FUZZY COLLOCATION METHODS FOR SECOND- ORDER FUZZY ABEL-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we intend to offer new numerical methods to solve the second-order fuzzy Abel-Volterra integro-differential equations under the generalized H-differentiability. The existence and uniqueness of the solution and convergence of the proposed methods are proved in details and the efficiency of the methods is illustrated through a numerical example.

1. Introduction

The fuzzy integral equations and the fuzzy integro-differential equations are very useful for solving many problems in several applied fields like mathematical economics, electrical engineering, medicine and biology and optimal control theory. Since these equations usually can not be solved explicitly, so it is required to obtain approximate solutions. There are numerous numerical methods which have been focusing on the solution of these equations [16, 14, 28, 7, 2, 3, 1, 20, 4, 5, 21, 23, 24, 18, 25, 26, 6, 27].

In this work, we develope the Jacobi polynomials and the Airfoil polynomials fuzzy collocation methods to solve the second- order fuzzy Abel-Volterra integro-differential equation of the second kind as follows:

$$\widetilde{u}^{''}(x) = \widetilde{f}(x) + \mu \int_a^x k(x,t) \frac{1}{\sqrt{x-t}}\ \widetilde{u}(t)\ dt, \tag{1}$$

with fuzzy initial conditions:

$$\widetilde{u}^r(a) = \widetilde{b_r}, \quad r = 0, 1. \tag{2}$$

Where a and μ are crisp constant values and $k(x,t)$ is function that has derivatives on an interval $a \le t \le x \le b$ and b_r are fuzzy constant values and $\widetilde{f}(x)$ is fuzzy function. The second- order Abel-Volterra integro-differential equations occur in various physical applications including heat transfer and viscoelasticity [10] .

Here is an outline of the paper. In section 2, the basic notations and definitions in fuzzy calculus are briefly presented . Section 3 describes how to find an approximate solution of the given second-order fuzzy Abel-Volterra integro-differential equations by using proposed methods. The existence and uniqueness of the solution and convergence of the proposed methods are proved in Section 4 respectively. Finally

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in section 5, we apply the proposed methods by an example to show the simplicity and efficiency of the methods, and a brief conclusion is given in Section 6.

2. **Preliminaries**

Here basic definitions of a fuzzy number are given as follows , [15, 22, 17, 29, 12, 8, 11]

Definition 2.1. An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:
(i) $\underline{u} : r \to u_r^- \in \mathbb{R}$ is a bounded left-continuous non-decreasing function over $[0, 1]$,
(ii) $\overline{u} : r \to u_r^+ \in \mathbb{R}$ is a bounded left-continuous non-increasing function over $[0, 1]$,
(iii) $\underline{u} \leq \overline{u}, \quad 0 \leq r \leq 1$.

Definition 2.2. For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E$, we use the distance (Hausdorff metric) [17]
$D(u(r), v(r)) = \max\{\sup_{r\in[0,1]} |\underline{u}(r) - \underline{v}(r)|, \sup\ |\overline{u}(r) - \overline{v}(r)|\}$,
and it is shown [8] that $(E$, D) is a complete metric space and the following properties are well known:

$$D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\ \widetilde{u}, \widetilde{v} \in E,$$
$$D(k\widetilde{u}, k\widetilde{v}) =| k \mid D(\widetilde{u}, \widetilde{v}), \forall\ k \in \mathbb{R}, \widetilde{u}, \widetilde{v} \in E,$$
$$D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}), \forall\ \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E.$$

Definition 2.3. Consider $\widetilde{x}$, $\widetilde{y} \in E$. If there exists $\widetilde{z} \in E$ such that $\widetilde{x} = \widetilde{y} + \widetilde{z}$ then $\widetilde{z}$ is called the H- difference of $\widetilde{x}$ and $\widetilde{y}$, and is denoted by $\widetilde{x} \ominus \widetilde{y}$, [8].

Definition 2.4. Let $\widetilde{f} : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $\widetilde{f}'(x_0) \in E$, such that [8]:

i) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0), \exists \widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{\widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)}{h} = \widetilde{f}'(x_0)$$

or

ii) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0) \ominus \widetilde{f}(x_0 + h), \exists \widetilde{f}(x_0 - h) \ominus \widetilde{f}(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{\widetilde{f}(x_0) \ominus \widetilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\widetilde{f}(x_0 - h) \ominus \widetilde{f}(x_0)}{-h} = \widetilde{f}'(x_0)$$

or

iii) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0), \exists \widetilde{f}(x_0 - h) \ominus \widetilde{f}(x_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{\widetilde{f}(x_0 + h) \ominus \widetilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\widetilde{f}(x_0 - h) \ominus \widetilde{f}(x_0)}{-h} = \widetilde{f}'(x_0)$$

or

iv) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0) \ominus \widetilde{f}(x_0 + h), \exists \widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)$ and the following limits hold:

$$\lim_{h \to 0} \frac{\widetilde{f}(x_0) \ominus \widetilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\widetilde{f}(x_0) \ominus \widetilde{f}(x_0 - h)}{h} = \widetilde{f}'(x_0).$$

Definition 2.5. Let $\widetilde{f} : (a, b) \to E$. We say $\widetilde{f}$ is (i)-differentiable on (a, b) if $\widetilde{f}$ is differentiable in the sense (i) of Definition (2.4) and similarly for (ii), (iii) and (iv) differentiability [11].

