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(T) FUZZY INTEGRAL OF MULTI-DIMENSIONAL FUNCTION WITH RESPECT TO MULTI-VALUED MEASURE

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Abstract. Introducing more types of integrals will provide more choices to deal with various types of objectives and components in real problems. Firstly, in this paper, a (T) fuzzy integral, in which the integrand, the measure and the integration result are all multi-valued, is presented with the introduction of $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm. Then, some classical results of the integral are obtained based on the properties of $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm mainly. The presented integral can act as an aggregation tool which is especially useful in many information fusing and data mining problems such as classification and programming.

1. Introduction

Triangular norms (briefly t-norms i.e., t-norm and t-conorm) were introduced by Schweizer and Sklar [21] as $[0,1]^2 \to [0,1]$ commutative nondecreasing associative mappings with the neutral element 1 to model the triangle inequality in the framework of statistical (probabilistic) metric spaces. Later, some scholars made a further research on its properties [4, 10-13, 15] and generalized the definition of triangular norms [3, 24, 33]. To resolve some multi-dimensional problems, Liu, Song and Zhang [14] first presented $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm.

In practice, Zadeh's conventional t-norm and t-conorm, $\wedge$ and $\vee$, have been used in almost every design for fuzzy logic controllers and even in the modeling of other decision-making processes. For example, Sugeno integral which was first introduced in fuzzy integral thorems used the operators $\wedge$ and $\vee$. However, some theoretical and experimental studies seem to indicate that other types of t-norm and t-conorm may work better in some situations, especially in the context of decision-making processes. Based on this, many authors generalized the Sugeno integral by using some other operators to replace the special operator(s) $\wedge$ and $\vee$ (see, e.g., [16, 17, 28, 33-35]). The integrand functions of all these integrals are one-dimensional. Nevertheless, these can not meet many practical application, especially in economics fields, control and many other fields. In this aspect, since

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Aumann [2] brought out the concept of the integrals of set-valued functions in 1965, the theory had been made deeper by many other scholars. Among them, Zhang et al. [30, 31] built up a relatively complete theory of fuzzy integral of set-valued functions, Jang et al. [9] and Zhang et al. [32] studied set-valued Choquet integrals, Yang et al. [29] introduced the Choquet integral with a measurable interval-valued integrand, A. Gavrilut et al. [5-7, 19] discussed a Gould type integral with respect to multimeasure and multisubmeasure. Liu, Song and Zhang [14] investigated (T) Fuzzy integral of multi-dimensional function, in which the integrand is multi-valued and the measure is scalar-valued.

Any specified fuzzy integral can be used as an aggregation tool in information fusion, so introducing more types of integrals will provide more choices to deal with various types of objectives and components in real problems. This paper aims at defining and investigating a new type of fuzzy integral, called (T) fuzzy integral based on the concepts of $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm presented by Liu, Song and Zhang [14]. For this integral, the integrand, the measure and the integration result are all multi-valued. Moreover, different integration results can be got based on different kinds of component of multi-valued measure. Here, we will discuss some results in case the component of multi-valued measure is generalized fuzzy measure, fuzzy measure and generalized T measure.

The paper is organized as follows. In section 2, some preparatory knowledge on multi-dimensional function, t-norm and t-conorm are given. Section 3 presents (T) fuzzy integral of multi-dimensional function, in which the integrand is multi-valued and the measure is scalar-valued. The major work on (T) fuzzy integral of multi-dimensional function with respect to multi-valued measure is shown in section 4 and section 5. In section 4, we define the new integral and give two simple examples to support the reasonableness of it. Then many basic properties of the integral are obtained in section 5. Finally, some conclusions are given in section 6.

2. **Preliminaries**

This section reviews some basic definitions on multi-dimensional function and generalized triangular norms which can be found in [1, 3, 4, 20, 22, 23, 33].

Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space and d be a natural number $(d > 1)$.

$\mathbf{L}^d = \{f : f : \mathbf{\Omega} \to \mathbf{R}^d\}$.

$\mathbf{L}(\mathscr{F})$ =$\{f : f : \mathbf{\Omega} \to \mathbf{R}$ is a measurable function on $(\mathbf{\Omega}, \mathscr{F})\}$.

$\mathbf{L}^d(\mathscr{F}) = \{f : f = (f^1, \cdots, f^d) \in \mathbf{L}^d,\ f^i \in \mathbf{L}(\mathscr{F}),\ i = 1, \cdots, d\}$.

Let the linear operation of $\mathbf{L}^d$ be the usual linear operation.

Suppose, $f = (f^1, \cdots, f^d)$, $f_n = (f_n^1, \cdots, f_n^d)$, $g = (g^1, \cdots, g^d) \in \mathbf{L}^d$, $k \in \mathbf{R}$, then

$$\begin{aligned}
&f^+ = ((f^1)^+, \cdots, (f^d)^+), \quad f^- = ((f^1)^-, \cdots, (f^d)^-),\\
&|f| = (|f^1|, \cdots, |f^d|), \ \ f \pm g = (f^1 \pm g^1, \cdots, f^d \pm g^d),\\
&kf = (kf^1, \cdots, kf^d).
\end{aligned}$$

Obviously, $\quad f = f^+ - f^-, \quad |f| = f^+ + f^-.$

f is called greater than or equal to g, if $f^i \geq g^i$, $i = 1, \cdots, d$, denoted by $f \geq g$. Especially, f is called non-negative if $f^i \geq 0$, $i = 1, \cdots, d$, denoted by $f \geq 0$.

$\{f_n\}$ is called pointwise convergence to f, if $\forall \varepsilon > 0$, $\exists N$, for all $n > N$, $||f(x) - f_n(x)|| < \varepsilon$, where $||f|| = \{\sum_{i=1}^{d} |f^i|^2\}^{1/2}$.

$\{f_n\}$ is called bounded, if $\exists c \geq 0$, such that $||f_n|| \leq c$, $n = 1, 2, \cdots$.

$\{f_n\}$ is called non-decreasing, if $f_1 \leq f_2 \leq \cdots$.

