $\sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{[1+k+(1-\alpha)\Delta_4]^t}$ and $\sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{[1+k-(1-\alpha)\Delta_3]^t}$ are increasing sequences with respect to n and continuous over $0 \leq \alpha \leq 1$ respectively, $\frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4}$ and $\frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3}$ are also continuous functions with respect to α respectively. Hence (14), (15) converge in the interval $0 \leq \alpha \leq 1$ respectively. According to the THEOREM of DINI. [2], (14) and (15) are uniformly convergent in the interval $0 \leq \alpha \leq 1$ respectively. Therefore, for each $\alpha \in [0, 1]$ and any choice of $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ without respect to α such that (16) and (17) are hold if $n > N(\varepsilon)$:

$$\left| \sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{[1+k+(1-\alpha)\Delta_4]^t} - \frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4} \right| < \varepsilon; \quad (16)$$

$$\left| \sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{[1+k-(1-\alpha)\Delta_3]^t} - \frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3} \right| < \varepsilon. \quad (17)$$

By (16),(17) and Definition 2.5, for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ (as above), such that the following conduct is hold for all $\lambda \in (0, 1)$ and $n > N(\varepsilon)$.

$$\begin{aligned} & \rho_\lambda \left(\sum_{t=1}^n (\widetilde{D}(\div) \widetilde{M}^t), \widetilde{D}(\div) \widetilde{k} \right) \\ &= \left| \int_0^1 \left[\lambda \sum_{t=1}^n (\widetilde{D}(\div) \widetilde{M}^t)_L(\alpha) + (1-\lambda) \sum_{t=1}^n (\widetilde{D}(\div) \widetilde{M}^t)_U(\alpha) - \lambda (\widetilde{D}(\div) \widetilde{k})_L(\alpha) \right. \right. \\ & \quad \left. \left. - (1-\lambda) (\widetilde{D}(\div) \widetilde{k})_U(\alpha) \right] d\alpha \right| \\ &\leq \lambda \int_0^1 \left| \sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{(1+k+(1-\alpha)\Delta_4)^t} - \frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4} \right| d\alpha \end{aligned}$$

$$+(1-\lambda) \int_0^1 \left| \sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{(1+k-(1-\alpha)\Delta_3)^t} - \frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3} \right| d\alpha < \varepsilon$$

Next, by Definition 2.6 (b), $\lim_{n \rightarrow \infty} \sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t) = \tilde{D}(\div) \tilde{k}$, hence Theorem 4.1 is proved. \square

Theorem 4.2. *In Theorem 4.1, for each $\lambda \in (0, 1)$, using λ -signed distance method to defuzzify the fuzzy sets $\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t)$, $n = 1, 2, 3, \dots$ and $\tilde{D}(\div) \tilde{k}$, then the estimate of the intrinsic stock value $\hat{P}_{0\lambda}^*$ of zero dividend growth model in the fuzzy sense corresponding to (3) can be derived as*

$$\begin{aligned} \hat{P}_{0\lambda}^* &= \lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t), \tilde{0}; \lambda\right) = d(\tilde{D}(\div) \tilde{k}, \tilde{0}; \lambda) \\ &= \left(\frac{D}{k}\right) \left[\lambda \left(\frac{k}{\Delta_4} + \frac{\Delta_1 k^2}{\Delta_4^2 D}\right) \ln\left(1 + \frac{\Delta_4}{k}\right) - \lambda \frac{\Delta_1 k}{\Delta_4 D}\right. \\ &\quad \left. - (1-\lambda) \left(\frac{k}{\Delta_3} + \frac{\Delta_2 k^2}{\Delta_3^2 D}\right) \ln\left(1 - \frac{\Delta_3}{k}\right) - (1-\lambda) \frac{\Delta_2 k}{\Delta_3 D}\right]. \end{aligned}$$