Definition 2.6. A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \geq 0, \beta \geq 0$ with

$$\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha}), & x \leq a, \\ R(\frac{x-b}{\beta}), & x \geq a, \end{cases}$$

the mean value of $\tilde{A}, a$ is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.7. Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,
$(1) : \lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$
$(2) : -\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$
$(3) : \tilde{M} \oplus \tilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$

$$(4) : \tilde{M} \odot \tilde{N} \simeq \begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}, & \tilde{M}, \tilde{N} > 0, \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR}, & \tilde{M} > 0, \tilde{N} < 0, \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR}, & \tilde{M}, \tilde{N} < 0. \end{cases}$$

Definition 2.8. A triangular fuzzy number is defined as a fuzzy set in E, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $[u]^r = [u_-^r, u_+^r]$ are the endpoints of r-level sets for all $r \in [0, 1]$, where $u_-^r = a + (b - a)r$ and $u_+^r = c - (c - b)r$. Here, $u_-^0 = a, u_+^0 = c, u_-^1 = u_+^1 = b$, which is denoted by u^1. The set of triangular fuzzy numbers will be denoted by E.

Definition 2.9. The mapping $\widetilde{f} : T \to E$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:

$$[f(t)]^r = [f_-^r(t), f_+^r(t)], \quad t \in T, \quad r \in [0, 1].$$

Definition 2.10. Let $\widetilde{f} : T \to E$ be Hukuhara differentiable and denote by $[f(t)]^r = [f_-^r, f_+^r]$. Then, the boundary function f_-^r and f_+^r are differentiable (or Seikkala differentiable) and

$$[f'(t)]^r = [(f_-^r)'(t), (f_+^r)'(t)], \quad t \in T, \quad r \in [0, 1].$$

3. Description of the Methods

In this section we are going to solve the equation (1) by using the Jacobi polynomials and the Airfoil polynomials fuzzy collocation methods.

To obtain the approximation solution of equation (1), based on Definition (2.4) there are two possible cases for each derivation, so there are four possible cases for the second order derivation. The conditions for the existence of solution in each case can now be started.

Theorem 3.1. *Let $x_0 \in [a,b]$ and assume that $\widetilde{f}$ is continuous. A mapping $u : [a,b] \to E$ is a solution to the initial value equation (1) if and if only $u^{'}$ and $u^{''}$ are continuous and satisfy on the following conditions*

(a)

$$\begin{aligned}&\widetilde{u}(x) = \int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\ &\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds + \widetilde{b}_1(x-x_0)+\widetilde{b}_0,\end{aligned} \tag{3}$$

where $\widetilde{u}^{'}$ and $\widetilde{u}^{''}$ are (i)-differentials, or

(b)

$$\begin{aligned}&\widetilde{u}(x) = \ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\ &\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]]+\widetilde{b}_0,\end{aligned} \tag{4}$$

where $\widetilde{u}^{'}$ and $\widetilde{u}^{''}$ are (ii)-differentials, or

(c)

$$\begin{aligned}&\widetilde{u}(x) = \ominus(-1)[\widetilde{b}_1(x-x_0)+\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\ &\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]+\widetilde{b}_0,\end{aligned} \tag{5}$$

where $\widetilde{u}^{'}$ is the (i)-differential and $\widetilde{u}^{''}$ is the (ii)-differential, or

(d)

$$\begin{aligned}&\widetilde{u}(x) = [\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\ &\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]]+\widetilde{b}_0\end{aligned} \tag{6}$$

where $\widetilde{u}^{'}$ is the (ii)-differential and $\widetilde{u}^{''}$ is the (i)-differential.

Proof. Since $\widetilde{f}$ is continuous, it must be integrable [9]. So,

$$\begin{aligned}&\widetilde{u}^{''}(x) = \widetilde{f}(x)+\mu\int_{a}^{x} k_1(x,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt,\\ &\widetilde{u}(x_0)=\widetilde{b}_0,\quad \widetilde{u}^{'}(x_0)=\widetilde{b}_1,\end{aligned} \tag{7}$$

can be written in each as follows:

(a) Let $\widetilde{u}^{'}$ and $\widetilde{u}^{''}$ be (i)-differentiable. So, for equation (3) we have equivalently

$$\widetilde{u}^{'}(x) = \int_{x_0}^{x} \widetilde{f}(s)\ ds + \mu\int_{x_0}^{x}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds + \widetilde{b}_1,$$

and thus,

$$\begin{aligned}&\widetilde{u}(x) = \int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds + \mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds\\ &+\widetilde{b}_1(x-x_0)+\widetilde{b}_0.\end{aligned}$$

(b) Let $\widetilde{u}^{'}$ and $\widetilde{u}^{''}$ be (ii)-differentiable. So, for equation(3) we have

$$\widetilde{u}^{'}(x) = \ominus(-1)[\int_{x_0}^{x} \widetilde{f}(s)\ ds + \mu\int_{x_0}^{x}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds] + \widetilde{b}_1,$$

and equivalently,

$$\begin{aligned}&\widetilde{u}(x) = \ominus(-1)[\widetilde{b}_1(x-x_0) \ominus (-1)[\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\&\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]] + \widetilde{b}_0.\end{aligned}$$

(c) Let $\widetilde{u}^{'}$ be (ii)-differentiable and $\widetilde{u}^{''}$ be (i)-differentiable. Then for equation (3), we have equivalently

$$\widetilde{u}^{'}(x) = \int_{x_0}^{x} \widetilde{f}(s)\ ds + \mu \int_{x_0}^{x}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds + \widetilde{b}_1,$$

so,

$$\begin{aligned}&\widetilde{u}(x) = \ominus(-1)[\widetilde{b}_1(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds\\&+\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds] + \widetilde{b}_0.\end{aligned}$$

(d) Let $\widetilde{u}^{'}$ be (i)-differentiable and $\widetilde{u}^{''}$ be (ii)-differentiable. Then for equation (3) we have equivalently

$$\widetilde{u}^{'}(x) = \ominus(-1)[\int_{x_0}^{x} \widetilde{f}(s)\ ds + \mu \int_{x_0}^{x}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds] + \widetilde{b}_1,$$

so,

$$\begin{aligned}&\widetilde{u}(x) = [\widetilde{b}_1(x-x_0) \ominus (-1)[\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds\\&+\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]] + \widetilde{b}_0,\end{aligned}$$

which completes the proof.