For $\alpha = (\alpha^1, \cdots, \alpha^d) \in \mathbf{R}^d$, the sign $\alpha \in \mathbf{L}^d$ denotes a constant function which is identically equal to α. Let

$$\mathbf{R}_+^d = \{\alpha \in \mathbf{R}^d : \alpha = (\alpha^1, \cdots, \alpha^d), \alpha^i \geq 0, i = 1, \cdots, d\}.$$

Definition 2.1. [4,33,22] Let $\mathbf{R}^+ = [0, +\infty]$. A generalized *triangular norm* (briefly t-norm) on $\mathbf{R}^+$ is a binary operation T: $\mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ such that for all $a, b, c \in \mathbf{R}^+$ the following axioms are satisfied:

(I)$T(a, 0) = 0$, and $\exists e \in \mathbf{R}^+$ such that $T(a, e) = a$;

(II) $T(a, b) = T(b, a)$;

(III) $T(a, T(b, c)) = T(T(a, b), c)$;

(IV) $T(a_1, b_1) \leq T(a_2, b_2)$, if $a_1 \leq a_2$, $b_1 \leq b_2$ $(a_1, a_2, b_1, b_2 \in \mathbf{R}^+)$;

(V) $\lim_{n\to\infty} T(a_n, b_n) = T(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$, if $a_n, b_n \in \mathbf{R}^+$, $n = 1, 2, \cdots$, and $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

A generalized *triangular conorm* (t-conorm for short) on $\mathbf{R}^+$ is a binary operation S: $\mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ satisfying **(II)** $-$ **(V)** and $S(a, 0) = a$.

For $\{x_1, \cdots, x_n\} \subset \mathbf{R}^+$, we define

$$T_{i=1}^n x_i = T(T(x_1, \cdots, x_{n-1}), x_n);$$
$$S_{i=1}^n x_i = S(S(x_1, \cdots, x_{n-1}), x_n).$$

Generally, for $\{x_1, \cdots, x_n, \cdots\}$, we define

$$T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^n x_i, \quad S_{i=1}^\infty x_i = \lim_{n\to\infty} S_{i=1}^n x_i$$

if the limits of the right-hand side exist.

Casasnovas and Riera presented a more general definition on t-norms in [3]. Here, we discuss the specialization defined above.

Definition 2.2. Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space,

(I) if $E \in \mathscr{F}$, a real function

$$\chi_E(x) = \begin{cases} e & , \quad x \in E \\ 0 & , \quad x \notin E \end{cases}$$

is called a characteristic function of E;

(II) for $\alpha_i \in \mathbf{R}^+$, if $E_i \cap E_j = \emptyset$ $(i \neq j)$, $\cup_{i=1}^n E_i = \mathbf{\Omega}$, $(E_i, E_j \in \mathscr{F}, i, j = 1, \cdots, n)$, then we call the real function $h(x) = S_{i=1}^n(T(a_i, \chi_{E_i}(x)))$ a non-negative simple function.

Definition 2.3. A function $S(x) = (S^1(x), \cdots, S^d(x))(d > 1)$ is called a multi-dimensional non-negative simple function if $S^i(x)(i = 1, \cdots, d)$ is a one-dimensional non-negative simple function defined above.

Let $\mathbf{L}_s(\mathscr{F})$ and $\mathbf{L}_s^d(\mathscr{F})$ denote all multi-dimensional non-negative simple functions in $\mathbf{L}(\mathscr{F})$ and $\mathbf{L}^d(\mathscr{F})$, respectively.

Theorem 2.4. *Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space and d be a natural number $(d > 1)$, if $f(x) \geq 0$ and $f(x) \in \mathbf{L}^d(\mathscr{F})$, then there exists an increasing sequence of non-negative simple function $\{S_n(x)\}$ which converges to $f(x)$ point by point, where $S_n(x) \in \mathbf{L}_s^d(\mathscr{F})$.*

The proof of Theorem 2.4 can be seen in [14].

3. (T) Fuzzy Integral of Multi-dimensional Function

In this section, we give a representation theorem of multi-dimensional non-negative simple function and a definition of (T) fuzzy integral of multi-dimensional non-negative measurable function with respect to scalar-valued measure based on the concepts of $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm [14].

With the concepts of multi-dimensional functions, $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm are defined as follows.

Definition 3.1. [14] A generalized $\mathfrak{T}-norm$ is a function $\mathfrak{T}$: $\mathbf{R}_+^d \times \mathbf{R}^+ \to \mathbf{R}_+^d$ satisfying

$$\mathfrak{T}((a_1, \cdots, a_d), b) = (T(a_1, b), \cdots, T(a_d, b)),$$

where $a_i, b \in \mathbf{R}^+, i = 1, \cdots, d$

A generalized $\mathfrak{T}-conorm$ is a function $\mathfrak{S}$: $\mathbf{R}_+^d \times \mathbf{R}_+^d \to \mathbf{R}_+^d$ satisfying

$$\mathfrak{S}((a_1, \cdots, a_d), (b_1, \cdots, b_d)) = (S(a_1, b_1), \cdots, S(a_d, b_d)).$$

By Definition 2.1 and Definition 3.1, we get the following properties at once:

(I) $\exists e \in \mathbf{R}^+$ such that $\mathfrak{T}(\alpha, e) = \alpha$ and $\mathfrak{T}(a, \mathbf{e}) = \mathbf{a}$, $a \in \mathbf{R}^+$; $\alpha \in \mathbf{R}_+^d$, $\mathbf{e} = (e, \cdots, e)$and $\mathbf{a} = (a, \cdots, a)$ are d-dimensional vectors;

(II) $\mathfrak{T}(\alpha, 0) = \mathfrak{T}(a, \mathbf{0}) = \mathbf{0}$, where $\mathbf{0} = (0, \cdots, 0)$ is a d-dimensional vector, $\alpha \in \mathbf{R}_+^d$, $a \in \mathbf{R}^+$;

(III) $\mathfrak{T}(\alpha, b) = \mathfrak{T}(b, \alpha)$, $\alpha \in \mathbf{R}_+^d$, $b \in \mathbf{R}^+$;

