# A BI-OBJECTIVE PROGRAMMING APPROACH TO SOLVE MATRIX GAMES WITH PAYOFFS OF ATANASSOV'S TRIANGULAR INTUITIONISTIC FUZZY NUMBERS

D. F. LI, J. X. NAN, Z. P. TANG, K. J. CHEN, X. D. XIANG AND F. X. HONG

Abstract. The intuitionistic fuzzy set has been applied to game theory very rarely since it was introduced by Atanassov in 1983. The aim of this paper is to develop an effective methodology for solving matrix games with payoffs of Atanassov's triangular intuitionistic fuzzy numbers (TIFNs). In this methodology, the concepts and ranking order relations of Atanassov's TIFNs are defined. A pair of bi-objective linear programming models for matrix games with payoffs of Atanassov's TIFNs is derived from two auxiliary Atanassov's intuitionistic fuzzy programming models based on the ranking order relations of Atanassov's TIFNs defined in this paper. An effective methodology based on the weighted average method is developed to determine optimal strategies for two players. The proposed method in this paper is illustrated with a numerical example of the market share competition problem.

## 1. Introduction

Mathematical objects introduced by Atanassov [1,2] and "studied under the name intuitionistic fuzzy set" (IFS) have become a popular topic of investigation in the fuzzy set community. The Atanassov's IFS is characterized by a membership function and a non-membership function. Burillo and Bustince [5] proved that the vague set [20] is the same as the Atanassov's IFS. Deschrijver and Kerre [17] established the relationship between the Atanassov's IFS, the interval-valued IFS [3], the L-fuzzy set [19] and the interval-valued fuzzy set [48]. Moreover, they showed that the interval-valued fuzzy set is equivalent to the Atanassov's IFS, which was also emphasized by several other scholars [3,14,16,18,21]. Recently, there is some debate about the appropriateness of the terminology of the Atanassov's IFS [4,14]. Dubois et al. [14] pointed out that there is a terminological clash between the Atanassov's IFS and intuitionistic logic [43]. They differ both by their motivations and their underlying mathematical structure. In reply to Dubois et al. [14], Atanassov [4] discussed the relationship between the notion of the Atanassov's IFS and the notion proposed by Takeuti and Titani [43] under the same name. But here we are not involved in this discussion and use the name "Atanassov's IFS" to express the intuitionistic fuzzy set introduced by Atanassov [1,2]. The Atanassov's IFS has been

Received: March 2011; Revised: July 2011 and September 2011; Accepted: November 2011
*Key words and phrases:* Uncertainty, Fuzzy set, Atanassov's intuitionistic fuzzy set, Fuzzy number, Matrix game, Mathematical programming.

proven to be highly useful to deal with uncertainty and vagueness in decision making problems [26,37,45,46], medical diagnosis [13] and pattern recognitions [30]. In the fuzzy set theory developed by Zadeh [47], the concept of a fuzzy number [15,23] is of very importance to application in the framework of expert systems and decision making. Roughly speaking, a fuzzy number may be considered as a representation for an ill-defined quantity. The Atanassov's IFS is a generalization of the fuzzy set [47]. Therefore, an ill-defined quantity may also be expressed with an intuitionistic fuzzy number (IFN) in the sense of Atanassov [1,2]. By adding a degree of non-membership, the Atanassov's IFN may be defined in a similar way to the fuzzy number introduced by Dubois and Prade [15]. Also it is a generalization of the fuzzy number. The Atanassov's IFN is an ill-defined quantify in the sense of the Atanassov's IFS and differs from the interval-valued fuzzy set although Atanassov's IFN is a special set of the Atanassov's IFS on the set of real numbers. The reason is further explained in section 2. The fuzzy set theory was used to solve several types of fuzzy game problems [6-12,22, 24,25,32,33,35-39,41-43]. The fuzzy set uses only a membership function, which assigns to each element $x$ of the universe of discourse a real number $\mu(x) \subseteq [0,1]$ to indicate the degree of belongingness to the fuzzy set under consideration. The degree of non-belongingness is just automatically equal to $1-\mu(x)$. However, a human being who expresses the degree of membership of a given element in a fuzzy set very often does not express corresponding degree of non-membership as the complement to 1. Sometimes it seems to be more natural to describe imprecise and uncertain opinions not only by membership functions. It is due to the fact that in some situations players also describe their negative feelings, i.e., degrees of dissatisfaction about the outcomes of the game. On the other hand, players could only estimate the payoff values approximately with some imprecise degree. But it is possible that he/she is not so sure about it. In other words, there may be hesitation about the approximate payoff values. Thus, the Atanassov's IFS theory may provide players a natural tool for modeling such uncertain and vague situations. The idea of the Atanassov's IFS is useful in matrix games in that the Atanassov's IFS can indicate the players' preference information in terms of support, opposition and neutralization. An example is the market share competition problem in which different strategies of two competing companies may lead to different market shares. Due to a lack of information or imprecision of the available information, the managers of the two companies usually are not able to exactly forecast the sales amounts of the companies. They may estimate the sales amount. But it is possible that they are not so sure about it. Thus, there may be some hesitation about the estimation of the sales amount. Up to the present, there exists less investigation on game problems using the Atanassov's IFS. Li and Nan [31] studied the matrix games with payoffs of IFSs. Nayak and Pal [34] studied the bi-matrix games with Atanassov's intuitionistic Fuzzy goals. The Atanassov's triangular intuitionistic fuzzy number (TIFN) is a special case of the Atanassov's IFS and easily used in matrix game problems in that it has appealing interpretations and can be easily specified and implemented by the players. Thus, in this paper, the definition and arithmetic operations of TIFNs are proposed through adding a degree of non-membership in the definition of triangular fuzzy numbers

[15]. Matrix games with payoffs of Atanassov's TIFNs are formulated. To obtain optimal strategies for players, a pair of linear programming models is derived from the constructed pair of bi-objective linear programming problems.

This paper is organized as follows. Section 2 introduces the concept and arithmetic operations of Atanassov's TIFNs. The concepts of an average index of the membership function and an average index of the non-membership function for an Atanassov's TIFN are defined. Hereby ranking order relations between Atanassov's TIFNs are defined. In section 3, the concept of solutions of matrix games with payoffs of Atanassov's TIFNs is defined. A pair of auxiliary Atanassov's intuitionistic fuzzy programming models for players are constructed and transformed into linear programming models by using the weighted average method. An effective methodology is developed to solve the matrix games with payoffs of Atanassov's TIFNs accordingly. A numerical example of market share competition problem is given in section 4. Section 5 concludes this paper.

## 2. **Definitions and Notations**

How to express an ill-defined quantify using the Atanassov's IFS is an important problem in decision making and game problems. In a similar way to the definition of a fuzzy number [15], the concept of an Atanassov's TIFN is defined through adding a degree of non-membership as follows.

**Definition 2.1.** [28] An Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a convex Atanassov's IFS on the set $R$ of real numbers, whose membership function and non-membership function are defined as follows:

$$u_{\tilde{a}}(x) = \begin{cases} \frac{a-x+u_{\tilde{a}}(x-\underline{a})}{a-\underline{a}} & \text{if } \underline{a} \leq x < a \\ u_{\tilde{a}} & \text{if } x = a \\ \frac{x-a+u_{\tilde{a}}(\overline{a}-x)}{\overline{a}-a} & \text{if } a \leq x < \overline{a} \\ 0 & \text{if } x < \underline{a} \text{ or} x > \overline{a} \end{cases} \tag{1}$$

and

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-\underline{a}}{a-\underline{a}} w_{\tilde{a}} & \text{if } \underline{a} \leq x < a \\ w_{\tilde{a}} & \text{if } x = a \\ \frac{\overline{a}-x}{\overline{a}-a} w_{\tilde{a}} & \text{if } a \leq x < \overline{a} \\ 0 & \text{if } x < \underline{a} \text{ or}, x > \overline{a} \end{cases} \tag{2}$$

respectively, depicted as in Figure 1.