□

3.1. **Description of the Jacobli Polynomials Fuzzy Collocation Method.** To obtain the approximation solution of equation (1), according to the Jacobli polynomials method [19] we can write:

$$\widetilde{u}_n(x) = w(x)\sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(x), \quad \alpha,\beta > -1, \tag{8}$$

where,

$$\begin{aligned}&w(x) = \frac{(1-x)^{\alpha}}{(1+x)^{\beta}},\\&p_i^{\alpha,\beta}(x) = \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{(-2)^n\ n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}],\\&(p_i^{\alpha,\beta})^{'}(s) = \frac{1}{2}(n+\alpha+\beta+1)p_{n-1}^{(\alpha+1,\beta+1)}(s).\end{aligned} \tag{9}$$

Now from this method, we have four cases as follows:

Case (a):

$$\begin{aligned}&w(x)\sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(x) = \int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+\\&\mu \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds + \widetilde{b}_1(x-x_0) + \widetilde{b}_0.\end{aligned} \tag{10}$$

We can write equation (10) as follows:

$$\begin{aligned}&\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds + \mu\sum_{i=0}^{n} \widetilde{a}_i \int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} w(t)k(s,t)\frac{1}{\sqrt{s-t}} p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds+\\&\widetilde{b}_1(x-x_0) + \widetilde{b}_0 = w(x)\sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(x) + \widetilde{R}_n(x).\end{aligned} \tag{11}$$

So, we have

$$\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}w(t)k(s,t)\frac{1}{\sqrt{s-t}}p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds+ \\ \widetilde{b}_1(x-x_0)+\widetilde{b}_0\ominus w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)=\widetilde{R}_n(x). \tag{12}$$

$$\widetilde{R}_n(x_j)=\widetilde{0}. \tag{13}$$

It means,

$$\underline{R}_n^r(x_j)=0,\quad \overline{R}_n^r(x_j)=0,\quad \forall r\in[0,1],$$

where x_j $(j=1,...,n)$ are collocation points.
Therefore we can write,

$$\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds+ \\ \widetilde{b}_1(x_j-x_0)+\widetilde{b}_0\ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x_j)=\widetilde{0}. \tag{14}$$

Equation (14) can be written in the following operator form

$$\widetilde{F}\oplus A\widetilde{a}\ominus B\widetilde{a}=\widetilde{0}. \tag{15}$$

$$\widetilde{F}_j\oplus\sum_{a_{ij}\geq 0}a_{ij}\widetilde{a}_j\oplus\sum_{a_{ij}<0}a_{ij}\ominus(\sum_{b_{ij}\geq 0}b_{ij}\widetilde{a}_j\oplus\sum_{b_{ij}<0}b_{ij}\widetilde{a}_j)=\widetilde{0}.$$

Where,

$$(\widetilde{F})_j=\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\widetilde{b}_1(x_j-x_0)+\widetilde{b}_0, \\ (A)_{ij}=\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds, \\ (B)_{ij}=w(x_j)p_i^{\alpha,\beta}(x_j). \tag{16}$$

Case (b):

$$w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)=\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \\ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0. \tag{17}$$

We can write equation (17) as follows:

$$\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \\ \mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0= \\ w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)+\widetilde{R}_n(x). \tag{18}$$

So, we have

$$\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \\ \mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0 \\ \ominus w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)=\widetilde{R}_n(x). \tag{19}$$

Therefore we can write,

$$\ominus(-1)[\widetilde{b}_1(x_j-x_0)\ominus(-1)[\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \\ \mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0 \\ \ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x_j)=\widetilde{0}. \tag{20}$$

Equation (20) can be written in the following operator form

$$\widetilde{F} \oplus A\widetilde{a} \ominus B\widetilde{a} = \widetilde{0}. \tag{21}$$

$$\widetilde{F}_j \oplus \sum_{a_{ij}\geq 0} a_{ij}\widetilde{a}_j \oplus \sum_{a_{ij}<0} a_{ij} \ominus (\sum_{b_{ij}\geq 0} b_{ij}\widetilde{a}_j \oplus \sum_{b_{ij}<0} b_{ij}\widetilde{a}_j) = \widetilde{0}.$$

Where,

$$\begin{aligned}
&(\widetilde{F})_j = \ominus(-1)[\widetilde{b}_1(x-x_0) \ominus (-1)[\textstyle\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds] + \widetilde{b}_0,\\
&(A)_{ij} = \mu \textstyle\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j} w(t)k(s,t)\frac{1}{\sqrt{s-t}} p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds,\\
&(B)_{ij} = w(x_j)p_i^{\alpha,\beta}(x_j).
\end{aligned} \tag{22}$$

Case (c)

$$\begin{aligned}
&w(x)\textstyle\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x) = \ominus(-1)[\widetilde{b}_1(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds +\\
&\mu\textstyle\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds] + \widetilde{b}_0.
\end{aligned} \tag{23}$$

We can write equation (23) as follows:

$$\begin{aligned}
&\ominus(-1)[\widetilde{b}_1(x-x_0) + \textstyle\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds +\\
&\mu\textstyle\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds] + \widetilde{b}_0 =\\
&w(x)\textstyle\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x) + \widetilde{R}_n(x).
\end{aligned} \tag{24}$$

So, we have

$$\begin{aligned}
&\ominus(-1)[\widetilde{b}_1(x-x_0) + \textstyle\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds +\\
&\mu\textstyle\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds] + \widetilde{b}_0\\
&\ominus w(x)\textstyle\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x) = \widetilde{R}_n(x).
\end{aligned} \tag{25}$$

