CHARACTERIZATION OF L-FUZZIFYING MATROIDS BY L-FUZZIFYING CLOSURE OPERATORS

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Abstract. An L-fuzzifying matroid is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a map from 2^E to L satisfying three axioms. In this paper, the notion of closure operators in matroid theory is generalized to an L-fuzzy setting and called L-fuzzifying closure operators. It is proved that there exists a one-to-one correspondence between L-fuzzifying matroids and their L-fuzzifying closure operators.

1. Introduction

Matroid theory plays an important role in combinatorial optimization problems [14]. In fact, matroids are precisely the structures for which the very simple and efficient greedy algorithm works.

In the classical problem, it is assumed that all weights are precisely known. However this assumption may be a serious restriction, since in many practical applications the exact values of the weights are not known in advance. One of the simplest methods of modeling imprecise weights is to define them as closed intervals. It is then assumed that the value of each weight may fall within a given range, independently of the values taken by other weights. Classical intervals can be generalized into fuzzy intervals or fuzzy numbers, which are richer in information.

In [3, 5, 6, 7], when all values of a weight function are closed intervals or fuzzy numbers, the optimality of solutions and elements are characterized by means of a family of interval matroids $\{(E, \mathrm{I}_a) : a \in [0, 1]\}$. In some cases, such a family of interval matroids $\{(E, \mathrm{I}_a) : a \in [0, 1]\}$ is precisely a fuzzifying matroid in the sense of [12].

In [12], when L is a complete lattice, an L-fuzzifying matroid is defined as the pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a map from 2^E to L satisfying three axioms. Thus each subset of E can be regarded as independent to some degree.

The aim of this paper is to generalize the notion of closure operators of matroids to an L-fuzzy setting. We shall prove that there exists a one-to-one correspondence between L-fuzzifying matroids and their L-fuzzifying closure operators.

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2. Preliminaries

Let (E, I) be a crisp matroid. For any $A \in 2^E$, if we define

$$\mathrm{cl}(A) = \{x \in E : \rho(A \cup \{x\}) = \rho(A)\},$$

where ρ denotes the rank function of (E, I), then we know [14] that the closure operator $\mathrm{cl} : 2^E \to 2^E$ satisfies the following conditions:

(C1) If $A \in 2^E$, then $A \subseteq \mathrm{cl}(A)$.

(C2) If $A, B \in 2^E$ and $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

(C3) If $A \in 2^E$, then $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

(C4) If $A \in 2^E$ and $x, y \in E$, $y \in \mathrm{cl}(A \cup x) - \mathrm{cl}(A)$, then $x \in \mathrm{cl}(A \cup y)$.

Conversely, let $\mathrm{cl} : 2^E \to 2^E$ be a function which satisfies the conditions (C1)–(C4). Define the collection I of subsets of E by

$$A \in I \Leftrightarrow x \notin \mathrm{cl}(A - x) \text{ for any } x \in A.$$

Then (E, I) is a matroid.

Throughout this paper L will denote a completely distributive lattice and $\perp$ and $\top$ respectively will denote the smallest and largest element in L.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [2, 9]. $\{a \in L : a \prec b\}$ is called the greatest minimal family of b in the sense of [13]and denoted by $\beta(b)$. In a completely distributive lattice L, there exists $\beta(b)$ for each $b \in L$, and $b = \bigvee \beta(b)$ [13]. It is easy to verify that $\beta(\perp) = \emptyset$.

For an L-fuzzy set $A \in L^X$ and $a \in L$, we define

$$A_{[a]} = \{x \in X : A(x) \geqslant a\}, \quad A_{(a)} = \{x \in X : a \in \beta(A(x))\}.$$

Some properties of these "cut" sets can be found in [4, 10, 11, 12].

We often abbreviate $X - \{e\}$ to $X - e$.

Definition 2.1. [12] Let $\mathbb{N}$ denote the set of all natural numbers. An L-fuzzy natural number is an antitone map $\lambda : \mathbb{N} \to L$ satisfying

$$\lambda(0) = \top, \quad \bigwedge_{n \in \mathbb{N}} \lambda(n) = \perp.$$

The set of all L-fuzzy natural numbers is denoted by $\mathbb{N}(L)$.