The values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ respectively represent the maximum membership degree and the minimum non-membership degree which satisfy the conditions: $0 \leq w_{\tilde{a}} \leq 1$, $0 \leq u_{\tilde{a}} \leq 1$ and $0 \leq w_{\tilde{a}} + u_{\tilde{a}} \leq 1$.

Let

$$\chi_{\tilde{a}(x)} = 1 - \mu_{\tilde{a}(x)} - \nu_{\tilde{a}(x)}, \tag{3}$$

which is called the Atanassov's intuitionistic fuzzy index of element $x$ in $\tilde{a}$. It is the degree of indeterminacy membership of the element $x$ to $\tilde{a}$.

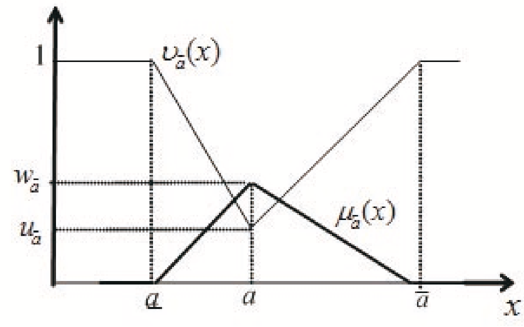


FIGURE 1. An Atanassovs TIFN $\tilde{\alpha}$

An Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ may express an ill-defined quantify, which is approximately equal to $a$ . Namely, the ill-defined quantify is expressed using any value between $\underline{a}$ and $\bar{a}$ with different membership degree and non-membership degree. In other words, the most possible value is $a$ with the membership degree $w_{\tilde{a}}$ and the non-membership degree $u_{\tilde{a}}$; the pessimistic value is $\underline{a}$ with the membership degree 0 and the non-membership degree 1; the optimistic value is $\bar{a}$ with the membership degree 1 and the non-membership degree 0; other values are any $x$ in the open interval $(\underline{a}, \bar{a})$ with the membership degree $\mu_{\tilde{a}}(x)$ and the non-membership degree $\nu_{\tilde{a}}(x)$.

An Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ mainly gives an approximately estimated range of an ill-defined quantify, i.e., an interval $[\underline{a}, \bar{a}]$. For any fixed $x$ in the closed interval $[\underline{a}, \bar{a}]$, the membership degree $\mu_{\tilde{a}}(x)$ and the non-membership degree $\nu_{\tilde{a}}(x)$ are real numbers rather than intervals. However, for the interval-valued fuzzy set $B$, the membership degree $\mu_B(x)$ for any fixed $x \in B$ is an interval rather than a real number [47]. Similarly, for any fixed $x \in B$ , the non-membership degree $\nu_B(x) = 1 - \mu_B(x)$ is an interval rather than a real number. In other words, the Atanassov's TIFN focuses on measurement uncertainty of the ill-defined quantify estimated while the interval-valued fuzzy set $B$ focuses on uncertainty of the belongingness degree to $B$ . Therefore, the Atanassov's TIFN differs from the interval-valued fuzzy set although Atanassov's TIFN is a special case of the Atanassov's IFS on the set of real numbers and the interval-valued fuzzy set is equivalent to the Atanassov's IFS [17].

Obviously, if $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$ then $\mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) = 1$ for any $x \in R$ and hence the Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ degenerates to $\tilde{a} =< (\underline{a}, a, \bar{a}); 1, 0 >$, which is just a triangular fuzzy number [15].Therefore, the concept of the Atanassov's TIFN is a generalization of that of the triangular fuzzy number [15].

Two new parameters $w_{\tilde{a}}$ and $u_{\tilde{a}}$ are introduced to reflect the confidence level and non-confidence level of the Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ , respectively. Compared with triangular fuzzy numbers, Atanassov's TIFNs may express more uncertainty.

According to the extension principle of the Atanassov's IFS proposed by Li [29], the arithmetic operations over Atanassov's TIFNs can be defined as follows [27].

**Definition 2.2.** [28] Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ and $\tilde{b} =< (\underline{b}, b, \bar{b}); w_{\tilde{b}}, u_{\tilde{b}} >$ be two Atanassov's TIFNs and $\gamma$ be a real number. The arithmetic operations are defined as follows:

$$\tilde{a} + \tilde{b} =< (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > \tag{4}$$

$$\tilde{a} - \tilde{b} =< (\underline{a} - \bar{b}, a - b, \bar{a} - \underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > \tag{5}$$

$$\tilde{a}\tilde{b} = \begin{cases} < (\underline{a}\underline{b}, ab, \bar{a}\bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\underline{a}\bar{b}, ab, \bar{a}\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}\bar{b}, ab, \underline{a}\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \tag{6}$$

$$\tilde{a}/\tilde{b} = \begin{cases} < (\underline{a}/\bar{b}, a/b, \bar{a}/\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\underline{b}, a/b, \underline{a}/\bar{b}); \min\{w_{\tilde{a}}, w_{\tilde{b}}\}, \max\{u_{\tilde{a}}, u_{\tilde{b}}\} > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \tag{7}$$

$$\gamma\tilde{a} = \begin{cases} < (\gamma\underline{a}, a, \gamma\bar{a}); w_{\tilde{a}}, u_{\tilde{a}} > & \text{if } \gamma > 0 \\ < (\gamma\bar{a}, a, \gamma\underline{a}); w_{\tilde{a}}, u_{\tilde{a}} > & \text{if } \gamma < 0 \end{cases} \tag{8}$$

$$\tilde{a}^{-1} =< (1/\bar{a}, 1/a, 1/\underline{a}); w_{\tilde{a}}, u_{\tilde{a}} > \text{if } \tilde{a} > 0 \tag{9}$$

It was proven that the results from multiplication and division are not Atanassov's TIFNs. But, we often use Atanassov's TIFNs to express these operational results approximately.

Obviously, if $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$ , i.e., $\tilde{a} =< (\underline{a}, a, \bar{a}); 1, 0 >$, and $\tilde{b} =< (\underline{b}, b, \bar{b}); 1, 0 >$, are triangular fuzzy numbers, then equations (4)-(9) degenerate to equations (10)-(15) as follows:

$$\tilde{a} + \tilde{b} =< (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b}); 1, 0 > \tag{10}$$

$$\tilde{a} - \tilde{b} =< (\underline{a} - \bar{b}, a - b, \bar{a} - \underline{b}); 1, 0 > \tag{11}$$

$$\tilde{a}\tilde{b} = \begin{cases} < (\underline{a}\underline{b}, ab, \bar{a}\bar{b}); 1, 0 > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\underline{a}\bar{b}, ab, \bar{a}\underline{b}); 1, 0 > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}\bar{b}, ab, \underline{a}\underline{b}); 1, 0 > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \tag{12}$$

$$\tilde{a}/\tilde{b} = \begin{cases} < (\underline{a}/\bar{b}, a/b, \bar{a}/\underline{b}); 1, 0 > & \text{if } \tilde{a} > 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\bar{b}, a/b, \underline{a}/\underline{b}); 1, 0 > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} > 0 \\ < (\bar{a}/\underline{b}, a/b, \underline{a}/\bar{b}); 1, 0 > & \text{if } \tilde{a} < 0 \text{ and } \tilde{b} < 0 \end{cases} \tag{13}$$

$$\gamma\tilde{a} = \begin{cases} < (\gamma\underline{a}, a, \gamma\bar{a}); 1, 0 > & \text{if } \gamma > 0 \\ < (\gamma\bar{a}, a, \gamma\underline{a}); 1, 0 > & \text{if } \gamma < 0 \end{cases} \tag{14}$$

$$\tilde{a}^{-1} =< (1/\bar{a}, 1/a, 1/\underline{a}); 1, 0 > \text{if } \tilde{a} > 0 \tag{15}$$

respectively. Equations (10)-(15) are just arithmetic operations of triangular fuzzy numbers [15]. Therefore, the arithmetic operations of Atanassov's TIFNs are generalizations of those of the triangular fuzzy numbers [15].