Therefore we can write,

$$\begin{aligned}
&\ominus(-1)[\widetilde{b}_1(x_j-x_0) + \textstyle\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds +\\
&\mu\textstyle\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds] + \widetilde{b}_0\\
&\ominus w(x_j)\textstyle\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x_j) = \widetilde{0}.
\end{aligned} \tag{26}$$

Equation (26) can be written in the following operator form

$$\widetilde{F} \ominus A\widetilde{a} \ominus B\widetilde{a} = \widetilde{0}. \tag{27}$$

$$\widetilde{F}_j \ominus (\sum_{a_{ij}\geq 0} a_{ij}\widetilde{a}_j \oplus \sum_{a_{ij}<0} a_{ij}) \ominus (\sum_{b_{ij}\geq 0} b_{ij}\widetilde{a}_j \oplus \sum_{b_{ij}<0} b_{ij}\widetilde{a}_j) = \widetilde{0}.$$

Where,

$$\begin{aligned}
&(\widetilde{F})_j = \ominus(-1)[\widetilde{b}_1(x-x_0) + \textstyle\int_{x_0}^{x}\int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds] + \widetilde{b}_0,\\
&(A)_{ij} = \mu \textstyle\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j} w(t)k(s,t)\frac{1}{\sqrt{s-t}} p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds,\\
&(B)_{ij} = w(x_j)p_i^{\alpha,\beta}(x_j).
\end{aligned} \tag{28}$$

Case (d)

$$\begin{aligned}
&w(x)\textstyle\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x) = [\widetilde{b}_1(x-x_0) \ominus (-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds +\\
&\mu\textstyle\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x} k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]] + \widetilde{b}_0.
\end{aligned} \tag{29}$$

We can write equation (29)

$$[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0=$$
$$w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)+\widetilde{R}_n(x). \tag{30}$$

So, we have

$$[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0$$
$$\ominus w(x)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x)=\widetilde{R}_n(x). \tag{31}$$

Therefore we can write,

$$[\widetilde{b}_1(x_j-x_0)\ominus(-1)[\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(t))\ dt\ ds\ ds]]+\widetilde{b}_0$$
$$\ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i p_i^{\alpha,\beta}(x_j)=\widetilde{0}. \tag{32}$$

Equation (32) can be written in the following operator form

$$\widetilde{F}\oplus A\widetilde{a}\ominus B\widetilde{a}=\widetilde{0}. \tag{33}$$

$$\widetilde{F}_j\oplus\sum_{a_{ij}\geq 0}a_{ij}\widetilde{a}_j\oplus\sum_{a_{ij}<0}a_{ij}\ominus(\sum_{b_{ij}\geq 0}b_{ij}\widetilde{a}_j\oplus\sum_{b_{ij}<0}b_{ij}\widetilde{a}_j)=\widetilde{0}.$$

Where,

$$(\widetilde{F})_j=\widetilde{b}_1(x-x_0)\ominus(-1)\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\widetilde{b}_0,$$
$$(A)_{ij}=\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}p_i^{\alpha,\beta}(t)\ \ dt\ ds\ ds,$$
$$(B)_{ij}=w(x_j)p_i^{\alpha,\beta}(x_j). \tag{34}$$

3.2. **Description of the Airfoil polynomials fuzzy collocation method.** To obtain the approximation solution of equation (1), according to the airfoil polynomials method of the first kind [13] we can write:

$$\widetilde{x}_n(x)=w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x), \tag{35}$$

where

$$w(x)=\sqrt{\frac{1+x}{1-x}},$$
$$t_i(x)=\frac{\cos[(\frac{1}{2})\arccos x]}{\cos(\frac{1}{2}\arccos x)},$$
$$u_i(x)=\frac{\sin[(\frac{1}{2})\arcsin x]}{\cos(\frac{1}{2}\arcsin x)},$$
$$(1+x)t_i'(x)=(i+\frac{1}{2})u_i(x)-\frac{1}{2}t_i(x). \tag{36}$$

Now from this method, we have four cases as follows:

Case (a):

$$w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x)=\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds+\widetilde{b}_1(x-x_0)+\widetilde{b}_0. \tag{37}$$

We can write equation (37) as follows:

$$\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds+ \widetilde{b}_1(x-x_0)+\widetilde{b}_0=w(x)\sum_{i=0}^{n}\widetilde{a}_ip_i^{\alpha,\beta}(x)+\widetilde{R}_n(x). \tag{38}$$

So, we have

$$\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds+ \widetilde{b}_1(x-x_0)+\widetilde{b}_0\ominus w(x)\sum_{i=0}^{n}\widetilde{a}_ip_i^{\alpha,\beta}(x)=\widetilde{R}_n(x), \tag{39}$$

$$\widetilde{R}_n(x_j)=\widetilde{0}. \tag{40}$$

It means,

$$\underline{R}_n^r(x_j)=0,\quad \overline{R}_n^r(x_j)=0,\quad \forall r\in[0,1],$$

where $x_j\ (j=1,...,n)$ are collocation points.

$$x_j=-\cos\frac{2i-1}{2n+3}positiveimplicative,\ j=0,1,...,n.$$

Therefore we can write,

$$\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds+ \widetilde{b}_1(x_j-x_0)+\widetilde{b}_0\ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_it_i(x_j)=\widetilde{0}. \tag{41}$$

Equation (41) can be written in the following operator form

$$\widetilde{F}\oplus A\widetilde{a}\ominus B\widetilde{a}=\widetilde{0}. \tag{42}$$

$$\widetilde{F}_j\oplus\sum_{a_{ij}\geq 0}a_{ij}\widetilde{a}_j\oplus\sum_{a_{ij}<0}a_{ij}\ominus(\sum_{b_{ij}\geq 0}b_{ij}\widetilde{a}_j\oplus\sum_{b_{ij}<0}b_{ij}\widetilde{a}_j)=\widetilde{0}.$$