Definition 2.2. [12] For any $\lambda, \mu \in \mathbb{N}(L)$, define the sum $\lambda + \mu$ of λ and μ as follows: for any $n \in \mathbb{N}$,

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

Remark 2.3. [12] For any $m \in \mathbb{N}$, define $\underline{m} \in \mathbb{N}(L)$ such that

$$\underline{m}(t) = \begin{cases} \top, & \text{if } t \leqslant m; \\ \perp, & \text{if } t \geqslant m + 1; \end{cases} \tag{2.3.1}$$

Then $\mathbb{N}$ can be regarded as a subset of $\mathbb{N}(L)$. In the sequel, we shall not distinguish

m and $\underline{m}$.

Theorem 2.4. [12] *For any $\lambda, \mu \in \mathbb{N}(L)$ and for any $a \in L$, we have*

$$(\lambda+\mu)_{(a)} \subseteq \lambda_{(a)} + \mu_{(a)} \subseteq \lambda_{[a]} + \mu_{[a]} \subseteq (\lambda+\mu)_{[a]},$$

where

$$\lambda_{(a)} + \mu_{(a)} = \{n \in \mathbb{N} : n = k+l, k \in \lambda_{(a)} \text{ and } l \in \mu_{(a)}\}$$

and

$$\lambda_{[a]} + \mu_{[a]} = \{n \in \mathbb{N} : n = k+l, k \in \lambda_{[a]} \text{ and } l \in \mu_{[a]}\}.$$

Definition 2.5. Let E be a finite set. A map $\mathcal{I} : 2^E \to L$ is called an L-fuzzifying family of independent subsets on E if it satisfies the following conditions:

- **(FI1)** $\mathcal{I}(\emptyset) = \top$.
- **(FI2)** For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathcal{I}(A) \geqslant \mathcal{I}(B)$.
- **(FI3)** If $A, B \in \mathbf{2}^E$ and $|B| > |A|$, then $\bigvee_{e \in B-A} \mathcal{I}(A \cup \{e\}) \geqslant \mathcal{I}(A) \wedge \mathcal{I}(B)$.

If $\mathcal{I}$ is an L-fuzzifying family of independent subsets on E, then the pair $(E, \mathcal{I})$ is called an L-fuzzifying matroid.

Remark 2.6. In definition 1 of [12], the condition **(FI3)** was given in the following form.

(FI3)* If $A, B \in \mathbf{2}^E$ and $|B| > |A|$, then there exists $e \in B - A$ such that $\mathcal{I}(A \cup \{e\}) \geqslant \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Obviously **(FI3)*** implies **(FI3)**, and when $L = [0,1]$, **(FI3)*** is equivalent to **(FI3)**. But, as the following example shows, **(FI3)** does not imply **(FI3)*** in general.

Example 2.7. Let $E = \{x, y, z\}$ and $L = \mathbf{2}^{\{1,2,3\}}$. Define $\mathcal{I} : \mathbf{2}^E \to L$ by

$$\mathcal{I}(A) = \begin{cases} \top, & A \in \{\emptyset, \{x\}, \{y\}, \{z\}\}; \\ \{2,3\}, & A = \{y, z\}; \\ \{1,3\}, & A = \{x, y\}; \\ \{1,2\}, & A = \{x, z\}; \\ \bot, & A = E. \end{cases} \tag{2.7.2}$$

It is easy to check that $\mathcal{I}$ satisfies the conditions **(FI1)**, **(FI2)** and **(FI3)**, but not **(FI3)***. In fact, for $\{x\}, \{y, z\} \in \mathbf{2}^E$, we have

$$\mathcal{I}(\{x\} \cup \{y\}) = \{1,3\} \not\geqslant \{2,3\} = \mathcal{I}(\{x\}) \wedge \mathcal{I}(\{y,z\})$$

and

$$\mathcal{I}(\{x\} \cup \{z\}) = \{1,2\} \not\geqslant \{2,3\} = \mathcal{I}(\{x\}) \wedge \mathcal{I}(\{y,z\}).$$

But

$$\mathcal{I}(\{x\} \cup \{y\}) \vee \mathcal{I}(\{x\} \cup \{z\}) = \top \geqslant \{2,3\} = \mathcal{I}(\{x\}) \wedge \mathcal{I}(\{y,z\}).$$

In the sequel, we shall use **(FI3)**. The following theorem, stated in [12], is true even with the Definition 2.5.