Proof. Using the λ -signed distance method to defuzzify $\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t)$ and $\tilde{D}(\div) \tilde{k}$ respectively, then by (13) we have

$$\begin{aligned} &d\left(\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t), \tilde{0}; \lambda\right) \\ &= \int_0^1 \left[\lambda \sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{(1+k+(1-\alpha)\Delta_4)^t} + (1-\lambda) \sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{(1+k-(1-\alpha)\Delta_3)^t} \right] d\alpha \end{aligned} \quad (18)$$

and

$$d(\tilde{D}(\div) \tilde{k}, \tilde{0}; \lambda) = \int_0^1 \left[\lambda \frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4} + (1-\lambda) \frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3} \right] d\alpha. \quad (19)$$

Employing the theorem of integration (the order of limit and integration can be exchanged) [2] and by (18),(19) to yield:

$$\begin{aligned} \hat{P}_{0\lambda}^* &= \lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t), \tilde{0}; \lambda\right) \\ &= \lambda \int_0^1 \left(\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D-(1-\alpha)\Delta_1}{(1+k+(1-\alpha)\Delta_4)^t} \right) d\alpha \\ &\quad + (1-\lambda) \int_0^1 \left(\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D+(1-\alpha)\Delta_2}{(1+k-(1-\alpha)\Delta_3)^t} \right) d\alpha \\ &= \lambda \int_0^1 \left(\frac{D-(1-\alpha)\Delta_1}{k+(1-\alpha)\Delta_4} \right) d\alpha + (1-\lambda) \int_0^1 \left(\frac{D+(1-\alpha)\Delta_2}{k-(1-\alpha)\Delta_3} \right) d\alpha \\ &= \left(\frac{D}{k}\right) \left[\lambda \left(\frac{k}{\Delta_4} + \frac{\Delta_1 k^2}{\Delta_4^2 D}\right) \ln\left(1 + \frac{\Delta_4}{k}\right) - \lambda \frac{\Delta_1 k}{\Delta_4 D}\right. \\ &\quad \left. - (1-\lambda) \left(\frac{k}{\Delta_3} + \frac{\Delta_2 k^2}{\Delta_3^2 D}\right) \ln\left(1 - \frac{\Delta_3}{k}\right) - (1-\lambda) \frac{\Delta_2 k}{\Delta_3 D}\right] \\ &= d(\tilde{D}(\div) \tilde{k}, \tilde{0}; \lambda), \end{aligned}$$

hence Theorem 4.2 is proved. \square

4.2. Fuzzification of Expected Growth Rate and Required Rate of Return in Crisp Case 3.2.

In practice, even if the company maintain a dividend policy with steady growth and the last dividend (D_0) is known, it is still difficult for the typical stockholders to precisely estimate growth rate and required rate of return by taking a certain value, but relatively easier to determine that by taking a possible interval. Hence, D_0 , g , and k can be simultaneously fuzzified as triangular fuzzy numbers $\tilde{D}_0 = (D_0 - \theta_1, D_0, D_0 + \theta_2)$, $\tilde{g} = (g - \theta_3, g, g + \theta_4)$, and $\tilde{k} = (k - \theta_5, k, k + \theta_6)$ corresponding to the intervals $[D_0 - \theta_1, D_0 + \theta_2]$, $[g - \theta_3, g + \theta_4]$, and $[k - \theta_5, k + \theta_6]$, respectively, where $\theta_3, \theta_4, \theta_5, \theta_6$ can be appropriately determined by stockholders to satisfy

$$0 < \theta_3 < g, \quad 0 < \theta_4 < k - g; \quad 0 < \theta_5 < k - g - \theta_4, \quad 0 < \theta_6 < k - g - \theta_3. \quad (20)$$

In this fuzzy case, since the last dividend (D_0) has been paid already by the company, \tilde{D}_0 should be regarded as a fuzzy point at D_0 (i.e., $\theta_1 = \theta_2 = 0$).