Where,

$$\begin{aligned}&(\widetilde{F})_j=\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\widetilde{b}_1(x_j-x_0)+\widetilde{b}_0,\\&(A)_{ij}=\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds,\\&(B)_{ij}=w(x_j)t_i(x_j).\end{aligned} \tag{43}$$

Case (b):

$$w(x)\sum_{i=0}^{n}\widetilde{a}_it_i(x)=\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_it_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0. \tag{44}$$

We can write equation (44) as follows:

$$\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)t_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0= w(x)\sum_{i=0}^{n}\widetilde{a}_it_i(x)+\widetilde{R}_n(x). \tag{45}$$

So, we have

$$\ominus(-1)[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_it_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0 \ominus w(x)\sum_{i=0}^{n}\widetilde{a}_it_i(x)=\widetilde{R}_n(x). \tag{46}$$

Therefore we can write,

$$\ominus(-1)[\widetilde{b}_1(x_j - x_0) \ominus (-1)[\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\sum_{i=0}^{n}\widetilde{a}_i\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds]] + \widetilde{b}_0$$
$$\ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i t_i(x_j) = \widetilde{0}. \quad (47)$$

Equation (47) can be written in the following operator form

$$\widetilde{F} \oplus A\widetilde{a} \ominus B\widetilde{a} = \widetilde{0}, \quad (48)$$

$$\widetilde{F}_j \oplus \sum_{a_{ij}\geq 0} a_{ij}\widetilde{a}_j \oplus \sum_{a_{ij}<0} a_{ij} \ominus (\sum_{b_{ij}\geq 0} b_{ij}\widetilde{a}_j \oplus \sum_{b_{ij}<0} b_{ij}\widetilde{a}_j) = \widetilde{0}.$$

Where,

$$(\widetilde{F})_j = \ominus(-1)[\widetilde{b}_1(x - x_0) \ominus (-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds] + \widetilde{b}_0,$$
$$(A)_{ij} = \mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j} w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds,$$
$$(B)_{ij} = w(x_j)p_i^{\alpha,\beta}(x_j). \quad (49)$$

Case (c)

$$w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x) = \ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds] + \widetilde{b}_0. \quad (50)$$

We can write equation (50) as follows:

$$\ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds] + \widetilde{b}_0 =$$
$$w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x) + \widetilde{R}_n(x). \quad (51)$$

So, we have

$$\ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds] + \widetilde{b}_0$$
$$\ominus w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x) = \widetilde{R}_n(x). \quad (52)$$

Therefore we can write,

$$\ominus(-1)[\widetilde{b}_1(x_j - x_0) + \int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+$$
$$\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds] + \widetilde{b}_0$$
$$\ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i t_i(x_j) = \widetilde{0}. \quad (53)$$

Equation (53) can be written in the following operator form

$$\widetilde{F} \ominus A\widetilde{a} \ominus B\widetilde{a} = \widetilde{0}, \quad (54)$$

$$\widetilde{F}_j \ominus (\sum_{a_{ij}\geq 0} a_{ij}\widetilde{a}_j \oplus \sum_{a_{ij}<0} a_{ij}) \ominus (\sum_{b_{ij}\geq 0} b_{ij}\widetilde{a}_j \oplus \sum_{b_{ij}<0} b_{ij}\widetilde{a}_j) = \widetilde{0}.$$

Where,

$$(\widetilde{F})_j = \ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds] + \widetilde{b}_0,$$
$$(A)_{ij} = \mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j} w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds,$$
$$(B)_{ij} = w(x_j)t_i(x_j). \quad (55)$$

Case (d)

$$w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x)=[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0. \quad (56)$$

We can write equation (56)

$$[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0= w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x)+\widetilde{R}_n(x). \quad (57)$$

So, we have

$$[\widetilde{b}_1(x-x_0)\ominus(-1)[\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0 \ominus w(x)\sum_{i=0}^{n}\widetilde{a}_i t_i(x)=\widetilde{R}_n(x). \quad (58)$$

Therefore we can write,

$$[\widetilde{b}_1(x_j-x_0)\ominus(-1)[\int_{x_0}^{x_j}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+ \mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}k(s,t)\frac{1}{\sqrt{s-t}}\ (w(t)\sum_{i=0}^{n}\widetilde{a}_i t_i(t))\ dt\ ds\ ds]]+\widetilde{b}_0 \ominus w(x_j)\sum_{i=0}^{n}\widetilde{a}_i t_i(x_j)=\widetilde{0}. \quad (59)$$

Equation (59) can be written in the following operator form

$$\widetilde{F}\oplus A\widetilde{a}\ominus B\widetilde{a}=\widetilde{0}, \quad (60)$$

$$\widetilde{F}_j\oplus\sum_{a_{ij}\geq 0}a_{ij}\widetilde{a}_j\oplus\sum_{a_{ij}<0}a_{ij}\ominus(\sum_{b_{ij}\geq 0}b_{ij}\widetilde{a}_j\oplus\sum_{b_{ij}<0}b_{ij}\widetilde{a}_j)=\widetilde{0}.$$

Where,

$$(\widetilde{F})_j=\widetilde{b}_1(x-x_0)\ominus(-1)\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\widetilde{b}_0,$$
$$(A)_{ij}=\mu\int_{x_0}^{x_j}\int_{x_0}^{s}\int_{a}^{x_j}w(t)k(s,t)\frac{1}{\sqrt{s-t}}t_i(t)\ \ dt\ ds\ ds,$$
$$(B)_{ij}=w(x_j)t_i(x_j). \quad (61)$$

4. Convergence Analysis

In this section we are going to prove the existence and uniqueness of the solution and convergence of the proposed methods by using the following assumptions,