Theorem 2.8. *Let E be a finite set and $\mathcal{I} : 2^E \to L$ be a map. If for any $a, b \in L$,*

$$\beta(a \wedge b) = \beta(a) \cap \beta(b), \tag{2.8.3}$$

then $(E, \mathcal{I})$ is an L-fuzzifying matroid if and only if, for each $a \in \beta(\top)$, $(E, \mathcal{I}_{(a)})$ is a matroid.

Proof. **Necessity**. Suppose that $(E, \mathcal{I})$ is an L-fuzzifying matroid. By $\mathcal{I}(\emptyset) = \top$ we have $\emptyset \in \mathcal{I}_{(a)}$ for any $a \in \beta(\top)$.

For $A, B \in 2^E$ with $A \subseteq B$, if $B \in \mathcal{I}_{(a)}$, then $a \in \beta(\mathcal{I}(B))$ and by $\mathcal{I}(A) \geqslant \mathcal{I}(B)$, we obtain $a \in \beta(\mathcal{I}(A))$, i.e., $A \in \mathcal{I}_{(a)}$.

If $A, B \in \mathcal{I}_{(a)}$ and $|B| > |A|$, then $\bigvee_{e \in B-A} \mathcal{I}(A \vee \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$. By $A, B \in \mathcal{I}_{(a)}$ we have

$$\begin{array}{rcl} a \in & \beta(\mathcal{I}(A)) \cap \beta(\mathcal{I}(B)) & = \beta(\mathcal{I}(A) \wedge \mathcal{I}(B)) \\ \subseteq & \beta\left(\bigvee_{e \in B-A} \mathcal{I}(A \vee \{e\})\right) & = \bigcup_{e \in B-A} \beta\left(\mathcal{I}(A \vee \{e\})\right). \end{array}$$

This implies that there exists $e \in B - A$ such that $A \vee \{e\} \in \mathcal{I}_{(a)}$.

Therefore $(E, \mathcal{I}_{(a)})$ is a matroid.

Sufficiency. Suppose that $(E, \mathcal{I}_{(a)})$ is a matroid for each $a \in \beta(\top)$. We shall prove that $\mathcal{I}$ satisfies **(FI1), (FI2)** and **(FI3)**.

(FI1) For each $a \in \beta(\top)$, by $\emptyset \in \mathcal{I}_{(a)}$ we know that $a \in \beta\left(\mathcal{I}(\emptyset)\right)$. Therefore $\mathcal{I}(\emptyset) = \top$.

(FI2) For $A, B \in 2^E$ with $A \subseteq B$, let $\mathcal{I}(B) = b$. Then $B \in \mathcal{I}_{(a)}$ for each $a \in \beta(b)$. Thus, since $\mathcal{I}_{(a)}$ is a matroid, we have t $A \in \mathcal{I}_{(a)}$. This shows that $a \in \beta(\mathcal{I}(A))$ for each $a \in \beta(b)$. Therefore $\mathcal{I}(B) = b = \bigvee \beta(b) \leqslant \bigvee \beta(\mathcal{I}(A)) = \mathcal{I}(A)$.