For each $\lambda \in (0, 1)$, using λ -signed distance method to defuzzify \tilde{D}_0 , \tilde{g} and \tilde{k} , we have the estimate values of dividend, growth rate and required rate of return in the fuzzy sense based on λ -signed distance as

$$D_{0\lambda}^* \equiv d(\tilde{D}_0, \tilde{0}; \lambda) = D_0; \quad (21)$$

$$g_\lambda^* \equiv d(\tilde{g}, \tilde{0}; \lambda) = g + \frac{1}{2}[(1 - \lambda)\theta_4 - \lambda\theta_3]; \quad (22)$$

$$k_\lambda^* \equiv d(\tilde{k}, \tilde{0}; \lambda) = k + \frac{1}{2}[(1 - \lambda)\theta_6 - \lambda\theta_5]. \quad (23)$$

Let $\tilde{R} = \tilde{1}(+) \tilde{g}$, then using the same operations in (9) and (10) to yield:

$$\begin{aligned} & \sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t) \\ &= \bigcup_{0 \leq \alpha \leq 1} \left[\sum_{t=1}^n \frac{D_0[(1 + g - (1 - \alpha)\theta_3)]^t}{[1 + k + (1 - \alpha)\theta_6]^t}, \sum_{t=1}^n \frac{D_0[(1 + g + (1 - \alpha)\theta_4)]^t}{[1 + k - (1 - \alpha)\theta_5]^t}; \alpha \right]. \end{aligned} \quad (24)$$

Theorem 4.3. *By fuzzifying D_0, g, k in (6), the fuzzy intrinsic stock value \tilde{P}_0^* of constant dividend growth dividend model corresponding to (6) is*

$$\tilde{P}_0^* = \lim_{n \rightarrow \infty} \sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t) = \tilde{D}_0(\times) (\tilde{1}(+) \tilde{g})(\div) (\tilde{k}(-) \tilde{g}).$$

Proof. Similar to the proof procedure of Theorem 4.1, Theorem 4.3 can be proved. \square

For simplifying the formulation, let $K(a, b, u, v) = \int_0^1 \frac{a + (1 - \alpha)b}{u + (1 - \alpha)v} d\alpha$, and $\delta = 1 - \alpha$, $m = u + v\delta$, then

$$K(a, b, u, v) = \int_0^1 \frac{a + b\delta}{u + v\delta} d\delta = \frac{1}{v^2} \int_u^{u+v} \frac{av - bu + bm}{m} dm$$

$$= \frac{1}{v^2}[(av - bu) \ln \frac{u+v}{u} + bv]. \quad (25)$$

Theorem 4.4. *In Theorem 4.3, for each $\lambda \in (0, 1)$, employing λ -signed distance method to defuzzify $\sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t)$, $t = 1, 2, 3, \dots$ and $\tilde{D}_0(\times) (\tilde{I}(+) \tilde{g})(\div) (\tilde{k}(-) \tilde{g})$, then the estimate of intrinsic stock value $\hat{P}_{0\lambda}^*$ of constant dividend growth model in the fuzzy sense can be derived as*

$$\begin{aligned} \hat{P}_{0\lambda}^* &= \lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t), \tilde{0}; \lambda\right) = d(\tilde{D}_0(\times) (\tilde{I}(+) \tilde{g})(\div) \tilde{k}(-) \tilde{g}, \tilde{0}; \lambda) \\ &= \lambda K(D_0(1+g), -D_0\theta_3, k-g, \theta_3+\theta_6) \\ &\quad + (1-\lambda)K(D_0(1+g), D_0(1+g), D_0\theta_4, k-g, -\theta_4-\theta_5). \end{aligned}$$

Proof. Similar to the proof procedure of Theorem 4.2 and by the operation of (25), we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t), \tilde{0}; \lambda\right) \\ &= \int_0^1 \left[\lambda \lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D_0[1+g-(1-\alpha)\theta_3]^t}{[1+k+(1-\alpha)\theta_6]^t} \right] d\alpha \\ &\quad + \int_0^1 \left[(1-\lambda) \lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D_0[1+g+(1-\alpha)\theta_4]^t}{[1+k-(1-\alpha)\theta_5]^t} \right] d\alpha \\ &= \int_0^1 \frac{\lambda D_0[1+g-(1-\alpha)\theta_3]}{k+(1-\alpha)\theta_6-[g-(1-\alpha)\theta_3]} d\alpha + \int_0^1 \frac{(1-\lambda)D_0[1+g+(1-\alpha)\theta_4]}{k-(1-\alpha)\theta_5-[g+(1-\alpha)\theta_4]} d\alpha \\ &= \lambda K(D_0(1+g), -D_0\theta_3, k-g, \theta_3+\theta_6) + (1-\lambda)K(D_0(1+g), D_0\theta_4, k-g, -\theta_4-\theta_5) \\ &= d(\tilde{D}(\times) (\tilde{I}(+) \tilde{g})(\div) \tilde{k}(-) \tilde{g}, \tilde{0}; \lambda), \end{aligned}$$