Let,
$$|\ \mu\frac{k(x,t)}{\sqrt{x-t}}\ |\leq M_1.$$
$$\alpha_1=(b-a)^3M_1.$$

According to the equation (13) we have $\underline{R}_n^r=\overline{R}_n^r=0$, so, $D(\widetilde{R}_n,\widetilde{0})\to 0$. Therefore we can write

$$D(\widetilde{a},\widetilde{a}^{(n)})\leq c_1 n^{-z},\quad z>1.$$

Lemma 4.1. *If $\widetilde{u},\widetilde{v},\widetilde{w}\in E^n$ and $\lambda\in R$, then,*
(i) $D(\widetilde{u}\ominus\widetilde{v},\widetilde{u}\ominus\widetilde{w})=D(\widetilde{v},\widetilde{w})$,
(ii) $D(\ominus\lambda\widetilde{u},\ominus\lambda\widetilde{v})=|\ \lambda\ |\ D(\widetilde{u},\widetilde{v})$.

Proof. By the definition of D, we have,

$$D(\widetilde{u}\ominus\widetilde{v},\widetilde{u}\ominus\widetilde{w})=\max\{\sup_{r\in[0,1]}\mid\underline{u(r)-v(r)}-\underline{u(r)-w(r)}\mid,\\ \sup_{r\in[0,1]}\mid\overline{u(r)-v(r)}-\overline{u(r)-w(r)}\mid\}=\\ \max\{\sup_{r\in[0,1]}\mid(\underline{u}(r)-\underline{v}(r))-(\underline{u}(r)-\underline{w}(r)\mid,\\ \sup_{r\in[0,1]}\mid(\overline{u}(r)-\overline{v}(r))-(\overline{u}(r)-\overline{w}(r))\mid\}=\\ \max\{\sup_{r\in[0,1]}\mid\underline{w}(r)-\underline{v}(r)\mid,\\ \sup_{r\in[0,1]}\mid\overline{w}(r)-\overline{v}(r)\mid\}=\\ \max\{\sup_{r\in[0,1]}\mid\underline{v}(r)-\underline{w}(r)\mid,\sup_{r\in[0,1]}\mid\overline{v}(r)-\overline{w}(r)\mid\}=D(\widetilde{v},\widetilde{w}).$$

□

Proof.

$$D(\ominus\lambda\widetilde{u},\ominus\lambda\widetilde{v})=\max\{\sup_{r\in[0,1]}\mid\underline{\lambda u(r)}-\underline{\lambda v(r)}\mid,\\ \sup_{r\in[0,1]}\mid\overline{\lambda u(r)}-\overline{\lambda v(r)}\mid\}=\\ \max\{\sup_{r\in[0,1]}\mid\underline{\lambda u(r)}-\underline{\lambda v(r)}\mid,\\ \sup_{r\in[0,1]}\mid\overline{\lambda u(r)}-\overline{\lambda v(r)}\mid\}=D(\lambda\widetilde{u},\lambda\widetilde{v})=\\ \mid\lambda\mid D(\widetilde{u},\widetilde{v}).$$

□

Theorem 4.2. *Let $0<\alpha_1<1$, then problem (1), has a unique solution when $\widetilde{u}^{'}(x)$ is (ii)-differentiable and $\widetilde{u}^{''}(x)$ is (i)-differentiable.*

Proof. Let $\widetilde{u}$ and $\widetilde{u}^*$ be two different solutions of equation (1) then

$$D(\widetilde{u}(x),\widetilde{u}^*(x))=D(\ominus(-1)[\widetilde{b}_1(x-x_0)\\ +\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\\ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]+\widetilde{b}_0,\\ \ominus(-1)[\widetilde{b}_1(x-x_0)+\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds+\\ \mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}^*(t)\ dt\ ds\ ds]+\widetilde{b}_0)\\ =D(\ominus(-1)[\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds]\\ ,\ominus(-1)[\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\ \widetilde{u}^*(t)\ dt\ ds\ ds]\\ \le(b-a)^3M_1\ D(\widetilde{u}(x),\widetilde{u}^*(x))=\alpha_1\ D(\widetilde{u}(x),\widetilde{u}^*(x)).$$

From which we get $(1-\alpha_1)D(\widetilde{u}(x),\widetilde{u}^*(x))\le 0$. Since $0<\alpha_1<1$, then $D(\widetilde{u}(x),\widetilde{u}^*(x))=0$. Implies $\widetilde{u}(x)=\widetilde{u}^*(x)$ and completes the proof.

□

Theorem 4.3. *Suppose that $\widetilde{F}:[a,b]\to E$ with $D(\widetilde{F}(x),\widetilde{0})\le M$ and*

$$\mu\int_{x_0}^{x}\int_{x_0}^{s}\int_{a}^{x}k(s,t)\frac{1}{\sqrt{s-t}}\le L_1,\quad L_1<1,$$

$$F(x)=\ominus(-1)[\widetilde{b}_1(x-x_0)+\int_{x_0}^{x}\int_{x_0}^{s}\widetilde{f}(s)\ ds\ ds]+\widetilde{b}_0.$$

Then the solution of the equation (1) is bounded.