(FI3) Suppose that $A, B \in 2^E$, $|B| > |A|$. If $a \in \beta(\mathcal{I}(A) \wedge \mathcal{I}(B))$, then $a \in \beta(\mathcal{I}(A))$ and $a \in \beta(\mathcal{I}(B))$. This implies that $A, B \in \mathcal{I}_{(a)}$. Hence, since $\mathcal{I}_{(a)}$ is a matroid, there exists $e \in B - A$ such that $A \vee \{e\} \in \mathcal{I}_{(a)}$. This shows that $a \in \beta(\mathcal{I}(A \vee \{e\})) \subseteq \beta\left(\bigvee_{e \in B-A} \mathcal{I}(A \vee \{e\})\right)$. Therefore $\bigvee_{e \in B-A} \mathcal{I}(A \vee \{e\}) \geqslant \mathcal{I}(A) \wedge \mathcal{I}(B)$. □

Remark 2.9. For $L = [0, 1]$ and $a \in [0, 1]$, it is obvious that $\beta(a) = [0, a)$. Thus we have $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in [0, 1]$. Now, for any nonempty set X, let $L = 2^X$. Then for any $A \in L$, we have $\beta(A) = A$. So $\beta(A \cap B) = \beta(A) \cap \beta(B)$ holds for any $A, B \in L$.

In the sequel, we shall always suppose that L satisfies the condition $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in L$.

3. Properties of L-fuzzifying Rank Functions

Definition 3.1. [12] Let $(E, \mathcal{I})$ be an L-fuzzifying matroid. The map $R : 2^E \to \mathbb{N}(L)$ defined by

$$R_{\mathcal{I}}(A)(n) = \bigvee \{\mathcal{I}(B) : B \subseteq A, |B| \geqslant n\} \tag{3.1.4}$$

is called the L-fuzzifying rank function for $(E, \mathcal{I})$. If $A \in 2^E$, then $R_{\mathcal{I}}(A)$ is called the L-fuzzifying rank of A.

Example 3.2. Let $E = \{x, y, z\}$ and $I = [0, 1]$. Define $\mathcal{I} : 2^E \to I$ by

$$\mathcal{I}(A) = \begin{cases} 0, & A \in \{E, \{y, z\}\}; \\ 0.3, & A \in \{\{z\}, \{x, z\}\}; \\ 0.5, & A \in \{\{y\}, \{x, y\}\}; \\ 0.7, & A = \{x\}; \\ 1, & A = \emptyset. \end{cases} \tag{3.2.5}$$

Then $(E, \mathcal{I})$ is an I-fuzzifying matroid. By $R_{\mathcal{I}}(A)(n) = \bigvee\{\mathcal{I}(B) : B \subseteq A, |B| \geqslant n\}$, we Obtain the I-fuzzifying rank function $R_{\mathcal{I}} : 2^E \to \mathbb{N}(I)$ as follows:

$$R_{\mathcal{I}}(\emptyset)(n) = \begin{cases} 1, & n = 0; \\ 0, & n \neq 0, \end{cases} \qquad R_{\mathcal{I}}(\{x\})(n) = \begin{cases} 1, & n = 0; \\ 0.7, & n = 1; \\ 0, & n \geq 2, \end{cases}$$

$$R_{\mathcal{I}}(\{y\})(n) = \begin{cases} 1, & n = 0; \\ 0.5, & n = 1; \\ 0, & n \geq 2, \end{cases} \qquad R_{\mathcal{I}}(\{z\})(n) = \begin{cases} 1, & n = 0; \\ 0.3, & n = 1; \\ 0, & n \geq 2, \end{cases}$$

$$R_{\mathcal{I}}(\{x, y\})(n) = \begin{cases} 1, & n = 0; \\ 0.7, & n = 1; \\ 0.5, & n = 2; \\ 0, & n \geq 3, \end{cases} \qquad R_{\mathcal{I}}(\{x, z\})(n) = \begin{cases} 1, & n = 0; \\ 0.7, & n = 1; \\ 0.3, & n = 2; \\ 0, & n \geq 3, \end{cases}$$

$$R_{\mathcal{I}}(\{y, z\})(n) = \begin{cases} 1, & n = 0; \\ 0.5, & n = 1; \\ 0, & n \geq 2, \end{cases} \qquad R_{\mathcal{I}}(E)(n) = \begin{cases} 1, & n = 0; \\ 0.7, & n = 1; \\ 0.5, & n = 2; \\ 0, & n \geq 3. \end{cases}$$

In order to investigate properties of L-fuzzifying rank functions, we need the following lemma.