hence Theorem 4.4 is proved. \square

5. Numerical Examples

Think of the stock-related information declared by a listed company as the following ranges: $\tilde{D} \in (0, 10)$, $\tilde{k} \in (0, 10\%)$ and $\tilde{g} \in (0, 10\%)$, the stockholders can employ the proposed Theorems 1 to 4 to evaluate the intrinsic stock value under a certain λ level that can be analyzed graphically as a surface. The further discussion with respect to the results of using different λ levels is shown in Appendix A.3.

By Theorems 2 and 4, the stockholders can compute the intrinsic values of their own stocks in the fuzzy sense corresponding the crisp cases (5) and (6), and Δ_1 , Δ_2 , Δ_3 , Δ_4 , θ_3 , θ_4 , θ_5 , θ_6 can be appropriately determined by the stockholders' subjective judgment to satisfy the conditions of $0 < \Delta_1 < D$, $0 < \Delta_2$, $0 < \Delta_3 < k$, $0 < \Delta_4$, and (20). The comparison of the behavior of fuzzy variables (e.g., dividend, discount rate, and growth rate) in the numerically acceptable range with the different classical DDMs are showed as Figures 1, 3, 4, and 6. Note that the D_0 is a certain value (i.e., $\theta_1 = \theta_2 = 0$). Furthermore, in order to compare the relative errors between fuzzy cases ($\hat{P}_{0\lambda}^*$) and crisp cases (P_0^*), we set $D = 2$, $k = 6\%$ in crisp case (5) and $D_0 = 2$, $k = 6\%$, and $g = 3\%$ in crisp case (6). And then, the

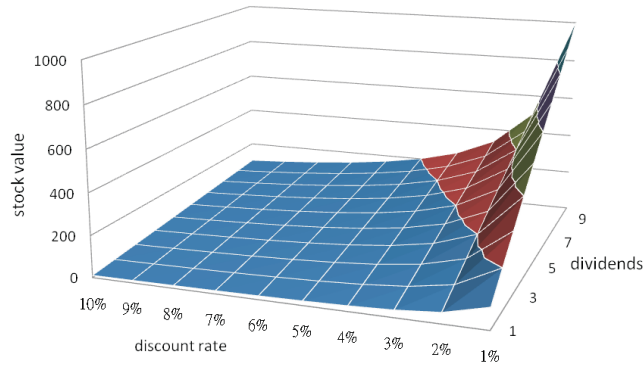


FIGURE 1. The Fuzzy Stock Values with Zero Dividend Growth

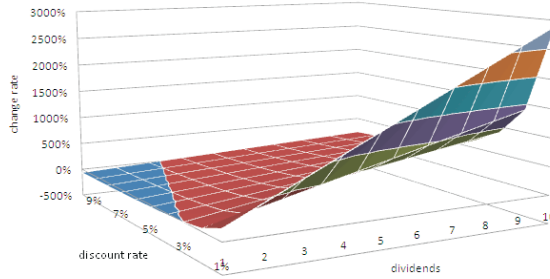


FIGURE 2. The Relative Error Between Fuzzy and Crisp Case with Zero Dividend Growth

change rate $r = \frac{\hat{P}_{0\lambda}^* - P_0^*}{P_0^*} \times 100\%$ of the relative errors between $\hat{P}_{0\lambda}^*$ and P_0^* during the ranges $\tilde{k} \in (0, 10\%)$ and $\tilde{g} \in (0, 10\%)$ can be shown as Figures 2 and 5.