**Definition 2.3.** [2] A $(\alpha, \beta)$-cut set of an Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of $R$ which is defined as follows:

$$\tilde{a}_{\alpha,\beta} = \{x|\mu_{\tilde{a}}(x) \geq \alpha, \nu_{\tilde{a}}(x) \leq \beta\}, \tag{16}$$

where $0 \leq \alpha \leq w_{\tilde{a}}$, $u_{\tilde{a}} \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$.

**Definition 2.4.** [2] A $\alpha$-cut set of an Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of $R$ which is defined as follows:

$$\tilde{a}_{\alpha} = \{x|\mu_{\tilde{a}}(x) \geq \alpha\}, \tag{17}$$

where $0 \leq \alpha \leq w_{\tilde{a}}$.

Using equation (1) and Definition 2.4, $\tilde{a}_{\alpha}$ is a closed interval and calculated as follows:

$$\tilde{a}_{\alpha} = \left[\underline{a} + \frac{\alpha}{w_{\tilde{a}}}(a - \underline{a}), a - \frac{\alpha}{w_{\tilde{a}}}(\bar{a} - a)\right]. \tag{18}$$

**Definition 2.5.** [2] A $\beta$-cut set of an Atanassov's TIFN $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ is a crisp subset of $R$ which is defined as follows:

$$\tilde{a}_{\beta} = \{x|\nu_{\tilde{a}}(x) \leq \beta\}, \tag{19}$$

where $u_{\tilde{a}} \leq \beta \leq 1$.

Using equation (2) and Definition 2.5, $\tilde{a}_{\beta}$ is a closed interval and calculated as follows:

$$\tilde{a}_{\beta} = \left[\frac{(1-\beta)a + (\beta - u_{\tilde{a}})\underline{a}}{1 - u_{\tilde{a}}}, \frac{(1-\beta)a + (\beta - u_{\tilde{a}})\bar{a}}{1 - u_{\tilde{a}}}\right]. \tag{20}$$

**Theorem 2.6.** [28] *Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ be any Atanassov's TIFN. For any $\alpha \in [0, w_{\tilde{a}}]$ and $\beta \in [u_{\tilde{a}}, 1]$, where $0 \leq \alpha + \beta \leq 1$, the following equality is valid*

$$\tilde{a}_{\alpha,\beta} = \tilde{a}_{\alpha} \bigcap \tilde{a}_{\beta}. \tag{21}$$

Let $m(\tilde{a}_{\alpha})$ and $m(\tilde{a}_{\beta})$ be mean values of the elements in the intervals $\tilde{a}_{\alpha}$ and $\tilde{a}_{\beta}$, respectively, i.e.,

$$m(\tilde{a}_{\alpha}) = \frac{2\alpha a + (w_{\tilde{a}} - \alpha)(\underline{a} + \bar{a})}{2w_{\tilde{a}}}, \tag{22}$$

and

$$m(\tilde{a}_{\beta}) = \frac{2(1-\beta)a + (\beta - u_{\tilde{a}})(\underline{a} + \bar{a})}{2(1 - u_{\tilde{a}})}. \tag{23}$$

Then an average index of the membership function $\mu_{\tilde{a}}(x)$ and an average index of the non-membership function $\nu_{\tilde{a}}(x)$ for the Atanassov's TIFN are defined as follows:

$$s_{\mu}(\tilde{a}) = \int_0^{w_{\tilde{a}}} m(\tilde{a}_{\alpha})d\alpha = \frac{2a + \underline{a} + \bar{a}}{4} w_{\tilde{a}}, \tag{24}$$

and

$$s_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^{1} m(\tilde{a}_\beta) d\beta = \frac{2a + \underline{a} + \bar{a}}{4}(1 - u_{\tilde{a}}). \tag{25}$$

Due to the condition $0 \leq w_a + u_a \leq 1$, it is directly derived from equations (24) and (25) that

$$\frac{2a + \underline{a} + \bar{a}}{4} w_{\tilde{a}} \leq \frac{2a + \underline{a} + \bar{a}}{4}(1 - u_{\tilde{a}}),$$

i.e.,$s_\mu(\tilde{a}) \leq s_\nu(\tilde{a})$.

Obviously, the average index of the membership function $s_\mu(\tilde{a})$ and the average index of the non-membership function $s_\nu(\tilde{a})$ synthetically reflect information on membership degrees and non-membership degrees at all levels, respectively. The resulting values may be used to rank Atanassov's TIFNs. In other words, $s_\mu(\tilde{a})$ and $s_\nu(\tilde{a})$ may be chosen as ranking indices of the Atanassov's TIFN $\tilde{a}$. The larger $s_\mu(\tilde{a})$ and $s_\nu(\tilde{a})$, the larger $\tilde{a}$ . Thus, a ranking order relation between two Atanassov's TIFNs may be defined as follows.

**Definition 2.7.** Let $\tilde{a} =< (\underline{a}, a, \bar{a}); w_{\tilde{a}}, u_{\tilde{a}} >$ and $\tilde{b} =< (\underline{b}, b, \bar{b}); w_{\tilde{b}}, u_{\tilde{b}} >$ be two Atanassov's TIFNs. $s_\mu(\tilde{a}) = \frac{2a+\underline{a}+\bar{a}}{4} w_{\tilde{a}}$ and $s_\mu(\tilde{b}) = \frac{2b+\underline{b}+\bar{b}}{4} w_{\tilde{b}}$ are the average indices of the membership functions for $\tilde{a}$ and $\tilde{b}$, respectively. $s_\nu(\tilde{a}) = \frac{2a+\underline{a}+\bar{a}}{4}(1 - u_{\tilde{a}})$ and $s_\nu(\tilde{b}) = \frac{2b+\underline{b}+\bar{b}}{4}(1 - u_{\tilde{b}})$ are the average indices of the non-membership functions $\tilde{a}$ and $\tilde{b}$, respectively. Then

(1) If $s_\mu(\tilde{a}) < s_\mu(\tilde{b})$ and $s_\nu(\tilde{a}) < s_\nu(\tilde{b})$ then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} \tilde{<}_{IF} \tilde{b}$;

(2) If $s_\mu(\tilde{a}) > s_\mu(\tilde{b})$ and $s_\nu(\tilde{a}) > s_\nu(\tilde{b})$ then $\tilde{a}$ is larger than $\tilde{b}$, denoted by $\tilde{a} \tilde{>}_{IF} \tilde{b}$;

(3) If $s_\mu(\tilde{a}) = s_\mu(\tilde{b})$ and $s_\nu(\tilde{a}) = s_\nu(\tilde{b})$ then $\tilde{a}$ and $\tilde{b}$ represent the same amount, i.e., $\tilde{a}$ is equal to $\tilde{b}$, denoted by $\tilde{a} \tilde{=}_{IF} \tilde{b}$.

The symbol "$\tilde{<}_{IF}$" is an Atanassov's intuitionistic fuzzy version of the order relation " $<$ "on the set of real numbers and has the linguistic interpretation "essentially smaller than". Similarly, the symbols "$\tilde{>}_{IF}$" and "$\tilde{=}_{IF}$" are Atanassov's intuitionistic fuzzy versions of the order relations " $>$ " and " $=$ " on the set of real numbers and have the linguistic interpretations "essentially larger than" and"essentially equal to", respectively. A numerical example is presented to show applicability of the aforementioned ranking order relation contrasting to the ratio ranking method of Atanassov's TIFNs given by Li [28].