(IV) $\mathfrak{T}(a, \mathfrak{T}(\beta, c)) = \mathfrak{T}(\mathfrak{T}(a, \beta), c)$, $\beta \in \mathbf{R}_+^d$, a, $c \in \mathbf{R}^+$;

(V) $\mathfrak{T}(\alpha_1, b_1) \leq \mathfrak{T}(\alpha_2, b_2)$, if $\alpha_1 \leq \alpha_2$, $b_1 \leq b_2$ $(\alpha_1, \alpha_2 \in \mathbf{R}_+^d, b_1, b_2 \in \mathbf{R}^+)$;

(VI) $\lim_{n\to\infty} \mathfrak{T}(\alpha_n, b_n) = \mathfrak{T}(\lim_{n\to\infty} \alpha_n, \lim_{n\to\infty} b_n)$, if $\alpha_n \in \mathbf{R}_+^d$, $b_n \in \mathbf{R}^+$, $n = 1, 2, \cdots$, $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} b_n$ exist;

(VII) $\mathfrak{S}(\alpha, \mathbf{0}) = \alpha$; $\mathfrak{S}_{i=1}^n \alpha_i = \mathfrak{S}(\mathfrak{S}(\alpha_1, \cdots, \alpha_{n-1}), \alpha_n)$, $\alpha_i \in \mathbf{R}_+^d, i = 1, \cdots, n$; $\mathfrak{S}_{i=1}^\infty = \lim_{n\to\infty} \mathfrak{S}_{i=1}^n \alpha_i$, if the right-hand side exists.

Suppose, t-norm T and t-conorm S satisfy the first distributive law:

$$T(S(a,b),c) = S(T(a,c),T(b,c)),$$

where $a,b,c \in \mathbf{R}^+$.
Similarly, we suppose $\mathfrak{T}$-norm $\mathfrak{T}$ and $\mathfrak{T}$-conorm $\mathfrak{S}$ satisfy:

$$\mathfrak{T}(S(a,b),\alpha) = \mathfrak{S}(\mathfrak{T}(a,\alpha),\mathfrak{T}(b,\alpha)),$$

$$\mathfrak{T}(\mathfrak{S}(\alpha,\beta),c) = \mathfrak{S}(\mathfrak{T}(\alpha,c),\mathfrak{T}(\beta,c)),$$

where $\alpha,\beta \in \mathbf{R}^d_+$, $a,b,c \in \mathbf{R}^+$.

Theorem 3.2. *A d-dimensional non-negative simple function $S(x)$ can be denoted by*

$$S(x) = \mathfrak{S}^n_{k=1}(\mathfrak{T}(\gamma_k,\chi_{C_k}(x))),$$

where $\gamma_k \in \mathbf{R}^d_+$, $C_k \in \mathscr{F}$, $C_i \cap C_j = \emptyset$ for all $i \neq j$ and $\cup^n_{k=1} C_k = \mathbf{\Omega}$.

Proof. Let $S(x) = (S^1(x),\cdots,S^d(x)) \in \mathbf{L}^d_s(\mathscr{F})$, suppose

$$S^1(x) = S^{n_1}_{i=1}(T(a_i,\chi_{E_i}(x))), \qquad S^2(x) = S^{n_2}_{j=1}(T(b_j,\chi_{F_j}(x))),$$

where $a_i, b_j \in \mathbf{R}^+$, $E_i \cap E_m = \emptyset$ $(i \neq m)$, $\cup^{n_1}_{i=1} E_i = \mathbf{\Omega}$, $(E_i, E_m \in \mathscr{F}$, $i,m = 1,\cdots,n_1)$, $F_j \cap F_n = \emptyset$ $(j \neq n)$, $\cup^{n_2}_{j=1} F_j = \mathbf{\Omega}$, $(F_j, F_n \in \mathscr{F}$, $j,n = 1,\cdots,n_2)$, then

$$S^1(x) = S^{n_1}_{i=1} S^{n_2}_{j=1}(T(a_i,\chi_{E_i \bigcap F_j}(x))),$$

$$S^2(x) = S^{n_1}_{i=1} S^{n_2}_{j=1}(T(b_j,\chi_{E_i \bigcap F_j}(x))),$$

recursively, we can get

$$S^i(x) = S^n_{k=1}(T(c^i_k,\chi_{C^i_k}(x))), \quad i = 1,\cdots,d$$

where each $c^i_k \in \mathbf{R}^+, C^i_k \in \mathscr{F}$.
From the linear operation of L^d, we know that $S(x)$ can be denoted by

$$S(x) = \mathfrak{S}^n_{k=1}(\mathfrak{T}(\gamma_k,\chi_{C_k}(x))),$$

where $\gamma_k \in \mathbf{R}^d_+$, $C_k \in \mathscr{F}$, $C_i \cap C_j = \emptyset$ for all $i \neq j$ and $\cup^n_{k=1} C_k = \mathbf{\Omega}$.
The theorem is proved. □

From Theorem 2.4 and Theorem 3.2, we reach a conclusion below.

Remark 3.3. For each $f(x) \in \mathbf{L}^d(\mathscr{F})$ and $f(x) \geq 0$, there exists an increasing sequence of non-negative simple function $\{S_n(x)\}$ which converges to $f(x)$ point by point, where $S_n(x)$ is denoted by

$$S_n(x) = \mathfrak{S}^{n_k}_{i=1}(\mathfrak{T}(\alpha_{ni},\chi_{E_{ni}}(x))),\ \alpha_{ni} \in \mathbf{R}^d_+,\ E_{ni} \in \mathscr{F},\ n_k \in N.$$

Following, all multi-dimensional non-negative simple functions are expressed like this.