According to Figures 2 and 5, stockholders can easily distinct the stock values in the fuzzy sense from the crisp stock values, and then judge the reasonableness of the current stock price. From Figures 3 and 6, the figures of “top-of-the-mountain“ view with a bird’s eye, they show that each α -cut should be related to one of the layers in the surface. More specifically, that can help stockholders visualize the room/width/space available at various α -cut plateaus.

6. Conclusion

Through the conscientious mathematical derivation, this paper has proposed two FDDMs with different assumptions for dividend growth. The relations among the proposed Theorems and crisp cases and the useful defuzzification method proposed

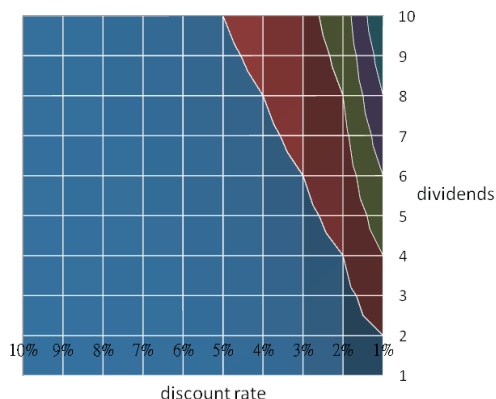


FIGURE 3. The Bird'S Eye View of Fuzzy Stock Value with Zero Dividend Growth

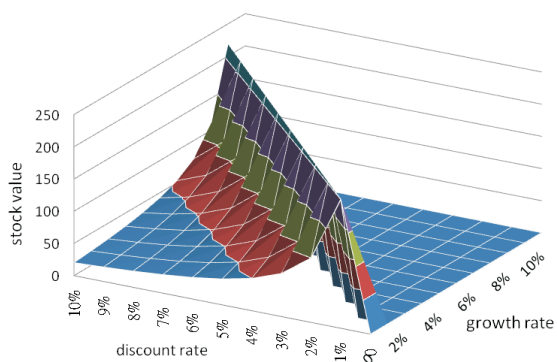


FIGURE 4. The Fuzzy Stock Values with Constant Dividend Growth

in FDDMs have been further explained in Appendix, and it may also be referred to Figures 1 to 7. In practice, since the parameters in DDMs, such like real distributions of dividends, required rate of return, growth rate of listed company in each year, may vary with the shifting economy, the real stock prices estimated by the stockholders might not be exactly equal to their former expectations. Therefore, the FDDMs will more fit in with real situation to capture these uncertain parameters by estimating a possible interval instead of point estimation. Also, we have emphasized the suitability of using fuzzy model and pointed out the defects of using probabilistic model which requires the fulfillment of some assumptions for probabilistic distribution, especially needs more complete information.

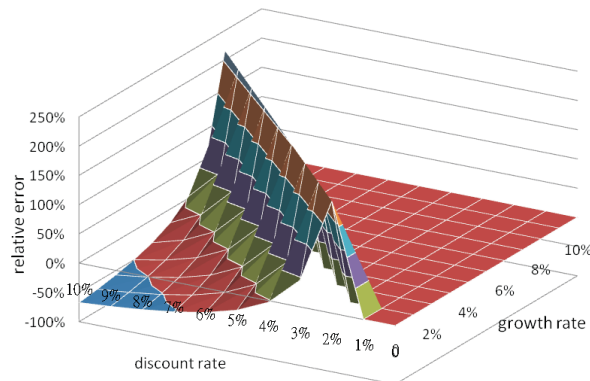


FIGURE 5. The Relative Error Between Fuzzy and Crisp Case with Constant Dividend Growth

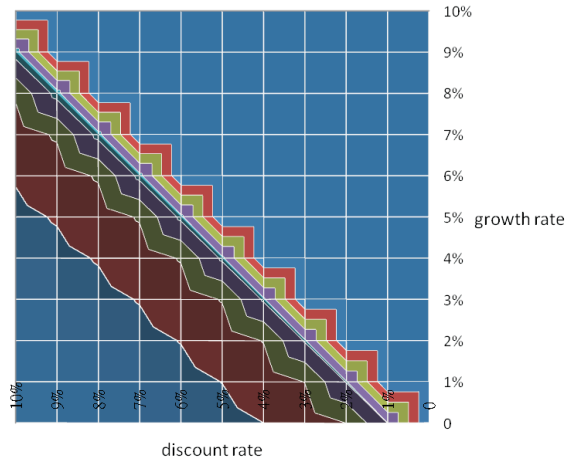


FIGURE 6. The Bird'S Eye Review of Fuzzy Stock Values with Constant Dividend Growth