Proof. Let $\widetilde{u}(x)$ be an unbounded solution of equation (1). Then for every $r>0$, there exists an element $x_1\in[a,b]$, such that

$$D(\widetilde{u}(x),\widetilde{0})<r,\quad \forall t\in[a,x_1],\\ D(\widetilde{u}(x_1),\widetilde{0})=r. \tag{62}$$

Clearly, we can find a positive number r with

$$M + r(L_1 + L_2) < r. \tag{63}$$

By equation (62) and equation (63), and the assumptions of the theorem we have

$$\begin{aligned}
&r = D(\widetilde{u}(x_1), \widetilde{0}) = D(\ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x} \int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds+ \\
&\mu \int_{x_0}^{x} \int_{x_0}^{s} \int_{a}^{x} k(s,t) \tfrac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds] + \widetilde{b}_0, \widetilde{0}) \\
&\leq D(\ominus(-1)[\widetilde{b}_1(x - x_0) + \int_{x_0}^{x} \int_{x_0}^{s} \widetilde{f}(s)\ ds\ ds] + \widetilde{b}_0, \widetilde{0})+ \\
&D(\ominus(-1)[\mu \int_{x_0}^{x} \int_{x_0}^{s} \int_{a}^{x} k(s,t) \tfrac{1}{\sqrt{s-t}}\ \widetilde{u}(t)\ dt\ ds\ ds], \widetilde{0}) \leq M + r(L_1 + L_2) < r,
\end{aligned}$$

which is a contraction. Thus $u(x)$ is bounded. Therefore, we can write $D(\widetilde{a}^{(n)}, \widetilde{0}) \leq c_2\ j^{-z},\ z > 1$.

□

Remark 4.4. The proof of other cases is similar to the previous theorems.

Theorem 4.5. *The maximum absolute truncation error of the solution* $\widetilde{u}_n(x) = w(x)\sum_{i=0}^{n} \widetilde{a}_i p_i^{\alpha,\beta}(x)$ *to equation (1) by using the Jacobi polynomials fuzzy collocation method is estimated to be*

$$D(E_j(x), \widetilde{0}) \leq c_2 n^{-z+1} + c_1 \frac{n^{-z+1}}{1-z}.$$

Proof. We have,

$$\begin{aligned}
&D(\widetilde{u}(x), \widetilde{u}_n(x)) = D(\textstyle\sum_{j=0}^{\infty} \widetilde{a}_j p_j^{\alpha,\beta}, \sum_{j=0}^{n} \widetilde{a}_j^{(n)} p_j^{\alpha,\beta}) = \\
&D(\textstyle\sum_{j=0}^{n} \widetilde{a}_j p_j^{\alpha,\beta} + \sum_{j=0}^{n} \widetilde{a}_j^{(n)} p_j^{\alpha,\beta}, \sum_{j=n+1}^{\infty} \widetilde{a}_j p_j^{\alpha,\beta}) \\
&\leq D(\textstyle\sum_{j=0}^{n} \widetilde{a}_j p_j^{\alpha,\beta}, \sum_{j=0}^{n} \widetilde{a}_j^{(n)} p_j^{\alpha,\beta}) + D(\sum_{j=n+1}^{\infty} \widetilde{a}_j p_j^{\alpha,\beta}, \widetilde{0}) \\
&\leq \textstyle\sum_{j=0}^{n} \mid p_j^{\alpha,\beta} \mid D(\widetilde{a}_j, \widetilde{a}_j^{(n)}) + \sum_{j=n+1}^{\infty} \mid p_j^{\alpha,\beta} \mid D(\widetilde{a}_j, \widetilde{0}) \\
&\leq \textstyle\sum_{j=0}^{n} D(\widetilde{a}_j, \widetilde{a}_j^{(n)}) + \sum_{j=n+1}^{\infty} D(\widetilde{a}_j, \widetilde{0}) \leq c_2 n^{-z+1} + c_1 \frac{n^{-z+1}}{1-z}.
\end{aligned}$$

□

Remark 4.6. The maximum absolute truncation error of the solution $\widetilde{u}_n(x) = w(x)\sum_{i=0}^{n} \widetilde{a}_i t_i(x)$ to equation (1) by using the Airfoil polynomials fuzzy collocation method is similar to the Jacobi polynomials fuzzy collocation method.

5. **Numerical Example**

In this section, we solve the second-order fuzzy Abel-Volterra integro-differential equation by using the Jacobi polynomials and the Airfoil polynomials fuzzy collocation methods. The program has been provided with Mathematica 6.

Algorithm 5.1. Step 1. Set $n \leftarrow 0$.

Step 2. Solve the systems (15) or (21) or (27) or (33) or (42) or (48) or (55) or (60).

Step 3. If $D(\widetilde{x}_{n+1}(s), \widetilde{x}_n(s)) < \varepsilon$ then go to step 4,
else $n \leftarrow n + 1$ and go to step 2.

Step 4. Print $\widetilde{x}_n(s)$ as the approximation of the exact solution.

r	$(\underline{u}, n = 6, x = 0.4)$	$(\overline{u}, n = 6, x = 0.4)$
0.0	0.1935645	0.5421619
0.1	0.2033578	0.5328423
0.2	0.2137553	0.5254231
0.3	0.2356409	0.4937533
0.4	0.2535644	0.4876537
0.5	0.2627612	0.4617815
0.6	0.2849328	0.4565963
0.7	0.2955877	0.4463437
0.8	0.3269507	0.4372477
0.9	0.3436238	0.4166234
1.0	0.3539719	0.3539719

TABLE 1. Case 1: Numerical Results for Example 5.2 by Using the Jacobi Polynomials Fuzzy Collocation Method

$$D(E_j(s), \widetilde{0}) \leq 0.000617.$$

r	$(\underline{u}, n = 5, x = 0.4)$	$(\overline{u}, n = 5, x = 0.4)$
0.0	0.1577312	0.4943681
0.1	0.1633476	0.4839655
0.2	0.1754673	0.4766359
0.3	0.1928417	0.4627358
0.4	0.2127327	0.4544752
0.5	0.2255881	0.4309799
0.6	0.2459925	0.4147448
0.7	0.2587896	0.4064228
0.8	0.2886562	0.3808357
0.9	0.2953816	0.3762249
1.0	0.3159559	0.3159559