Lemma 3.3. *For any $\lambda, \mu \in \mathbb{N}(L)$ and for any $a \in \beta(\top)$, we have $(\lambda + \mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$.*

Proof. By Theorem 2.4, we only need to prove that $(\lambda+\mu)_{(a)} \supseteq \lambda_{(a)}+\mu_{(a)}$. Suppose that $n \in \lambda_{(a)}+\mu_{(a)}$. Then there exist $k, l \in \mathbb{N}$ with $n = k+l$ such that $k \in \lambda_{(a)}$ and $l \in \mu_{(a)}$. This implies that $a \in \beta(\lambda(k))$ and $a \in \beta(\mu(l))$, i.e., $a \in \beta(\lambda(k)) \cap \beta(\mu(l)) = \beta(\lambda(k) \wedge \mu(l)) \subseteq \beta\left(\bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)\right) = \beta((\lambda + \mu)(n))$. Thus $n \in (\lambda + \mu)_{(a)}$ and hence $(\lambda + \mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$. $\square$

Theorem 3.4. *Let $(E,\mathcal{I})$ be an L-fuzzifying matroid and $R_{\mathcal{I}}$ be the L-fuzzifying rank function for $(E,\mathcal{I})$. Then $R_{(a)}(A) = (R_{\mathcal{I}}(A))_{(a)}$ for each $a \in \beta(\top)$ and for each $A \in 2^E$, where $R_{(a)}$ denotes the rank function for $(E,\mathcal{I}_{(a)})$.*

Proof. By the definition of $R_{\mathcal{I}}(A)$, we have

$$\begin{aligned}
n \notin (R_{\mathcal{I}}(A))_{(a)} &\Leftrightarrow a \notin \beta(R_{\mathcal{I}}(A)(n)) \\
&\Leftrightarrow a \notin \beta\left(\bigvee_{\substack{B\subseteq A\\ |B|\geqslant n}} \mathcal{I}(B)\right) = \bigcup_{\substack{B\subseteq A\\ |B|\geqslant n}} \beta(\mathcal{I}(B)) \\
&\Leftrightarrow \forall B \subseteq A \text{ with } |B| \geqslant n,\ a \notin \beta(\mathcal{I}(B)) \\
&\Leftrightarrow \forall B \subseteq A \text{ with } B \in \mathcal{I}_{(a)},\ |B| < n \\
&\Leftrightarrow \max\left\{|B| : B \subseteq A \text{ and } B \in \mathcal{I}_{(a)}\right\} < n \\
&\Leftrightarrow n \notin R_{(a)}(A).
\end{aligned}$$

It follows that $R_{(a)}(A) = (R(A))_{(a)}$. □

Corollary 3.5. *Let $(E,\mathcal{I})$ be an L-fuzzifying matroid, $R_{\mathcal{I}} : 2^E \to \mathbb{N}(L)$ the L-fuzzifying rank function of $(E,\mathcal{I})$, and $R_{(a)}$ the rank function for $(E,\mathcal{I}_{(a)})$. Then*

$$R_{\mathcal{I}}(A)(n) = \bigvee \left\{a \in \beta(\top) : n \in R_{(a)}(A)\right\}, \tag{3.5.6}$$

$$R_{\mathcal{I}}(A) = \bigvee_{a\in\beta(\top)} \left\{a \wedge R_{(a)}(A)\right\}. \tag{3.5.7}$$

The following theorem is from [12]. In fact, wemay prove it using the properties of the rank functions of crisp matroids, Lemma 3.3 and Theorem 3.4.