**Example 2.8.** Let us consider two Atanassov's TIFNs $\tilde{a} =< (0.769, 0.903, 1); 0.4, 0.5 >$ and $\tilde{b} =< (0.653, 0.849, 0.956); 0.5, 0.2 >$ . According to equations (24) and (25), the average indices of membership function and non-membership function for $\tilde{a}$ and $\tilde{b}$ are calculated as follows:

$$s_\mu(\tilde{a}) = 0.3575, s_\nu(\tilde{a}) = 0.4469$$

and

$$s_\mu(\tilde{b}) = 0.413, s_\nu(\tilde{b}) = 0.6614,$$

respectively.

Obviously, $s_\mu(\tilde{a}) < s_\mu(\tilde{b})$ and $s_\nu(\tilde{a}) < s_\nu(\tilde{b})$. According to Definition 2.7, the ranking order is $\tilde{a}\tilde{<}_{IF}\tilde{b}$, which is the same as that obtained by the ratio ranking method given by Li [28]. This shows that the proposed ranking order relation is effective. It is easily seen that the ratio ranking method [28] is complicated and has tedious calculations whereas the proposed ranking order relation in this paper is easy to be implemented. In addition, the ratio ranking method [28] involves in choice of the parameter $\lambda \in [0,1]$ , which is very difficult.

## 3. Auxiliary Bi-objective Linear Programming Models and Method for Matrix Games with Payoffs of Atanassov's TIFNs

Let us consider the matrix games with payoffs of Atanassov's TIFNs. Assume that $S_1 = \{\delta_1, \delta_2, \cdots, \delta_m\}$ and $S_2 = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ are sets of pure strategies for two players I and II, respectively. The vectors $\mathbf{y} = (y_1, y_2, \cdots, y_m)^T$ and $\mathbf{z} = (z_1, z_2, \cdots, z_n)^T$ are mixed strategies for players I and II, respectively, where $y_i(i = 1, 2, \cdots, m)$ and $z_j(j = 1, 2, \cdots, n)$ are probabilities in which I and II choose their pure strategies $\delta_i \in S_1(i = 1, 2, \cdots, m)$ and $\sigma_j \in S_2(j = 1, 2, \cdots, n)$, respectively. Sets of all mixed strategies for players I and II are denoted by $\mathbf{Y}$ and $\mathbf{Z}$ as follows:

$$\mathbf{Y} = \left\{\mathbf{y}\middle| \sum_{i=1}^{m} y_i = 1, y_i \geq 0(i = 1, 2, \cdots, m)\right\}, \tag{26}$$

and

$$\mathbf{Z} = \left\{\mathbf{z}\middle| \sum_{j=1}^{n} z_j = 1, y_j \geq 0(j = 1, 2, \cdots, n)\right\}, \tag{27}$$

respectively. Without loss of generality, payoffs of player I is concisely expressed in the matrix format as follows:

$$\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$$

which is called as the payoff matrix for player I, where all elements

$$\tilde{a}_{ij} =< (\underline{a}_{ij}, a_{ij}, \bar{a}_{ij}); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} > (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$$

are Atanassov's TIFNs. Then, a matrix game with payoffs of Atanassov's TIFNs is expressed as the triplet $(\mathbf{Y}, \mathbf{Z}, \tilde{\mathbf{A}})$ . In the sequel, such a matrix game with payoffs of Atanassov's TIFNs is often called as a matrix game with payoffs of Atanassov's TIFNs for short. According to Definition 2.2, the Atanassov's intuitionistic fuzzy expectation payoff for player I can be computed as follows:

$$\begin{aligned}\tilde{E}(\tilde{\mathbf{A}}) &= \mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z} = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{a}_{ij}y_iz_j\\ &=< (\sum_{i=1}^{m}\sum_{j=1}^{n} \underline{a}_{ij}y_iz_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}y_iz_j, \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{a}_{ij}y_iz_j);\\ &\min_{1\leq i\leq m;1\leq j\leq n}\{w_{\tilde{a}_{ij}}\}, \max_{1\leq i\leq m;1\leq j\leq n}\{u_{\tilde{a}_{ij}}\} >,\end{aligned} \tag{28}$$

which is still an Atanassov's TIFN. It is derived from Definition 2.1 that

$$\min_{1\leq i\leq m;1\leq j\leq n}\{w_{\tilde{a}_{ij}}\} \leq 1 - \max_{1\leq i\leq m;1\leq j\leq n}\{u_{\tilde{a}_{ij}}\}.$$

As the matrix game $\tilde{\mathbf{A}}$ with payoffs of Atanassov's TIFNs is zero-sum, according to Definition 2.2, the Atanassov's intuitionistic fuzzy expectation payoff for player II is obtained as follows:

$$\begin{aligned}&\tilde{E}(\text{-}\tilde{\mathbf{A}}) = \mathbf{y}^T(\text{-}\tilde{\mathbf{A}})\mathbf{z} = \sum_{i=1}^{m}\sum_{j=1}^{n} -\tilde{a}_{ij}y_iz_j\\ &=<(-\sum_{i=1}^{m}\sum_{j=1}^{n}\bar{a}_{ij}y_iz_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}y_iz_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}\underline{a}_{ij}y_iz_j);\\ &\min_{1\le i\le m;1\le j\le n}\{w_{\tilde{a}_{ij}}\}, \max_{1\le i\le m;1\le j\le n}\{u_{\tilde{a}_{ij}}\}>,\end{aligned} \tag{29}$$

which is still an Atanassov's TIFN. It is derived from Definition 2.1 that

$$\min_{1\le i\le m;1\le j\le n}\{w_{\tilde{a}_{ij}}\} \le 1 - \max_{1\le i\le m;1\le j\le n}\{u_{\tilde{a}_{ij}}\}.$$

Thus, in general, player I's gain-floor and player II's loss-ceiling should be Atanassov's TIFNs, denoted by $\tilde{v} =< (\underline{v}, v, \bar{v}); w_{\tilde{v}}, u_{\tilde{v}} >$ and $\tilde{\omega} =< (\underline{\omega}, \omega, \bar{\omega}); w_{\tilde{\omega}}, u_{\tilde{\omega}} >$, respectively. Now, the concept of a solution of the matrix game $\tilde{\mathbf{A}}$ with payoffs of Atanassov's TIFNs is defined as follows.

**Definition 3.1.** If there exist $\mathbf{y}^* \in \mathbf{Y}$ and $\mathbf{z}^* \in \mathbf{Z}$ so that

$$\mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z}^* \tilde{\le}_{IF} \mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}^*, \tag{30}$$

and

$$\mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z} \tilde{\ge}_{IF} \mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}^*, \tag{31}$$

for any $\mathbf{y} \in \mathbf{Y}$ and $\mathbf{z} \in \mathbf{Z}$, then $\mathbf{v}^* = \mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}^*$ is called as the value of the matrix game $\tilde{\mathbf{A}}$ with payoffs of Atanassov's TIFNs, $\mathbf{y}^*$ and $\mathbf{z}^*$ are called as optimal (mixed) strategies for players I and II, respectively. Obviously, $\mathbf{v}^* = \mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}^*$ is an Atanassov's TIFN.