TABLE 2. Case 2: Numerical Results for Example 5.2 by Using the Jacobi Polynomials Fuzzy Collocation Method

$$D(E_j(s), \widetilde{0}) \leq 0.000623.$$

Example 5.2. Consider the fuzzy Abel-Volterra integro-differential equation as follows:

$$u^{''}(x) = \widetilde{f}(x) + \frac{1}{7}\int_0^x \frac{(s^2 + t^3)}{\sqrt{x - t}}\widetilde{u}(t)\ dt,$$

where,

$$\begin{aligned}
&\widetilde{u}(0) = (0.01, 0.03, 0.06),\\
&\widetilde{u}^{'}(0) = (0.05, 0.07, 0.08).\\
&\widetilde{f}(x) = (x^2, x^2 + 3, x^2 + 4),\\
&\alpha_1 = 0.64574,\\
&\varepsilon = 10^{-4},\\
&\alpha = \frac{-1}{4},\\
&\beta = \frac{-1}{3}.
\end{aligned}$$

r	$(\underline{u}, n=6, x=0.4)$	$(\overline{u}, n=6, x=0.4)$
0.0	0.4124319	0.7565522
0.1	0.4227476	0.7933488
0.2	0.4317337	0.7821096
0.3	0.4546829	0.7682413
0.4	0.4774387	0.7525178
0.5	0.4966477	0.7374507
0.6	0.5049772	0.7131734
0.7	0.5183665	0.7048578
0.8	0.5596697	0.6581885
0.9	0.5647674	0.6676291
1.0	0.5848555	0.5848555

TABLE 3. Case 3: Numerical Results for Example 5.1 by Using the Jacobi Polynomials Fuzzy Collocation Method $D(E_j(s), \widetilde{0}) \leq 0.000615$.

r	$(\underline{u}, n=5, x=0.4)$	$(\overline{u}, n=5, x=0.4)$
0.0	0.4536459	0.7633915
0.1	0.4853429	0.7746625
0.2	0.4948536	0.7635628
0.3	0.5135407	0.7467317
0.4	0.5343805	0.7326215
0.5	0.5464581	0.7167859
0.6	0.5643269	0.6936831
0.7	0.5754317	0.6886649
0.8	0.6183408	0.6627322
0.9	0.6254705	0.6473396
1.0	0.6367584	0.6367584

TABLE 4. Case 4: Numerical Results for Example 5.1 by Using the Jacobi Polynomials Fuzzy Collocation Method $D(E_j(s), \widetilde{0}) \leq 0.000619$.

r	$(\underline{u}, n=5, x=0.4)$	$(\overline{u}, n=5, x=0.4)$
0.0	0.4536659	0.8237753
0.1	0.4639654	0.8353875
0.2	0.4725576	0.8268798
0.3	0.4915672	0.8005547
0.4	0.5129312	0.7965816
0.5	0.5244175	0.7742209
0.6	0.5436553	0.7579328
0.7	0.5549394	0.7465385
0.8	0.5923296	0.7254818
0.9	0.6061419	0.7078913
1.0	0.6149259	0.6149259

TABLE 5. Case 5: Numerical results for Example 5.1 by using the Airfoil polynomials fuzzy collocation method $D(E_j(s), \widetilde{0}) \leq 0.000605$.

r	$(\underline{u}, n=7, x=0.4)$	$(\overline{u}, n=7, x=0.4)$
0.0	0.2727442	0.6237379
0.1	0.2838695	0.6114823
0.2	0.2946542	0.6065117
0.3	0.3122321	0.5813541
0.4	0.3372304	0.5752621
0.5	0.3440264	0.5551278
0.6	0.3644813	0.5334554
0.7	0.3743292	0.5267315
0.8	0.3889636	0.5117462
0.9	0.4125638	0.4869326
1.0	0.4346738	0.4346738

TABLE 6. Case 6: Numerical Results for Example 5.1 by Using the Airfoil Polynomials Fuzzy Collocation Method $D(E_j(s), \widetilde{0}) \leq 0.000627$.

r	$(\underline{u}, n=6, x=0.4)$	$(\overline{u}, n=6, x=0.4)$
0.0	0.4323658	0.7939852
0.1	0.4435544	0.8021505
0.2	0.4544919	0.7965833
0.3	0.4756853	0.7749469
0.4	0.4967187	0.7652672
0.5	0.5045259	0.7432655
0.6	0.5258227	0.7268543
0.7	0.5372188	0.7195358
0.8	0.5687408	0.6912687
0.9	0.5772653	0.6779573
1.0	0.5947186	0.5947186

Table 7. Case 7: Numerical results for Example 5.1 by using the Airfoil polynomials fuzzy collocation method $D(E_j(s), \widetilde{0}) \leq 0.000615$.

r	$(\underline{u}, n=5, x=0.4)$	$(\overline{u}, n=5, x=0.4)$
0.0	0.5632436	92456558
0.1	0.5729682	0.9357607
0.2	0.5831844	0.9252543
0.3	0.6051448	0.9045749
0.4	0.6247616	0.8578513
0.5	0.6365807	0.8742658
0.6	0.6549709	0.8552168
0.7	0.6657538	0.8465313
0.8	0.6888638	0.8176859
0.9	0.7083558	0.7933837
1.0	0.7273019	0.7273019

Table 8. Case 8: Numerical Results for Example 5.1 by Using the Airfoil Polynomials Fuzzy Collocation Method $D(E_j(s), \widetilde{0}) \leq 0.000602$.

6. **Conclusion**

In this study, the Jacobi polynomials and the Airfoil polynomials fuzzy collocation method have been presented to solve the fuzzy Abel- Volterra integro-differential equation. These methods have been successfully employed to obtain the approximate solution of the fuzzy Abel-Volterra integro-differential equation under generalized H-differentiability. We can use these methods to solve another nonlinear fuzzy problems such as fuzzy partial differential equations and fuzzy integral equations.

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