Although some potential limitations exist in the FDDMs (e.g., the use of the FDDMs might be limited to evaluate some specific type of stocks, such as preferred stocks), the FDDMs can still help stockholders efficiently consider the imprecision in financial market and relatively provide more choices before executing their investing decisions since the estimated stock value shows only an approximate value. In other words, as stockholders are interested in applying the fuzzy logic to substitute their rough and arbitrary estimates with more appropriate fuzzy formulations in order to deal with the imprecise dividends, growth rate and required rate of return,

the conscientious FDDMs with λ -signed distance method may well be the feasible models than the traditional DDMs for evaluating the intrinsic stock price. In conclusion, this paper has successfully extended the classical DDMs by constructing an easy-to-understand and more realistic fuzzy stock pricing model without losing the essence of original crisp stock pricing model.

7. Appendix

7.1. A.1. The Relations Among Theorems 4.1, 4.2 and Crisp Case 3.1.

- (a) In Theorem 4.1, let $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0$, then $\tilde{D} = (D - \Delta_1, D, D + \Delta_2)$ and $\tilde{k} = (k - \Delta_3, k, k + \Delta_4)$ become fuzzy points, and (13) becomes

$$\begin{aligned} \sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t) &= \bigcup_{0 \leq \alpha \leq 1} \left[\sum_{t=1}^n \frac{D}{(1+k)^t}, \sum_{t=1}^n \frac{D}{(1+k)^t}; \alpha \right], \\ \tilde{D}(\div) \tilde{k} &= \bigcup_{0 \leq \alpha \leq 1} \left[\frac{D}{k}, \frac{D}{k}; \alpha \right]. \end{aligned}$$

Using the λ -signed distance method to defuzzify $\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t)$ and $\tilde{D}(\div) \tilde{k}$, we have

$$d\left(\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t), \tilde{0}; \lambda\right) = \sum_{t=1}^n \frac{D}{(1+k)^t}$$

and

$$d(\tilde{D}(\div) \tilde{k}, \tilde{0}; \lambda) = \frac{D}{k} \quad (\text{by Definition 2.4 (b)}).$$

Thus, for all $\lambda \in (0, 1)$, by Theorem 4.2, we have

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D}{(1+k)^t} = \lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}(\div) \tilde{M}^t), \tilde{0}; \lambda\right) = d(\tilde{D}(\div) \tilde{k}, \tilde{0}; \lambda) = \frac{D}{k},$$

that is the same as (5).

- (b) Let $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 \equiv \Delta$, by Theorem 4.2, for each $\lambda \in (0, 1)$, we have .

$$\hat{P}_{0\lambda}^* = \left(\frac{D}{k}\right) \left[\left(k + \frac{k^2}{D}\right) \frac{1}{\Delta} \left(\lambda \ln\left(1 + \frac{\Delta}{k}\right) - (1-\lambda) \ln\left(1 - \frac{\Delta}{k}\right) \right) - \frac{k}{D} \right].$$

Because

$$\begin{aligned} &\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[\lambda \ln\left(1 + \frac{\Delta}{k}\right) - (1-\lambda) \ln\left(1 - \frac{\Delta}{k}\right) \right] \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\frac{d\Delta}{d\Delta}} \left[\frac{\partial}{\partial \Delta} \lambda \ln\left(1 + \frac{\Delta}{k}\right) - \frac{\partial}{\partial \Delta} (1-\lambda) \ln\left(1 - \frac{\Delta}{k}\right) \right] \\ &= \lim_{\Delta \rightarrow 0} \left(\frac{\frac{\lambda}{k}}{1 + \frac{\Delta}{k}} + \frac{\frac{1-\lambda}{k}}{1 - \frac{\Delta}{k}} \right) = \frac{1}{k}, \end{aligned}$$

when $\Delta \rightarrow 0$, then $\hat{P}_{0\lambda}^* \rightarrow \left(\frac{D}{k}\right) \left[\left(k + \frac{k^2}{D}\right) \frac{1}{k} - \frac{k}{D} \right] = \frac{D}{k}$, that is the same as the P_0^* in (5) (cf. Figures 1 and 2.).