According to Definition 3.1, the optimal strategy $\mathbf{y}^* \in \mathbf{Y}$ for player I and optimal strategy $\mathbf{z}^* \in \mathbf{Z}$ for player II can be generated by solving a pair of Atanassov's intuitionistic fuzzy mathematical programming models constructed as follows:

$$\begin{aligned}&\max\{\tilde{v}\}\\ &\text{s. t.}\begin{cases}\sum_{i=1}^{m}\tilde{a}_{ij}y_iz_j \tilde{\ge}_{IF} \tilde{v}(j=1,2,\cdots,n) \text{ for any } \mathbf{z}\in\mathbf{Z}\\ y_1+y_2+\cdots+y_m=1\\ y_i\ge 0(i=1,2,\cdots,m)\end{cases}\end{aligned} \tag{32}$$

and

$$\begin{aligned}&\min\{\tilde{\omega}\}\\ &\text{s. t.}\begin{cases}\sum_{j=1}^{n}\tilde{a}_{ij}y_iz_j \tilde{\le}_{IF} \tilde{\omega}(i=1,2,\cdots,m) \text{ for any } \mathbf{y}\in\mathbf{Y}\\ z_1+z_2+\cdots+z_n=1\\ z_j\ge 0(j=1,2,\cdots,n)\end{cases}\end{aligned} \tag{33}$$

respectively, where $\tilde{v}$ and $\tilde{\omega}$ are Atanassov's TIFNs unknown need to be determined. It makes sense to consider only the extreme points of the sets $\mathbf{Y}$ and $\mathbf{Z}$ in the constraints of equations (32) and (33) since "$\tilde{\le}_{IF}$" and "$\tilde{\ge}_{IF}$" preserve the ranking order relations when Atanassov's TIFNs are multiplied by positive numbers according to Definition 2.2. Therefore, equations (32) and (33) can be converted into the Atanassov's intuitionistic fuzzy mathematical programming models as follows:

$$\max\{\tilde{v}\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^{m}\tilde{a}_{ij}y_i \tilde{\geq}_{IF}\tilde{v}(j=1,2,\cdots,n)\\ y_1+y_2+\cdots+y_m=1\\ y_i\geq 0(i=1,2,\cdots,m)\end{cases} \tag{34}$$

and

$$\min\{\tilde{\omega}\}$$
$$\text{s. t.}\begin{cases}\sum_{j=1}^{n}\tilde{a}_{ij}z_j \tilde{\leq}_{IF}\tilde{\omega}(i=1,2,\cdots,m)\\ z_1+z_2+\cdots+z_n=1\\ z_j\geq 0(j=1,2,\cdots,n)\end{cases} \tag{35}$$

respectively.

In this study, Atanassov's intuitionistic fuzzy optimization is made in the sense of Definition 2.6. In the following, we will focus on how to solve equations (34) and (35).

Using Definition 2.6, equation (34) can be transformed into the bi-objective programming problem as follows:

$$\max\{s_\mu(\tilde{v}),s_\nu(\tilde{v})\}$$
$$\text{s. t.}\begin{cases}s_\mu(\sum_{i=1}^{m}\tilde{a}_{ij}y_i)\geq s_\mu(\tilde{v})\,(j=1,2,\cdots,n)\\ s_\nu(\sum_{i=1}^{m}\tilde{a}_{ij}y_i)\geq s_\nu(\tilde{v})\,(j=1,2,\cdots,n)\\ \sum_{i=1}^{m}y_i=1,y_i\geq 0(i=1,2,\cdots,m)\end{cases} \tag{36}$$

According to equations (24) and (25), equation (36) can be rewritten as follows:

$$\max\{\tfrac{1}{4}(\underline{v}+2v+\overline{v})w_{\tilde{v}},\tfrac{1}{4}(\underline{v}+2v+\overline{v})(1-u_{\tilde{v}}\}$$
$$\text{s. t.}\begin{cases}\sum_{i=1}^{m}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})\min_{1\leq i\leq m}\{w_{\tilde{a}_{ij}}\}y_i\geq\frac{1}{4}(\underline{v}+2v+\overline{v})w_{\tilde{v}}\,(j=1,2,\cdots,n)\\ \sum_{i=1}^{m}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})(1-\max_{1\leq i\leq m}\{u_{\tilde{a}_{ij}}\})y_i\geq\frac{1}{4}(\underline{v}+2v+\overline{v})(1-u_{\tilde{v}})(j=1,2,\cdots,n)\\ \underline{v}\leq v\leq\overline{v}\\ \sum_{i=1}^{m}y_i=1,y_i\geq 0\,(i=1,2,\cdots,m)\end{cases} \tag{37}$$

where $\underline{v}$, $v$, $\overline{v}$ and $y_i(i=1,2,\cdots,m)$ are variables.

In general, it is difficult to obtain the optimal value $\tilde{v}^*$ in that in equation (37) determining $\tilde{v}^*$ exactly needs to compute all three variables $\underline{v}$, $v$, and $\overline{v}$. Thus, let

$v_1 = \frac{1}{4}(\underline{v}+2v+\overline{v})w_{\tilde{v}}$ and $v_2 = \frac{1}{4}(\underline{v}+2v+\overline{v})(1-u_{\tilde{v}})$ . equation (37) is transformed into equation (38) as follows:

$$\max\{v_1, v_2\}$$
$$\text{s. t.} \begin{cases} \sum_{i=1}^{m} \frac{1}{4}(\underline{a}_{ij} + 2a_{ij} + \overline{a}_{ij}) \min_{1\le i\le m}\{w_{\tilde{a}_{ij}}\} y_i \ge v_1 \, (j = 1, 2, \cdots, n) \\ \sum_{i=1}^{m} \frac{1}{4}(\underline{a}_{ij} + 2a_{ij} + \overline{a}_{ij})(1 - \max_{1\le i\le m}\{u_{\tilde{a}_{ij}}\}) y_i \ge v_2 \, (j = 1, 2, \cdots, n) \\ v_2 \ge v_1 \\ \sum_{i=1}^{m} y_i = 1, y_i \ge 0 (i = 1, 2, \cdots, m) \end{cases} \tag{38}$$

In equation (38), the inequality $v_2 \ge v_1$ is derived from $v_1 = \frac{1}{4}(\underline{v} + 2v + \overline{v})w_{\tilde{v}}$ and $v_2 = \frac{1}{4}(\underline{v} + 2v + \overline{v})(1 - u_{\tilde{v}})$ because of $0 \le w_{\tilde{v}} \le 1 - u_{\tilde{v}}$.

Obviously, equation (38) is a bi-objective linear programming model with decision variables $v_1$,$v_2$ and $y_i (i = 1, 2, \cdots, m)$. There are few standard ways of defining a solution for multi-objective programming problems. Normally, the concept of Pareto optimal/efficient solutions is commonly-used. There exist several solution methods for them. However, in this paper we use the weighted average method to solve equation (38) in the sense of Pareto optimality.

According to the weighted average method, equation (38) is transformed into the linear programming as follows:

$$\max\{\lambda v_1 + (1 - \lambda) v_2\}$$
$$\text{s. t.} \begin{cases} \sum_{i=1}^{m} \frac{1}{4}(\underline{a}_{ij} + 2a_{ij} + \overline{a}_{ij}) \min_{1\le i\le m}\{w_{\tilde{a}_{ij}}\} y_i \ge v_1 \, (j = 1, 2, \cdots, n) \\ \sum_{i=1}^{m} \frac{1}{4}(\underline{a}_{ij} + 2a_{ij} + \overline{a}_{ij})(1 - \max_{1\le i\le m}\{u_{\tilde{a}_{ij}}\}) y_i \ge v_2 \, (j = 1, 2, \cdots, n) \\ v_2 \ge v_1 \\ \sum_{i=1}^{m} y_i = 1, y_i \ge 0 (i = 1, 2, \cdots, m) \end{cases} \tag{39}$$

where $\lambda$ is a weight which represents players' preference information.

Using the Simplex method for linear programming, an optimal solution of equation (39) is obtained, denoted by $(\mathbf{y}^*, v_1^*, v_2^*)$.