Definition 3.4. (*Mesiar and Mesiarová [16]*). Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space. If a set function $\mu : \mathscr{F} \to \mathbf{R}^+$ satisfies

(I) $\mu(\emptyset) = 0; \mu(A) \le \mu(B)$, for $A \subset B, A, B \in \mathscr{F}$;

(II) if $A, B \in \mathscr{F}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = S(\mu(A), \mu(B))$;

(III) if $\{A_n\} \subset \mathscr{F}$ and $A_n \nearrow A$, then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$; if $\{\mu(A_n)\}$ is bounded, non-increasing and $\lim_{n\to\infty} A_n$ exists, then $\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, then we say that μ is a generalized T measure and $(\mathbf{\Omega}, \mathscr{F}, \mu)$ is a generalized T measurable space.

Definition 3.5. [14] Let $(\mathbf{\Omega}, \mathscr{F}, \mu)$ be a generalized T measurable space. We have

(I) If $S(x) = \mathfrak{S}_{i=1}^n(\mathfrak{T}(\alpha_i, \chi_{E_i}(x)))$ is a multi-dimensional simple function on $\mathbf{\Omega}$, then the (T) fuzzy integral of $S(x)$ on E is defined by

$$(T)\int_E S(x)\, d\mu = \mathfrak{S}_{i=1}^n(\mathfrak{T}(\alpha_i, \mu(E_i \cap E))),$$

where $E_i \in \mathscr{F}, E_i \cap E_j = \emptyset (i \ne j)$, $\cup_{i=1}^n E_i = \mathbf{\Omega}$, $\alpha_i \in \mathbf{R}_+^d$.

(II) If $f(x) \in \mathbf{L}^d(\mathscr{F})$ and $f(x) \ge 0$, then the (T) fuzzy integral of $f(x)$ on E is defined by

$$(T)\int_E f(x)\, d\mu = \sup_{\substack{0 \le S(x) \le f(x) \\ x \in E}} (T)\int_E S(x)\, d\mu,$$

where $S(x) \in \mathbf{L}_s^d(x)$. If

$$||\sup_{\substack{0 \le S(x) \le f(x) \\ x \in E}} (T)\int_E S(x)\, d\mu||$$

is a finite number, then $f(x)$ is called T integrable.

4. **The New Integral with Respect to Multi-valued Measure**

For the integral defined above, the integrand is multi-valued and the measure is scalar-valued, T-additive and monotone. However, these conditions may arise limitations in many practical applications [8, 16, 25-27, 29].

Example 4.1. We still use the example shown in [25]. Here, let t-conorm S be the addition operation $+$. Three workers, x_1, x_2 and x_3 are engaged in producing the same kind of products. Their efficiencies of working alone are $5, 6$ and 8 products per day, respectively. Their joint efficiencies are not the simple sum of the corresponding efficiencies given above, but are listed in the following:

$\{x_1, x_2\}$ 14,	$\{x_1, x_3\}$ 7,
$\{x_2, x_3\}$ 16,	$\{x_1, x_2, x_3\}$ 18.

These efficiencies can be regarded as a nonnegative set function μ defined on the power set of $X = \{x_1, x_2, x_3\}$ with $\mu(\emptyset) = 0$. Here, $\mu(\{x_1, x_2\}) > \mu(\{x_1\}) + \mu(\{x_2\})$ means that x_1 and x_2 have good cooperation while $\mu(\{x_1, x_3\}) < \mu(\{x_1\}) + \mu(\{x_3\})$, and even $\mu(\{x_1, x_3\}) < \mu(\{x_3\})$ means that x_1 and x_3 have bad relationship and are not suitable for working together. Obviously, μ here is nonadditive and nonmonotone.

On the other hand, a single-factor (the number of products) for workers is considered. In fact, the factors which affect the total benefit are usually more than one. For example, the technology of a worker plays a key role in the quality of the products. Thus, another set function should be introduced to describe the technologies of the workers. Meanwhile, the weight arises based on the relative importance of different factors.

Definition 4.2. [29] Set function $\mu : \mathscr{F} \to [0, +\infty)$ is called a generalized fuzzy measure if $\mu(\emptyset) = 0$. Generalized fuzzy measure μ is called a fuzzy measure if it is monotone, that is,

$$\mu(A) \leq \mu(B) \quad \forall A \subseteq B \qquad \text{and} \qquad A, B \in \mathscr{F}.$$

Sometimes, a continuity is also required for fuzzy measures [18].

From Definition 3.4 and Definition 4.2, it is obvious that the generalized T measure can be looked as a specialization of the generalized fuzzy measure.

Definition 4.3. A measure $\overline{\mu}(A) = (\mu_1(A), \mu_2(A) \cdots, \mu_s(A))\,(s > 1)$, $A \in \mathscr{F}$ is called a multi-valued generalized fuzzy measure if $\mu_i(A)\,(i = 1, 2, \cdots, s)$ is a single-valued generalized fuzzy measure. If μ_i is a fuzzy measure or a generalized T measure, then $\overline{\mu}$ is called a multi-valued fuzzy measure or a multi-valued generalized T measure, respectively.

Following, all $\overline{\mu}$ possess the form $\overline{\mu} = (\mu_1, \mu_2, \cdots, \mu_s)$ $(s > 1)$.

Definition 4.4. Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space, $\overline{\mu}$ is a multi-valued generalized fuzzy measure. We have

(I) If $S(x) = \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, \chi_{E_i}(x)))$ is a multi-dimensional simple function on $\mathbf{\Omega}$, then the (T) fuzzy integral of $S(x)$ on E is defined by

$$(T)\int_E S(x)\, d\overline{\mu} = \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j, \mu_j(E_i \cap E))))),$$

where $E_i \in \mathscr{F}, E_i \cap E_j = \emptyset\,(i \neq j)$, $\cup_{i=1}^{n} E_i = \mathbf{\Omega}$, $\alpha_i \in \mathbf{R}_+^d\,(i = 1, 2, \cdots, n)$, $b_j \in [0, +\infty)$, $\overline{\mu} = (\mu_1, \mu_2, \cdots, \mu_s)\,(s > 1)$.