7.2. A.2. The Relations Among Theorems 4.3, 4.4 and crisp Case 3.2.

- (a) Similar to A.1. (a), in Theorem 4.4, let $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$, then we have

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \frac{D_0(1+g)^t}{(1+k)^t} = \frac{D_0(1+g)}{k-g},$$

that is the same as (6).

- (b) Let $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = \theta$, by Theorem 4.4, for each $\lambda \in (0, 1)$, we have

$$\begin{aligned} \hat{P}_{0\lambda}^* &= \lim_{n \rightarrow \infty} d\left(\sum_{t=1}^n (\tilde{D}_0(\times) \tilde{R}^t(\div) \tilde{M}^t), \tilde{0}; \lambda\right) \\ &= d(\tilde{D}_0(\times) (\tilde{1}(+) \tilde{g})(\div) (\tilde{k}(-) \tilde{g}), \tilde{0}; \lambda) \\ &= \lambda K(D_0(1+g) - (D_0+1+g)\theta, \theta^2, k-g, 2\theta) \\ &\quad + (1-\lambda)K(D_0(1+g), (D_0+1+g)\theta, \theta^2, k-g, -2\theta) \\ &= \frac{1}{8\theta} [4D_0(1+g) + 2(D_0+1+g)(k-g) + (k-g)^2] \\ &\quad \times [\lambda \ln(1 + \frac{2\theta}{k-g}) - (1-\lambda) \ln(1 - \frac{2\theta}{k-g})] \\ &\quad - \frac{1}{2}(D_0+1+g) - \frac{1}{4}(k-g) - \frac{1}{4}(1-2\lambda)\theta. \end{aligned}$$

Because

$$\begin{aligned} &\lim_{\theta \rightarrow 0} \frac{1}{\theta} [\lambda \ln(1 + \frac{2\theta}{k-g}) - (1-\lambda) \ln(1 - \frac{2\theta}{k-g})] \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{d\theta}{d\theta}} \left[\frac{\partial}{\partial \theta} \lambda \ln(1 + \frac{2\theta}{k-g}) - \frac{\partial}{\partial \theta} (1-\lambda) \ln(1 - \frac{2\theta}{k-g}) \right] \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\frac{2\lambda}{k-g}}{1 + \frac{2\theta}{k-g}} + \frac{\frac{2(1-\lambda)}{k-g}}{1 - \frac{2\theta}{k-g}} \right) = \frac{2}{k-g}, \end{aligned}$$

when $\theta \rightarrow 0$, then

$$\begin{aligned} \hat{P}_{0\lambda}^* &\rightarrow \frac{1}{8} [4D_0(1+g) + 2(D_0+1+g)(k-g) + (k-g)^2] \frac{2}{k-g} \\ &\quad - \frac{1}{2}(D_0+1+g) - \frac{1}{4}(k-g) = \frac{D_0(1+g)}{k-g} \end{aligned}$$

that is the same as the P_0^* in (6) (cf. Figures 3 and 4).

7.3. A.3. The Results Using λ -signed Distance Method to Defuzzify Fuzzy Dividend \tilde{D} , Fuzzy Required Rate of Return \tilde{k} , and Fuzzy Growth Rate \tilde{g} with Different λ Levels.