It is not difficult to prove that $(\mathbf{y}^*, v_1^*, v_2^*)$ is a Pareto optimal solution of equation (38). Thus, the optimal strategy $\mathbf{y}^*$ for player I is obtained as well as the average index of the membership function $v_1^*$ and the average index of the non-membership function $v_2^*$ for player I's gain-floor $\tilde{v}^*$.

Similarly, according to Definition 2.6, equation (35) can be transformed into the bi-objective programming problem as follows:

$$\max\{s_\mu(\tilde{\omega}), s_\nu(\tilde{\omega})\}$$
$$\text{s. t.}\begin{cases} s_\mu(\sum_{j=1}^{n}\tilde{a}_{ij}z_j) \le s_\mu(\tilde{\omega})\,(i=1,2,\cdots,m) \\ s_\nu(\sum_{j=1}^{n}\tilde{a}_{ij}z_i) \le s_\nu(\tilde{\omega})\,(i=1,2,\cdots,m) \\ \sum_{j=1}^{n} z_j = 1, z_j \ge 0 (j=1,2,\cdots,n) \end{cases} \tag{40}$$

According to equations (24) and (25), equation (40) can be rewritten as follows:

$$\min\{\tfrac{1}{4}(\underline{\omega}+2\omega+\overline{\omega})w_{\tilde{\omega}}, \tfrac{1}{4}(\underline{\omega}+2\omega+\overline{\omega})(1-u_{\tilde{\omega}}\}$$
$$\text{s. t.}\begin{cases} \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})\min_{1\le j\le n}\{w_{\tilde{a}_{ij}}\}z_j \le \frac{1}{4}(\underline{\omega}+2\omega+\overline{\omega})w_{\tilde{\omega}}\,(i=1,2,\cdots,m) \\ \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})(1-\max_{1\le j\le n}\{u_{\tilde{a}_{ij}}\})z_j \ge \frac{1}{4}(\underline{\omega}+2\omega+\overline{\omega})(1-u_{\tilde{\omega}})\,(i=1,2,\cdots,m) \\ \underline{\omega}\le\omega\le\overline{\omega} \\ \sum_{j=1}^{n} z_j = 1, z_j \ge 0 (j=1,2,\cdots,n) \end{cases} \tag{41}$$

where $\underline{\omega}$,$\omega$,$\overline{\omega}$ and $\mathbf{z}_j$ $(j = 1, 2, \cdots, n)$ are decision variables.

In a similar consideration to equation (37), equation (41) can be simply rewritten as follows:

$$\min\{\omega_1, \omega_2\}$$
$$\text{s. t.}\begin{cases} \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})\min_{1\le j\le n}\{w_{\tilde{a}_{ij}}\}z_j \le \omega_1\,(i=1,2,\cdots,m) \\ \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})(1-\max_{1\le j\le n}\{u_{\tilde{a}_{ij}}\})z_j \le \omega_2\,(i=1,2,\cdots,m) \\ \omega_1\le\omega_2 \\ \sum_{j=1}^{n} z_j = 1, z_j \ge 0 (j=1,2,\cdots,n) \end{cases} \tag{42}$$

In equation (42), the inequality $\omega_1 \le \omega_2$ is derived from $\frac{1}{4}(\underline{\omega} + 2\omega + \overline{\omega})w_{\tilde{\omega}} = \omega_1$ and $\frac{1}{4}(\underline{\omega} + 2\omega + \overline{\omega})(1 - u_{\tilde{\omega}}) = \omega_2$ because of $0 \le w_{\tilde{\omega}} \le 1 - u_{\tilde{\omega}}$. Using the weighted average method, equation (42) is transformed into equation (43) as follows:

$$\min\{\lambda\omega_1 + (1-\lambda)\omega_2\}$$
$$\text{s. t.}\begin{cases} \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})\min_{1\le j\le n}\{w_{\tilde{a}_{ij}}\}z_j \le \omega_1\,(i=1,2,\cdots,m) \\ \sum_{j=1}^{n}\frac{1}{4}(\underline{a}_{ij}+2a_{ij}+\overline{a}_{ij})(1-\max_{1\le j\le n}\{u_{\tilde{a}_{ij}}\})z_j \le \omega_2\,(i=1,2,\cdots,m) \\ \omega_1\le\omega_2 \\ \sum_{j=1}^{n} z_j = 1, z_j \ge 0 (j=1,2,\cdots,n) \end{cases} \tag{43}$$

Let an optimal solution of equation (43) be denoted by $(\mathbf{z}^*, \omega_1^*, \omega_1^*)$. It is not difficult to prove that $(\mathbf{z}^*, \omega_1^*, \omega_1^*)$ is a Pareto optimal solution of equation (42). Thus, the optimal strategy $\mathbf{z}^*$ can be obtained as well as the average index of the membership function $\omega_1^*$ and the average index of the non-membership function $\omega_2^*$ for player II's loss-ceiling $\omega^*$.

It is easily seen that all Atanassov's TIFNs $\tilde{a}_{ij} =< (\underline{a}_{ij}, a_{ij}, \overline{a}_{ij}); w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} >$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ reduce to triangular fuzzy numbers when all values $w_{\tilde{a}_{ij}} = 1$ and $u_{\tilde{a}_{ij}} = 0$. Hence,$v_1 = v_2$ and $\omega_1 = \omega_2$ . Thus, the linear programming models (i.e., equations (39) and (43)) of the matrix game $\tilde{\mathbf{A}}$ with payoffs of Atanassov's TIFNs degenerate to those of the matrix game with payoffs of triangular fuzzy numbers [10-12]. Therefore, the matrix game with payoffs of Atanassov's TIFNs is a natural generalization of the matrix game with payoffs of triangular fuzzy numbers.

## 4. A Numerical Example of the Market Share Competition Problem

Suppose that there are two companies $p_1$ and $p_2$ aiming to enhance the market share of a product in a targeted market under the circumstance that the demand amount of the product in the targeted market basically is fixed. In other words, the market share of one company increases while the market share of another company decreases. The two companies are considering about three strategies to increase the market share: strategy $\delta_1$ (advertisement), $\delta_2$ (reducing the price), $\delta_3$ (improving the package). The above problem may be regarded as a matrix game. Namely, the companies $p_1$ and $p_2$ are regarded as players I and II, respectively. They use strategies $\delta_1$ , $\delta_2$ and $\delta_3$ , respectively. Due to a lack of information, the managers are not able to evaluate the sales amounts of the companies exactly. In order to handle the uncertain situation, Atanassov's TIFNs are used to express the estimated sales amounts of the product. According to some prediction methods and statistic data, the payoff matrix $\tilde{\mathbf{A}}$ for the company $p_1$ is given by its manager as follows:

$$\tilde{A} = \begin{bmatrix} <(175,180,190);0.6,0.2> & <(150,156,158);0.6,0.1> & <(80,90,100);0.9,0.1> \\ <(80,90,100);0.9,0.1> & <(175,180,190);0.6,0.2> & <(180,185,190);0.5,0.3> \\ <(180,185,190);0.5,0.3> & <(80,100,120);0.7,0.2> & <(150,160,170);0.5,0.1> \end{bmatrix}$$

where $<(175, 180, 190); 0.6, 0.2>$ in the matrix $\tilde{\mathbf{A}}$ is an Atanassov's TIFN, which indicates that the manager of the company $p_1$ estimates the sales amount being approximately 180 with the maximum confidence degree 0.6 and the minimum non-confidence degree 0.2 when $p_1$ and $p_2$ use strategy $\delta_1$ (advertisement) simultaneously. In other words, the manager of the company $p_1$'s hesitation degree is 0.2. Other Atanassov's TIFNs in the matrix $\tilde{\mathbf{A}}$ are explained similarly. According to equation (39), the linear programming problem is obtained as follows:

$$\max\{\lambda v_1 + (1-\lambda)v_2\}$$
$$\text{s. t.} \begin{cases} 181.25 \times 0.5y_1 + 90 \times 0.5y_2 + 185 \times 0.5y_3 \geq v_1 \\ 155 \times 0.6y_1 + 181.25 \times 0.6y_2 + 100 \times 0.6y_3 \geq v_1 \\ 90 \times 0.5y_1 + 185 \times 0.5y_2 + 160 \times 0.5y_3 \geq v_1 \\ 181.25 \times 0.7y_1 + 90 \times 0.7y_2 + 185 \times 0.7y_3 \geq v_2 \\ 155 \times 0.8y_1 + 181.25 \times 0.8y_2 + 100 \times 0.8y_3 \geq v_2 \\ 90 \times 0.7y_1 + 185 \times 0.7y_2 + 160 \times 0.7y_3 \geq v_2 \\ v_2 \geq v_1 \\ y_1 + y_2 + y_3 = 1 \\ y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 \end{cases} \tag{44}$$