(II) If $f(x) \in \mathbf{L}^d(\mathscr{F})$ and $f(x) \geq 0$, then the (T) fuzzy integral of $f(x)$ with respect to $\overline{\mu}$ on E is defined by

$$(T)\int_E f(x)\, d\overline{\mu} = \sup_{\substack{0 \leq S(x) \leq f(x) \\ x \in E}} (T)\int_E S(x)\, d\overline{\mu},$$

where $S(x) \in \mathbf{L}_s^d(x)$. If

$$\Big|\Big| \sup_{\substack{0 \leq S(x) \leq f(x) \\ x \in E}} (T)\int_E S(x)\, d\overline{\mu} \Big|\Big|$$

is a finite number, then $f(x)$ is called integrable. If $\overline{\mu}$ is a multi-valued generalized T measure, then $f(x)$ is called T integrable. Here, b_j represents weight of μ_j.

x_i	$f(x_i)$
x_1	$(8, 7, 9)$
x_2	$(5, 6, 7)$
x_3	$(8, 12, 15)$
x_4	$(5, 6, 7)$
x_5	$(8, 7, 9)$

TABLE 1. The Values of Function f in Example 4.5

In information fusion, usually, the number of information sources is finite. For example, when the information comes from a finite databases, each attribute is regarded as an information source and, therefore, there are only finitely many information sources. Thus, we give an example of the new integral on a finite set. When X is finite, say $X = \{x_1, x_2, \cdots, x_n\}$, usually, we take its power set $\mathscr{P}(X)$ as $\mathscr{F}$. Moreover, since all singletons are included in the power set $\mathscr{P}(X)$, the supremum must be reached when the equality $f = \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, \chi_{E_i}(x)))$ holds. Consequently, the expression of the integral can be reduced to the following form:

$$\begin{aligned}&(T)\int_E f(x)\, d\overline{\mu}\\&= \sup\left\{\mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j, \mu_j(E_i \cap E)))))|f = \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, \chi_{E_i}(x)))\right\},\end{aligned}$$

where $E_i \in \mathscr{F}, E_i \cap E_j = \emptyset\,(i \neq j)$, $\cup_{i=1}^{n} E_i = \mathbf{\Omega}$, $\alpha_i \in \mathbf{R}_+^d\,(i = 1, 2, \cdots, n)$, $b_j \in [0, +\infty)$, $\overline{\mu} = (\mu_1, \mu_2, \cdots, \mu_s)\,(s > 1)$.

In real problems, the weights can be selected by experience.

Example 4.5. Let $X = \{x_1, x_2, \cdots, x_5\}$. The values of f, μ_1 and μ_2 are shown in Table 1 and Table 2, respectively. Here, let $T(a, b) = a \wedge b, S(a, b) = a + b$, we can see that μ_1, μ_2 are not additive and monotonic. Values of the weights are assigned as : $b_1 = 6, b_2 = 4$. After calculating, we obtain

$$(T)\int f(x)\, d\overline{\mu} = (185,\ 208,\ 256).$$

The corresponding supremum is reached at

$$f = (8, 7, 9) \wedge \chi_{\{x_1, x_5\}} + (5, 6, 7) \wedge \chi_{\{x_2\}} + (8, 12, 15) \wedge \chi_{\{x_3\}} + (5, 6, 7) \wedge \chi_{\{x_4\}}.$$

If $T(a, b) = a \wedge b, S(a, b) = a \vee b$, we have

$$(T)\int f(x)\, d\overline{\mu} = (48,\ 48,\ 60).$$

The corresponding supremum is reached at

$$f = (8, 7, 9) \wedge \chi_{\{x_1, x_5\}} + (5, 6, 7) \wedge \chi_{\{x_2\}} + (8, 12, 15) \wedge \chi_{\{x_3\}} + (5, 6, 7) \wedge \chi_{\{x_4\}}$$

and

$$f = (8, 7, 9) \wedge \chi_{\{x_1, x_5\}} + (5, 6, 7) \wedge \chi_{\{x_2, x_4\}} + (8, 12, 15) \wedge \chi_{\{x_3\}}.$$

set	value of μ_1	value of μ_2
$\emptyset$	0	0
$\{x_1\}$	1	3
$\{x_2\}$	3	3
$\{x_1, x_2\}$	4	5
$\{x_3\}$	1	6
$\{x_1, x_3\}$	5	7
$\{x_2, x_3\}$	4	8
$\{x_1, x_2, x_3\}$	6	1
$\{x_4\}$	5	2
$\{x_1, x_4\}$	3	8
$\{x_2, x_4\}$	4	6
$\{x_1, x_2, x_4\}$	7	7
$\{x_3, x_4\}$	6	5
$\{x_1, x_3, x_4\}$	8	4
$\{x_2, x_3, x_4\}$	8	9
$\{x_1, x_2, x_3, x_4\}$	9	4
$\{x_5\}$	3	2
$\{x_1, x_5\}$	7	6
$\{x_2, x_5\}$	3	5
$\{x_1, x_2, x_5\}$	6	3
$\{x_3, x_5\}$	6	8
$\{x_1, x_3, x_5\}$	8	9
$\{x_2, x_3, x_5\}$	5	2
$\{x_1, x_2, x_3, x_5\}$	9	5
$\{x_4, x_5\}$	7	6
$\{x_1, x_4, x_5\}$	7	10
$\{x_2, x_4, x_5\}$	6	8
$\{x_1, x_2, x_4, x_5\}$	8	1
$\{x_3, x_4, x_5\}$	5	3
$\{x_1, x_3, x_4, x_5\}$	6	4
$\{x_2, x_3, x_4, x_5\}$	9	8
X	8	8

TABLE 2. The Values of Set Functions μ_1 and μ_2 in Example 4.5

5. **Properties of the New Integral**

In this section, we study some properties of the new integral defined in section 4.

In the following discussion, we suppose: Generalized t-norm and t-conorm satisfy semi-additivity, i. e.

$$S_{i=1}^{n}[T(a_i + b_i, c_i)] \leq S_{i=1}^{n}[T(a_i, c_i)] + S_{i=1}^{n}[T(b_i, c_i)],$$

similarly, $\mathfrak{T}$-norm and $\mathfrak{T}$-conorm satisfy

$$\mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i + \beta_i, c_i)] \leq \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, c_i)] + \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\beta_i, c_i)],$$

where $\alpha^i, \beta^i \in \mathbf{R}_{+}^{d}$, $a^i, b^i, c^i \in \mathbf{R}^{+}$.