Theorem 3.6. [12] *The L-fuzzifying rank function $R : 2^E \to \mathbb{N}(L)$ for $(E,\mathcal{I})$ satisfies the following conditions:*

(FR1) *For any $A \in 2^E$, $\underline{0} \leqslant R(A) \leqslant \underline{|A|}$.*
(FR2) *If $A, B \in 2^E$ and $A \subseteq B$, then $R(A) \leqslant R(B)$.*
(FR3) *For any $A, B \in 2^E$, $R(A) + R(B) \geqslant R(A \cap B) + R(A \cup B)$.*

Theorem 3.7. *Let $(E,\mathcal{I})$ be an L-fuzzifying matroid and $R : 2^E \to \mathbb{N}(L)$ the L-fuzzifying rank function of $(E,\mathcal{I})$. For any $A, B \in 2^E$ and $y \in B - A$, if $R(A \cup \{y\}) = R(A)$, we have $R(A \cup B) = R(A)$.*

Proof. Suppose that $B - A \neq \emptyset$ and $B - A = \{b_1, b_2, \cdots, b_k\}$. We shall use induction on k. If $k = 1$, the result is immediate. Now suppose that it is true for $k = n$. We shall prove it for $k = n + 1$. By the induction assumption and (FR3), for any $a \in \beta(\top)$

$$\begin{aligned}
&R_{(a)}(A) + R_{(a)}(A) \\
= \;& R_{(a)}(A \cup \{b_1, b_2, \cdots, b_n\}) + R_{(a)}(A \cup \{b_{n+1}\}) \\
\supseteq \;& R_{(a)}((A \cup \{b_1, b_2, \cdots, b_n\}) \cup (A \cup \{b_{n+1}\})) \\
& + R_{(a)}((A \cup \{b_1, b_2, \cdots, b_n\}) \cap (A \cup \{b_{n+1}\})) \\
= \;& R_{(a)}(A \cup B) + R_{(a)}(A) \supseteq R_{(a)}(A) + R_{(a)}(A), \quad \text{by (R2)}.
\end{aligned}$$

This shows that $R_{(a)}(A \cup B) = R_{(a)}(A)$ for each $a \in \beta(\top)$. Thus $R(A \cup B) = R(A)$. □

Example 3.8. It is easy to see that Example 3.2 is a trivial example of Theorem 3.7.

4. L-fuzzifying Closure Operator

In this section we define an L-fuzzifying closure operator, which is a generalization of the notion of closure operators of matroids in an L-fuzzy setting, and show that L-fuzzifying matroids can be completely characterized by means of these L-fuzzifying closure operators.

Definition 4.1. Let E be a finite set. A map $\mathrm{cl} : 2^E \times \beta(\top) \to 2^E$ is called an L-fuzzifying closure operator on E, if, for all $A, B \in 2^E$, $x, y \in E$ and $a \in \beta(\top)$, it satisfies:

(FC1) $A \subseteq \mathrm{cl}(A, a)$.

(FC2) If $B \subseteq A$, then $\mathrm{cl}(B, a) \subseteq \mathrm{cl}(A, a)$.

(FC3) $\mathrm{cl}(\mathrm{cl}(A, a), a) = \mathrm{cl}(A, a)$.

(FC4) If $y \in \mathrm{cl}(A \cup \{x\}, a) - \mathrm{cl}(A, a)$, then $x \in \mathrm{cl}(A \cup \{y\}, a)$.

(FC5) $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A - x, a) \right) = \emptyset \Leftrightarrow \exists b \in \beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A - x, b) \right) = \emptyset$.

Example 4.2. Let $E = \{x, y, z\}$ and $I = [0, 1]$. Define a map $\mathrm{cl} : 2^E \times [0, 1) \to 2^E$ by

$$\mathrm{cl}(\emptyset, a) = \begin{cases} E, & a \in [0.7, 1); \\ \{y, z\}, & a \in [0.5, 0.7); \\ \{z\}, & a \in [0.3, 0.5); \\ \emptyset, & a \in [0, 0.3), \end{cases} \qquad \mathrm{cl}(\{x\}, a) = \begin{cases} E, & a \in [0.5, 1); \\ \{x, z\}, & a \in [0.3, 0.5); \\ \{x\}, & a \in [0, 0.3), \end{cases}$$

$$\mathrm{cl}(\{y\}, a) = \begin{cases} E, & a \in [0.7, 1); \\ \{y, z\}, & a \in [0, 0.7), \end{cases} \quad \mathrm{cl}(\{z\}, a) = \begin{cases} E, & a \in [0.7, 1); \\ \{y, z\}, & a \in [0.5, 0.7); \\ \{z\}, & a \in [0, 0.5), \end{cases}$$