For $\lambda = 1/2$ , solving equation (44) by using the simplex method for linear programming, an optimal solution $(\mathbf{y}^*, v_1^*, v_2^*)$ can be obtained, where

$$\mathbf{y}^* = (0.185, 0.311, 0.504)^T$$

and

$$v_1^* = 77.39, v_2^* = 108.35.$$

Thus, the optimal strategy and the average index of the membership function and the average index of the non-membership function for player I's gain-floor are obtained as follows:

$$\mathbf{y}^* = (0.185, 0.311, 0.504)^T$$

and

$$v_1^* = 77.39, v_2^* = 108.35,$$

respectively.

Similarly, according to equation (43), the linear programming problem can be obtained as follows:

$$\min\{\lambda\omega_1 + (1-\lambda)\omega_2\}$$

$$\text{s.t.} \begin{cases} 181.25 \times 0.6z_1 + 155 \times 0.6z_2 + 90 \times 0.6z_3 \le \omega_1 \\ 90 \times 0.5z_1 + 181.25 \times 0.5z_2 + 185 \times 0.5z_3 \le \omega_1 \\ 185 \times 0.5z_1 + 100 \times 0.5z_2 + 160 \times 0.5z_3 \le \omega_1 \\ 181.25 \times 0.8z_1 + 155 \times 0.8z_2 + 90 \times 0.8z_3 \le \omega_2 \\ 90 \times 0.7z_1 + 181.25 \times 0.7z_2 + 185 \times 0.7z_3 \le \omega_2 \\ 185 \times 0.7z_1 + 100 \times 0.7z_2 + 160 \times 0.7z_3 \le \omega_2 \\ \omega_2 \ge \omega_1 \\ z_1 + z_2 + z_3 = 1 \\ z_1 \ge 0, z_2 \ge 0, z_3 \ge 0 \end{cases} \tag{45}$$

Similarly, for$\lambda = 1/2$, solving equation (45) by using the simplex method for linear programming, an optimal solution $(\mathbf{z}^*, \omega_1^*, \omega_2^*)$ of equation (45) can be obtained, where

$$\mathbf{z}^* = (0.299, 0.194, 0.507)^T$$

and

$$\omega_1^* = 77.39, \omega_2^* = 109.10.$$

Therefore, the optimal strategy and the average index of the membership function and the average index of the non-membership function for player II's loss-ceiling are obtained as follows:

$$\mathbf{z}^* = (0.299, 0.194, 0.507)^T$$

and

$$\omega_1^* = 77.39, \omega_2^* = 109.10,$$

respectively. It is easily seen that the value of the matrix game $\tilde{\mathbf{A}}$ with payoffs of Atanassov's TIFNs is

$$\tilde{V} = \mathbf{y}^{*T}\tilde{\mathbf{A}}\mathbf{z}^* =< (142.27, 151.0, 160.16); 0.5, 0.3 >,$$

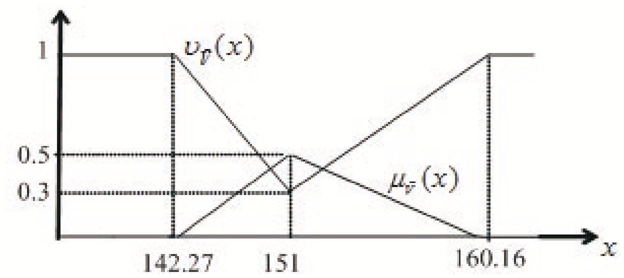


FIGURE 2. The Value of the Matrix Game with Payoffs of Atanassovs TIFNs

which is an Atanassov's TIFN and indicates that the sales amount of the company $p_1$ is approximately 151 when the companies $p_1$ and $p_2$ choose the mixed strategies

$$\mathbf{y}^* = (0.185, 0.311, 0.504)^T$$

and

$$\mathbf{z}^* = (0.299, 0.194, 0.507)^T,$$

respectively, depicted as in Fig. 2. For the manager of the company $p_1$ , the maximum confidence degree about the sales amount "approximately 151" is 0.5 while the minimum non-confidence degree about the sales amount "approximately 151" is 0.3, i.e., his/her hesitation degree about the sales amount "approximately 151" is 0.2. The results show that the Atanassov's IFS/IFN may express information more abundant and flexible than the fuzzy set when it is used to deal with uncertainty in game theory.

## 5. Conclusion

The matrix games with payoffs of Atanassov's TIFNs are formulated in the above. The arithmetic operations and ranking order relation of Atanassov's TIFNs are proposed. A pair of Atanassov's intuitionistic fuzzy optimization models for two players is derived from the definition of the solution for matrix games with payoffs of Atanassov's TIFNs and the arithmetic operations of Atanassov's TIFNs. Using the ranking order relation between Atanassov's TIFNs defined in this paper, the Atanassov's intuitionistic fuzzy optimization models for players are transformed into the bi-objective linear programming models, which are solved to determine optimal strategies for players and the value of the matrix game with payoffs of Atanassov's TIFNs. It is shown that the constructed linear programming models of matrix games with payoffs of Atanassov's TIFNs in this paper extend those of fuzzy matrix games with payoffs of triangular fuzzy numbers. Moreover, the proposed models and method are easy to be extended to matrix games with payoffs of trapezoidal intuitionistic fuzzy numbers. Although the models and method proposed in this paper are illustrated with the market share competition problem, it may also be applied to other similar game problems using the Atanassov's IFSs, which enable the satisfying strategies players can expect.

**Acknowledgements.** The authors would like to thank the valuable reviews and also appreciate the constructive suggestions from the anonymous referees. This research was supported by Program for New Century Excellent Talents in University (the Ministry of Education of China, NCET-10-0020), the Natural Science Foundation of China (Nos. 71171055, 71101033, 70871117, 70902041, 70901021 and 71001015), the Social Science Planning Project of Fujian (No. 2011C019), the Natural Science Guangxi of China (No. 2012GXNSFAA053013) and GLEU2001068.

## References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87-96.

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Springer-Verlag, Heidelberg, Germany, 1999.

[3] K. T. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **31** (1989), 343-349.

[4] K. T. Atanassov, *Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper Terminological difficulties in fuzzy set theory - the case of "intuitionistic fuzzy sets"*, Fuzzy Sets and Systems, **156** (2005), 496-499.

[5] C. R. Bector and S. Chandra, *Fuzzy mathematical programming and fuzzy matrix games*, Springer Verlag, Berlin, Germany, 2005.

[6] C. R. Bector, S. Chandra and V. Vijay, *Matrix games with fuzzy goals and fuzzy linear programming duality*, Fuzzy Optimization and Decision Making, **3** (2004), 255-269.

[7] C. R. Bector, S. Chandra and V. Vijay, *Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs*, Fuzzy Sets and Systems, **46(2)** (2004), 253-269.