Lemma 5.1. *For $k = (k_1, \cdots, k_d) \in \mathbf{R}^d_+$, $\overline{\mu}$ is a multi-valued generalized fuzzy measure,*

$$(T)\int_E k\, d\overline{\mu} = \mathfrak{T}(k, \dot{\overline{\mu}}(E)),$$

where $\dot{\overline{\mu}}(E) = S^s_{j=1}(T(b_j, \mu_j(E)))$.

Lemma 5.1 can be proved easily by Definition 2.1, 3.1 and 4.4.

Lemma 5.2. $\forall\, \varepsilon > 0$, $\exists \delta > 0$, $\forall\, E \in \mathscr{F}$ *and* $||\overline{\mu}(E)|| < \delta$,

$$||(T)\int_E f(x)\, d\overline{\mu}|| < \varepsilon,$$

where $\overline{\mu}$ is a multi-valued fuzzy measure.

Proof. for $\varepsilon > 0$, let $\delta = \varepsilon/\sqrt{d}$, when $||\overline{\mu}(E)|| < \delta = \varepsilon/\sqrt{d}$, $\mu_j(E) < \varepsilon/\sqrt{d}$, $j = 1, 2, \cdots, s$, from the monotonicity of μ_j we have $\mu_j(E_i \cap E) < \varepsilon/\sqrt{d}$, so

$$\begin{aligned}
||(T)\int_E f(x)\, d\overline{\mu}|| &= ||\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, S^s_{j=1}(T(b_j, \mu_j(E_i \cap E)))))|| \\
&\le ||\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, S^s_{j=1}(T(b_j, \varepsilon/\sqrt{d}))))|| \\
&\le ||\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, T(S^s_{j=1}b_j, \varepsilon/\sqrt{d})))|| \\
&\le ||\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, \varepsilon/\sqrt{d}))|| \\
&= ||\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{T}(\mathfrak{S}^n_{i=1}\alpha_i, \varepsilon/\sqrt{d})|| \\
&\le ||(\varepsilon/\sqrt{d}, \cdots, \varepsilon/\sqrt{d})|| = \sqrt{d(\varepsilon/\sqrt{d})^2} \\
&= \varepsilon.
\end{aligned}$$

The lemma is proved. □

Lemma 5.3. *Let $(\mathbf{\Omega}, \mathscr{F})$ be a measurable space, $\overline{\mu}$ be a multi-valued generalized T measure,*

(I) *If there are two expressions of $S(x) \in \mathbf{L}^d_s(\mathscr{F})$, i. e.*

$$\begin{aligned}
S(x) &= \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, \chi_{E_i}(x))) \\
&= \mathfrak{S}^m_{j=1}(\mathfrak{T}(\beta_j, \chi_{F_j}(x))),
\end{aligned}$$

then

$$\begin{aligned}
(T)\int_E S(x)\, d\overline{\mu} &= \mathfrak{S}^n_{i=1}(\mathfrak{T}(\alpha_i, S^s_{k=1}(T(b_k, \mu_k(E_i \cap E))))) \\
&= \mathfrak{S}^m_{j=1}(\mathfrak{T}(\beta_j, S^s_{k=1}(T(b_k, \mu_k(F_j \cap E))))),
\end{aligned}$$

where $\alpha_i, \beta_j \in \mathbf{R}^d_+$.

(II) *If $S_1(x) \le S_2(x)$, then*

$$(T)\int_E S_1(x)\, d\overline{\mu} \le (T)\int_E S_2(x)\, d\overline{\mu},$$

where $S_1(x), S_2(x) \in \mathbf{L}_s^d(\mathscr{F})$.

Lemma 5.3 can be proved easily based on Theorem 2 in [14].

Theorem 5.4. *Let* $(\mathbf{\Omega}, \mathscr{F})$ *be a measurable space,* $\overline{\mu}$ *be a multi-valued generalized* T *measure. If* $f(x) \leq g(x)$, *then*

$$(T)\int_E f(x)\,d\overline{\mu} \leq (T)\int_E g(x)\,d\overline{\mu},$$

where $f(x), g(x) \in \mathbf{L}^d(\mathscr{F})$ *and* $f(x), g(x) \geq 0$

Proof. Without loss of generality, we may assume that $E = X$. For any simple function $S_f(x)$ with expression $S_f(x) = \mathfrak{S}_{i=1}^n(\mathfrak{T}(\alpha_i, \chi_{E_i}(x)))$ satisfying $0 \leq S_f(x) \leq f(x)$, where $E_i \in \mathscr{F}$, $\alpha_i \in \mathbf{R}_+^d$ for all $i(i = 1, 2, \cdots n)$, from $S_f(x) \leq f(x) \leq g(x)$, we know that there exists simple function $S_g(x)$ with expression $S_g(x) = \mathfrak{S}_{i=1}^n(\mathfrak{T}(\alpha_i, \chi_{E_i}(x))) + \mathfrak{S}_{l=1}^m(\mathfrak{T}(\beta_l, \chi_{F_l}(x)))$ such that $S_g(x) \leq g(x)$, where $F_l \in \mathscr{F}$, $\beta_l \in \mathbf{R}_+^d$, m is a natural number. $S_g(x)$ can be still expressed as $\mathfrak{S}_{j=1}^s(\mathfrak{T}(\gamma_j, \chi_{G_j}(x)))$ which is a multidimensional simple function on X. Obviously, $0 \leq S_g(x) \leq g(x)$, so

$$\int S_g(x)d\overline{\mu} \in \left\{\int S(x)d\overline{\mu} \ \middle|\ 0 \leq S(x) \leq g(x),\, x \in E\right\}.$$

Based on Lemma 5.3, we get

$$\sup\left\{\int S(x)d\overline{\mu} \ \middle|\ 0 \leq S(x) \leq g(x),\, x \in E\right\} \geq \int S_g(x)d\overline{\mu} \geq \int S_f(x)d\overline{\mu}.$$