$$\mathrm{cl}(\{x, z\}, a) = \begin{cases} E, & a \in [0.5, 1); \\ \{x, z\}, & a \in [0, 0.5), \end{cases} \quad \mathrm{cl}(\{y, z\}, a) = \begin{cases} E, & a \in [0.7, 1); \\ \{y, z\}, & a \in [0, 0.7), \end{cases}$$

$$\mathrm{cl}(\{x, y\}, a) = \mathrm{cl}(E, a) = E \quad (a \in [0, 1)).$$

It is easy to verify that cl satisfies **(FC1)–(FC5)**. Therefore cl is an L-fuzzifying closure operator on E.

Theorem 4.3. *Let $(E, \mathcal{I})$ be an L-fuzzifying matroid. Define a map $\mathrm{cl}_{\mathcal{I}} : 2^E \times \beta(\top) \to 2^E$ by*

$$\mathrm{cl}_{\mathcal{I}}(A, a) = \left\{ x \in E : R_{(a)}(A) = R_{(a)}(A \cup \{x\}) \right\}, \tag{4.3.8}$$

where $R_{(a)}$ is the rank function of $(E, \mathcal{I}_{(a)})$. Then $\mathrm{cl}_{\mathcal{I}}$ is an L-fuzzifying closure operator on E.

Proof. It is easy to see that $\mathrm{cl}_{\mathcal{I}}|_{2^E\times\{a\}}$ is really the closure operator of $(E,\mathcal{I}_{(a)})$, and $A\in\mathcal{I}_{(a)}$ (or $R_{(a)}(A)=|A|$) if and only if $A\cap\left(\bigcap_{x\in A}\mathrm{cl}_{\mathcal{I}}(A-x,a)\right)=\emptyset$.

Thus **(FC1)**–**(FC4)** are immediate from the properties of the closure operator of a matroid. Moreover, by $\mathcal{I}_{(a)}=\bigcup_{a\in\beta(b)}\mathcal{I}_{(b)}$ we know that **(FC5)** holds for $\mathrm{cl}_{\mathcal{I}}$. □

Lemma 4.4. *An L-fuzzifying closure operator* $\mathrm{cl}:2^E\times\beta(\top)\to 2^E$ *satisfies the following conditions:* $\forall A\in 2^E$ *and* $\forall a\in\beta(\top)$,

(FC6) *There exists* $B\subseteq A$ *such that* $\mathrm{cl}(A,a)=\mathrm{cl}(B,a)$, $B\cap\left(\bigcap_{x\in B}\mathrm{cl}(B-x,a)\right)=\emptyset$, *and* $(B\cup\{y\})\cap\left(\bigcap_{x\in B\cup\{y\}}\mathrm{cl}(B\cup\{y\}-x,a)\right)\neq\emptyset$, *where* $y\in A-B$.

(FC7) *If* $A\cap\left(\bigcap_{x\in A}\mathrm{cl}(A-x,a)\right)=\emptyset$, *then* $(A\cup\{y\})\cap\left(\bigcap_{x\in A\cup\{y\}}\mathrm{cl}(A\cup\{y\}-x,a)\right)\neq\emptyset$ *if and only if* $y\in\mathrm{cl}(A,a)-A$.

(FC8) *If* $A\cap\left(\bigcap_{x\in A}\mathrm{cl}(A-x,a)\right)=\emptyset$, *then* $\mathrm{cl}(A,b)=\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)$.

Proof. Suppose that cl satisfies (FC1)-(FC5). Then there exists a crisp matroid $(E,\mathcal{I}_a)$ with $\mathrm{cl}|_{2^E\times\{a\}}$ as its closure operator for a fixed $a\in\beta(\top)$. And $A\in\mathcal{I}_a$ if and only if $A\cap\left(\bigcap_{x\in A}\mathrm{cl}(A-x,a)\right)=\emptyset$. Therefore (FC6) and (FC7) hold for $(E,\mathcal{I}_a)$.