[8] R. A. Borzooei and Y. B. Jun, *Intuitionistic fuzzy hyper bck-ideals of hyper bck-al gebras*, Iranian Journal of Fuzzy Systems, **1(1)** (2004), 65-78.

[9] P. Burillo and H. Bustince, *Vague sets are intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **79** (1996), 403-405.

[10] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems, **32** (1989), 275-289.

[11] L. Campos and A. Gonzalez, *Fuzzy matrix games considering the criteria of the players*, Kybernetes, **20** (1991), 17-23.

[12] L. Campos, A. Gonzalez and M. A. Vila, *On the use of the ranking function approach to solve fuzzy matrix games in a direct way*, Fuzzy Sets and Systems, **49** (1992), 193-203.

[13] S. K. De, R. Biswas and A. R. Roy, *An application of intuitionistic fuzzy sets in medical diagnosis*, Fuzzy Sets and Systems, **117** (2001), 209-213.

[14] G. Deschrijver, *Arithmetic operators in interval-valued fuzzy set theory*, Information Sciences, **177(14)** (2007), 2906-2924.

[15] G. Deschrijver and E. E. Kerre, *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems, **133** (2003), 227-235.

[16] G. Deschrijver and E. E. Kerre, *On the position of intuitionistic fuzzy set theory in the framework of theories modeling imprecision*, Information Sciences, **177** (2007), 1860-1866.

[17] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade, *Terminological difficulties in fuzzy set theory-the case of "Intuitionistic Fuzzy Sets"*, Fuzzy Sets and Systems, **156 (3)** (2005), 485-491.

[18] D. Dubois and H. Prade, *fuzzy sets and systems: theory and applications*, Mathematics in Science and Engineering, Academic Press, Berlin, Germany, **144** (1980).

[19] J. G. Garc and S. E. Rodabaugh, *Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies*, Fuzzy Sets and Systems, **156(3)** (2005), 445-484.

[20] W. L. Gau and D. J. Buehrer, *Vague sets*, IEEE Transaction on Systems, Man, and Cybernetics, **23** (1993), 610-614.

[21] J. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18** (1967), 145-174.

[22] E. E. Kerre and J. N. Mordeson, *A historical overview of fuzzy mathematicas*, New Mathematics and Natural Computation, **1(1)** (2005), 1-26.

[23] A. Khan, Y. B. Jun and M. Shabir, *Ordered semigroups characterized by their intuitionistic fuzzy bi-ideals,* Iranian Journal of Fuzzy Systems, **7(2)** (2010), 55-69.

[24] D. F. Li, *Fuzzy constrained matrix games with fuzzy payoffs*, The Journal of Fuzzy Mathematics, **7(4)** (1999), 873-880.

[25] D. F. Li, *A fuzzy multiobjective programming approach to solve fuzzy matrix games*, The Journal of Fuzzy Mathematics, **7(4)** (1999), 907-912.

[26] D. F. Li, *Multiattribute decision making models and methods using intuitionistic fuzzy sets*, Journal of Computer and System Sciences, **70** (2005), 73-85.

[27] D. F. Li, *A note on using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly*, Microelectronics Reliability, **48** (2008), 1741.

[28] D. F Li, *A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems*, Computers and Mathematics with Applications, **60(6)** (2010), 1557-1570.

[29] D. F. Li, *Representation of level sets and extension principles for Atanassov's intuitionistic fuzzy sets and algebraic operations*, Critical View, **4** (2010), 63-74.

[30] D. F. Li and C. T. Cheng, *New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions*, Pattern Recognition Letters, **23** (2002), 221-225.

[31] D. F. Li and J. X. Nan, *A nonlinear programming approach to matrix games with payoffs of Atanassov's intuitionistic fuzzy sets*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **17(4)** (2009), 585-607.

[32] S. T. Liu and C. Kao, *Solution of fuzzy matrix games: an application of the extension principle*, International Journal of Intelligent Systems, **22** (2007), 891-903.

[33] A. Maturo, *On some structures of fuzzy numbers*, Iranian Journal of Fuzzy Systems, **6(4)** (2009), 49-59.

[34] P. K. Nayak and M. Pal, *Bi-matrix games with intuitionistic fuzzy goals*, Iranian Journal of Fuzzy Systems, **7(1)** (2010), 65-79.

[35] I. Nishizaki and M. Sakawa, *Equilibrium solutions in multiobjective bimatrix games with fuzzy payoffs and fuzzy goals,* Fuzzy Sets and Systems, **111(1)** (2000), 99-116.

[36] I. Nishizaki and M. Sakawa, *Solutions based on fuzzy goals in fuzzy linear programming games*, Fuzzy Sets and Systems, **115(1)** (2000), 105-119.

[37] I. Nishizaki and M. Sakawa, *Fuzzy and multiobjective games for conflict resolution*, Physica-Verlag, Springer Verlag Company, Berlin, Germany, 2001.

[38] E. Pasha, A. Saiedifar and B. Asady, *The percentiles of fuzzy numbers and their applications*, Iranian Journal of Fuzzy Systems, **6(1)** (2009), 27-44.

[39] M. Sakawa and I. Nishizaki, *A lexicographical solution concept in an n-person cooperative fuzzy game*, Fuzzy Sets and Systems, **61** (1994), 265-275.

[40] M. Sakawa and I. Nishizaki, *Max-min solutions for fuzzy multiobjective matrix games*, Fuzzy Sets and Systems, **67** (1994), 53-69.

[41] E. Savas, *(A)-Double sequence spaces of fuzzy numbers via orlicz function*, Iranian Journal of Fuzzy Systems, **8(2)** (2011), 91-103.

[42] E. Szmidt and F. Baldwin, *Intuitionistic fuzzy set functions, mass assignment theory, possibility theory and histograms,* 2006 IEEE International Conference on Fuzzy Systems Sheraton Vancouver Wall Centre Hotel, Vancouver, BC, Canada, July 16-21, (2006), 35-41.

[43] G. Takeuti and S. Titani, *Intuitionistic fuzzy logic and intuitionistic fuzzy set theory*, The Journal of Symbolic Logic, **49(3)** (1984), 851-866.

[44] V. Vijay, S. Chandra and C. R. Bector, *Matrix games with fuzzy goals and fuzzy payoffs*, Omega, **33** (2005), 425-429.

[45] Z. S. Xu, *Models for multiple attribute decision making with intuitionistic fuzzy information*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **15** (2007), 285-297.
[46] Z. S. Xu, *Intuitionistic fuzzy aggregation operators*, IEEE Transactions on Fuzzy Systems, **15** (2007), 1179-1187.
[47] L. A. Zadeh, *Fuzzy sets,* Information and Control, **8** (1965), 338-356.
[48] L. A. Zadeh, *Outline of a new approach to the analysis of complex systems and decision processes interval-valued fuzzy sets*, IEEE Transactions on Systems, Man, and Cybernetics, **3** (1973), 28-44.

Deng-Feng Li, School of Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou 350108, Fujian, China
*E-mail address*: lidengfeng@fzu.edu.cn, dengfengli@sina.com

Jiang-Xia Nan*, School of Mathematics and Computing Sciences, Guilin University of Electronic Technology, Guilin, Guangxi 541004, China
*E-mail address*: nanjiangxia@guet.edu.cn

Zhen-Peng Tang, School of Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou 350108, Fujian, China
*E-mail address*: zhenpt@126.com

Ke-Jia Chen, School of Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou 350108, Fujian, China
*E-mail address*: kjchen@fzu.edu.cn

Xiao-Dong Xiang, School of Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou 350108, Fujian, China
*E-mail address*: xiangxiaodong2@yahoo.com.cn

Fang-Xuan Hong, School of Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou 350108, Fujian, China
*E-mail address*: hongfangxuan-2@163.com

*Corresponding author