From the arbitrariness of $S_f(x)$, we have

$$\sup\left\{\int S(x)d\overline{\mu} \ \middle|\ 0 \leq S(x) \leq g(x),\, x \in E\right\} \geq \sup\left\{\int S(x)d\overline{\mu} \ \middle|\ 0 \leq S(x) \leq f(x),\, x \in E\right\}$$

i.e.,

$$\int f(x)d\overline{\mu} \leq \int g(x)d\overline{\mu}.$$

The theorem is proved. □

Theorem 5.5. *Let* $(\mathbf{\Omega}, \mathscr{F})$ *be a measurable space,* $\overline{\mu}$ *be a multi-valued generalized* T *measure,* $f(x)$ *be a non-negative* T *integrable function and* $f(x) \in \mathbf{L}^d(\mathscr{F})$, *then*

$$(T)\int_{E\cup F} f(x)\,d\overline{\mu} = \mathfrak{S}[(T)\int_E f(x)\,d\overline{\mu}, (T)\int_F f(x)\,d\overline{\mu}],$$

where $E \cap F = \emptyset$.

Proof.

$$
\begin{aligned}
& (T)\int_{E\cup F} f(x)\,d\overline{\mu} \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap(E\cup F))))))] \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j((E_i\cap E)\cup(E_i\cap F))))))] \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,S(\mu_j(E_i\cap E),\mu_j(E_i\cap F))))))] \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, S_{j=1}^{s}(S(T(b_j,\mu_j(E_i\cap E)),T(b_j,\mu_j(E_i\cap F))))))] \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{T}(\alpha_i, S(S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap E))), \\
& \qquad\qquad\qquad\qquad S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap F))))))] \\
= & \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}[\mathfrak{S}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap E)))), \\
& \qquad\qquad\qquad\qquad \mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap F))))))] \\
= & \mathfrak{S}[\sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap E))))), \\
& \qquad\qquad \sup_{\substack{0\le S(x)\le f(x)\\ x\in E\cup F}} \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap F)))))] \\
= & \mathfrak{S}[\sup_{\substack{0\le S(x)\le f(x)\\ x\in E}} \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap E))))), \\
& \qquad\qquad \sup_{\substack{0\le S(x)\le f(x)\\ x\in F}} \mathfrak{S}_{i=1}^{n}(\mathfrak{T}(\alpha_i, S_{j=1}^{s}(T(b_j,\mu_j(E_i\cap F)))))] \\
= & \mathfrak{S}[(T)\int_E f(x)\,d\overline{\mu}, (T)\int_F f(x)\,d\overline{\mu}].
\end{aligned}
$$

The theorem is proved. □

Corollary 5.6. *Let $(\mathbf{\Omega},\mathscr{F})$ be a measurable space, $\overline{\mu}$ be a multi-valued generalized T measure, $f(x)$ be a non-negative T integrable function and $f(x)\in\mathbf{L}^d(\mathscr{F})$, then*

$$(T)\int_E f(x)\,d\overline{\mu} \le (T)\int_F f(x)\,d\overline{\mu} \qquad \textit{for} \qquad E\subseteq F.$$

Proof. From Theorem 5.5 and the monotonicity of $\mathfrak{S}$, we have

$$
\begin{aligned}
(T)\int_F f(x)\,d\overline{\mu} = (T)\int_{E\cup(F-E)} f(x)\,d\overline{\mu} &= \mathfrak{S}[(T)\int_E f(x)\,d\overline{\mu}, (T)\int_{F-E} f(x)\,d\overline{\mu}] \\
&\ge \mathfrak{S}[(T)\int_E f(x)\,d\overline{\mu}, \mathbf{0}] \\
&= (T)\int_E f(x)\,d\overline{\mu}.
\end{aligned}
$$

The corollary is proved. □

Lemma 5.7. *For* $k \in \mathbf{R}_+^d$, $f(x) \in \mathbf{L}^d(\mathscr{F}), f(x) \geq 0$,

$$(T)\int_E [f(x)+k]\,d\overline{\mu} \leq (T)\int_E f(x)\,d\overline{\mu} + \mathfrak{T}(k, S_{j=1}^s(T(b_j, \mu_j(E)))),$$

where $f(x)$ *is a* T *integrable function.*

It can be proved easily based on Lemma 6 in [14] and the former results.

Theorem 5.8. *Let* $(\mathbf{\Omega}, \mathscr{F})$ *be a measurable space,* $\overline{\mu}$ *be a multi-valued generalized* T *measure,* $f(x)$ *be a non-negative* T *integrable function and* $f(x) \in \mathbf{L}^d(\mathscr{F})$, $\{S_n(x)\}$ *which satisfies* $\lim_{n\to\infty} S_n(x) = f(x)$ *be an increasing sequence and* $S_n(x) \in \mathbf{L}_s^d(\mathscr{F})$, *then*

$$(T)\int_E f(x)\,d\overline{\mu} = \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu}.$$

Proof. Since $S_n(x)$ is monotonically increasing, hence for all n and p, we have $S_n(x) \leq S_{n+p}(x)$, let $p \to \infty$, then $S_n(x) \leq f(x)$, $x \in E$, thus

$$(T)\int_E f(x)\,d\overline{\mu} \geq (T)\int_E S_n(x)\,d\overline{\mu},$$

further

$$(T)\int_E f(x)\,d\overline{\mu} \geq \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu},$$

the limit of the right-hand side exists, because $\{(T)\int_E S_n(x)\,d\overline{\mu}\}$ is a monotone bounded sequence. Conversely, we will prove

$$(T)\int_E f(x)\,d\overline{\mu} \leq \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu}.$$