- (a) When $\lambda < 0.5$, $\lambda < 0.5 < (1-\lambda)$, for each $\alpha \in [0, 1]$, the point $\lambda \tilde{D}_L(\alpha) + (1-\lambda) \tilde{D}_U(\alpha)$ in $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$ will be closer to the right-end point $\tilde{D}_U(\alpha)$. Obviously, because $0 < \tilde{D}_L(\alpha) < \tilde{D}_U(\alpha)$ for all $\alpha \in [0, 1]$, we have

$$\begin{aligned} \lambda \tilde{D}_L(\alpha) + (1-\lambda) \tilde{D}_U(\alpha) &= \tilde{D}_U(\alpha) - \lambda(\tilde{D}_U(\alpha) - \tilde{D}_L(\alpha)) \\ &> \tilde{D}_U(\alpha) - 0.5(\tilde{D}_U(\alpha) - \tilde{D}_L(\alpha)) \\ &= 0.5\tilde{D}_L(\alpha) + 0.5\tilde{D}_U(\alpha) \end{aligned}$$

for all $\alpha \in [0, 1]$. By (7), .

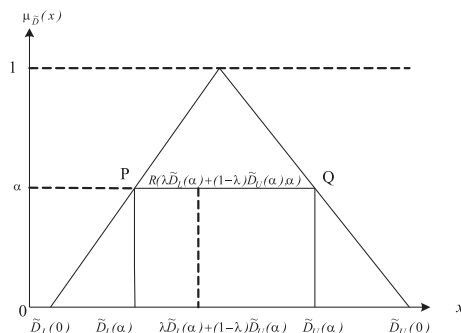


FIGURE 7. α -cut $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$ and Point $\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)$ in $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$

$$\begin{aligned}
 D_\lambda^* &= d(\tilde{D}, \tilde{0}; \lambda) = \int_0^1 [\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)]d\alpha \\
 &> \int_0^1 [0.5\tilde{D}_L(\alpha) + 0.5\tilde{D}_U(\alpha)]d\alpha \\
 &= d(\tilde{D}, \tilde{0}; 0.5) = D_{0.5}^*.
 \end{aligned}$$

Contrarily, when $\lambda > 0.5$, for each $\alpha \in [0, 1]$, then $D_\lambda^* < D_{0.5}^*$. Also, we can obtain the same relations corresponding to \tilde{D} , \tilde{k} and \tilde{g} .

(b) By (7)(8)(21), we have when $\lambda < 0.5$, then $D_\lambda^* > D + \frac{1}{2}(0.5\Delta_2 - 0.5\Delta_1) = D_{0.5}^*$, $k_\lambda^* > k_{0.5}^*$, $g_\lambda^* > g_{0.5}^*$ and $\hat{P}_{0\lambda}^* > \hat{P}_{0.5}^*$. On the contrary, when $\lambda > 0.5$, then $D_\lambda^* < D_{0.5}^*$, $k_\lambda^* < k_{0.5}^*$, $g_\lambda^* < g_{0.5}^*$ and $\hat{P}_{0\lambda}^* < \hat{P}_{0.5}^*$.

(c) The use of λ -signed distance method is based on the natural extension. Since a fuzzy set \tilde{D} is not a certain value, for a given $\lambda \in (0, 1)$, the weighted average of $\tilde{D}_L(\alpha)$ and $\tilde{D}_U(\alpha)$ denoted by $\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)$ is the inner point of α -level set $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$, and $R(\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha), \alpha)$ is also the inner point of line segment \overline{PQ} (see Figure 7). In other words, if λ is closer to "0", then the estimate of the intrinsic stock value in the fuzzy sense $\hat{P}_{0\lambda}^*$ will be greater; if λ is closer to "1", then $\hat{P}_{0\lambda}^*$ will be smaller. Based on this, λ level can also be simply regarded as an objective indicator describing investor's attitude toward the estimates of fuzzy numbers. That is, if $\lambda < 0.5$, it implies that such a investor is relative optimistic for estimating the values of fuzzy dividend (\tilde{D}), fuzzy required rate of return (\tilde{k}), and fuzzy growth rate of dividend (\tilde{g}). On the contrary, if $\lambda > 0.5$, the investor is relative pessimistic for estimating them. Also, if $\lambda = 0.5$, the investor's attitude is moderate. Therefore, the λ -signed distance method is an effective tool to deal with defuzzification of fuzzy models when their membership functions are hard to find out, and it is comparatively easier to be derived from mathematical operations.

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