For all $\varepsilon > 0$, let $E_n(\varepsilon) = E(||f - S_n|| \geq \varepsilon)$, then $\lim_{n\to\infty} ||\mu(E_n(\varepsilon))|| = 0$ (because $S_n(x)$ is a monotonically increasing sequence, then $E_n(\varepsilon)$ is a monotonically decreasing set sequence, from the continuity of the generalized T measure, we get $\lim_{n\to\infty}\mu_j(E_n(\varepsilon)) = \mu_j(\cap_{n=1}^{\infty}(E_n(\varepsilon))) = 0$, further, $\lim_{n\to\infty}||\mu(E_n(\varepsilon))|| = 0$), from Lemma 5.2, $||\lim_{n\to\infty}(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu}|| = 0$, so

$$\lim_{n\to\infty}(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu} = \mathbf{0}.$$

From Theorem 5.5 and Corollary 5.6, we get

$$\begin{aligned}
& (T)\int_E f(x)\,d\overline{\mu} \\
= \ & \mathfrak{S}[(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu}, (T)\int_{E(||f-S_n||<\varepsilon)} f(x)\,d\overline{\mu}] \\
\leq \ & \mathfrak{S}[(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu}, (T)\int_{E(||f-S_n||<\varepsilon)} (S_n(x)+\overline{\varepsilon})\,d\overline{\mu}] \\
\leq \ & \mathfrak{S}[(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu}, (T)\int_E (S_n(x)+\overline{\varepsilon})\,d\overline{\mu}] \\
\leq \ & \mathfrak{S}[(T)\int_{E_n(\varepsilon)} f(x)\,d\overline{\mu}, (T)\int_E S_n(x)\,d\overline{\mu} + \mathfrak{T}(\overline{\varepsilon}, S_{j=1}^s(T(b_j, \mu_j(E))))].
\end{aligned}$$

Let $n \to \infty$, then

$$\begin{aligned}(T)\int_E f(x)\,d\overline{\mu} &\leq \mathfrak{S}[\mathbf{0}, \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu} + \mathfrak{T}(\overline{\varepsilon}, S_{j=1}^s(T(b_j, \mu_j(E))))] \\ &= \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu} + \mathfrak{T}(\overline{\varepsilon}, S_{j=1}^s(T(b_j, \mu_j(E)))),\end{aligned}$$

since $\varepsilon > 0$ is arbitrary , let $\overline{\varepsilon} \to \mathbf{0}^+$, we have $\mathfrak{T}(\overline{\varepsilon}, S_{j=1}^s(T(b_j, \mu_j(E)))) \to \mathbf{0}$, so

$$(T)\int_E f(x)\,d\overline{\mu} \leq \lim_{n\to\infty}(T)\int_E S_n(x)\,d\overline{\mu},$$

where $\overline{\varepsilon} = (\varepsilon, \cdots, \varepsilon) \in \mathbf{R}_+^d$. □

Theorem 5.9. *Let $f_1(x), f_2(x)$ be non-negative T integrable functions and $f_1(x), f_2(x) \in \mathbf{L}^d(\mathscr{F})$, then*

$$(T)\int_E \mathfrak{S}[f_1(x), f_2(x)]d\overline{\mu} = \mathfrak{S}[(T)\int_E f_1(x)d\overline{\mu}, (T)\int_E f_2(x)d\overline{\mu}].$$

Proof. From the Remark, we know that there exists increasing sequences of non-negative simple function $S_n^1(x), S_n^2(x)$ such that

$$\lim_{n\to\infty} S_n^1(x) = f_1(x), \quad \lim_{n\to\infty} S_n^2(x) = f_2(x),$$

where $S_n^1(x), S_n^2(x) \in \mathbf{L}_s^d(\mathscr{F})$, so

$$\lim_{n\to\infty} \mathfrak{S}(S_n^1(x), S_n^2(x)) = \mathfrak{S}(f_1(x), f_2(x)).$$

Let

$$S_n^1(x) = \mathfrak{S}_{i=1}^{p_n}[\mathfrak{T}(\alpha_i^n, \chi_{E_i^n}(x))], \quad S_n^2(x) = \mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\beta_j^n, \chi_{F_j^n}(x))],$$

then

$$S_n^1(x) = \mathfrak{S}_{i=1}^{p_n}\mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\alpha_i^n, \chi_{E_i^n\cap F_j^n}(x))],\ S_n^2(x) = \mathfrak{S}_{i=1}^{p_n}\mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\beta_j^n, \chi_{E_i^n\cap F_j^n}(x))].$$

From the distributive law, we have

$$\begin{aligned}&(T)\int_E \mathfrak{S}[f_1(x), f_2(x)]d\overline{\mu} = \mathfrak{S}_{i=1}^{p_n}\mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\mathfrak{S}(\alpha_i^n, \beta_j^n), \dot{\overline{\mu}}(E_i^n \cap F_j^n))] \\ =\ &\mathfrak{S}\{\mathfrak{S}_{i=1}^{p_n}\mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\alpha_i^n, \dot{\overline{\mu}}(E_i^n \cap F_j^n))], \mathfrak{S}_{i=1}^{p_n}\mathfrak{S}_{j=1}^{q_n}[\mathfrak{T}(\beta_j^n, \dot{\overline{\mu}}(E_i^n \cap F_j^n))]\} \\ =\ &\mathfrak{S}[(T)\int_E S_n^1(x)d\overline{\mu}, (T)\int_E S_n^2(x)d\overline{\mu}],\end{aligned}$$

from the definition of the $\mathfrak{T}$-conorm and Theorem 5.8, let $n \to \infty$, we get

$$(T)\int_E \mathfrak{S}[f_1(x), f_2(x)]d\overline{\mu} = \mathfrak{S}[(T)\int_E f_1(x)d\overline{\mu}, (T)\int_E f_2(x)d\overline{\mu}],$$

where $\dot{\overline{\mu}}(E_i^n \cap F_j^n) = S_{j=1}^s(T(b_j, \mu_j(E_i^n \cap F_j^n)))$. □

6. **Conclusions**

We have introduced and studied a (T) fuzzy integral of multi-dimensional function with respect to multi-valued measure and obtained many basic properties of the integral. Here, the integral is defined by giving the definition of simple function firstly. However, this may arise limitations in some practical applications. Thus, it would be interesting to investigate a new integral defined by elementary function.

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