(FC8) For $A\in 2^E$ and $a\in\beta(\top)$, if $A\cap\left(\bigcap_{x\in A}\mathrm{cl}(A-x,a)\right)=\emptyset$, then $A\in\mathcal{I}_a$. Let R_a be the rank function of $(E,\mathcal{I}_a)$. Then $R_a(A)=|A|$. By (FC5) we know that $\mathcal{I}_b=\bigcup_{b\in\beta(a)}\mathcal{I}_a$, then $A\in\mathcal{I}_b$, i.e., $A\cap\left(\bigcap_{x\in A}\mathrm{cl}(A-x,b)\right)=\emptyset$.

In order to prove that $\mathrm{cl}(A,b)=\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)$, suppose that $x\in\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)-A$. Then for each $a\in\beta(\top)$ and $b\in\beta(a)$, we have $R_a(A\cup\{x\})=R_a(A)=|A|$. In this case we have $A\cup\{x\}\notin\mathcal{I}_a$. Hence $A\cup\{x\}\notin\mathcal{I}_b$. Therefore $(A\cup\{x\})\cap\left(\bigcap_{y\in A\cup\{x\}}\mathrm{cl}(A\cup\{x\}-y,b)\right)\neq\emptyset$. By (FC7) we obtain $x\in\mathrm{cl}(A,b)-A$. This shows that $\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)\subseteq\mathrm{cl}(A,b)$.

Conversely, if $x\in\mathrm{cl}(A,b)-A$, then $R_b(A)=R_b(A\cup\{x\})$, i.e., $A\cup\{x\}\notin\mathcal{I}_b$. By (FC5), we know that for each $a\in\beta(\top)$ with $b\in\beta(a)$, $A\cup\{x\}\notin\mathcal{I}_a$, i.e., $(A\cup\{x\})\cap\left(\bigcap_{y\in A\cup\{x\}}\mathrm{cl}(A\cup\{x\}-y,a)\right)\neq\emptyset$. Therefore $x\in\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)-A$. So $\mathrm{cl}(A,b)\subseteq\bigcap_{b\in\beta(a)}\mathrm{cl}(A,a)$. Thus (FC8) is proved. □

If $\mathrm{cl}: 2^E \to 2^E$ is the closure operator of a matroid (E, I), we know that $A \in I$ if and only if for any $x \in A$, $x \notin \mathrm{cl}(A-x)$, that is to say, $A \cap (\bigcap_{x \in A} \mathrm{cl}(A-x)) = \emptyset$. Hence we have the following theorem.

Theorem 4.5. *Let* $\mathrm{cl}: 2^E \times \beta(\top) \to 2^E$ *be an* L*-fuzzifying closure operator on* E*. Define a map* $\mathcal{I}_{\mathrm{cl}}: 2^E \to L$ *by*

$$\mathcal{I}_{\mathrm{cl}}(A) = \bigvee \left\{ a \in \beta(\top) : A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset \right\}. \tag{4.5.9}$$

Then $(E, \mathcal{I}_{\mathrm{cl}})$ *is an* L*-fuzzifying matroid. Furthermore,* $\mathrm{cl} = \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}$.

Proof. In order to check that $(E, \mathcal{I}_{\mathrm{cl}})$ is an L-fuzzifying matroid, we only need to prove that for any $a \in \beta(\top)$,

$$A \in (\mathcal{I}_{\mathrm{cl}})_{(a)} \Leftrightarrow A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset. \tag{4.5.10}$$

Now if $a \in \beta(\mathcal{I}_{\mathrm{cl}}(A))$, then there exists $b \in \beta(\top)$ such that $a \in \beta(b)$ and $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, b) \right) = \emptyset$. By (FC5) we have $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset$. On the other hand, if $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset$, then there exists $b \in \beta(\top)$ such that $a \in \beta(b)$ and by (FC5) $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, b) \right) = \emptyset$. Therefore $b \leqslant \mathcal{I}_{\mathrm{cl}}(A)$, $a \in \beta(b) \subseteq \beta(\mathcal{I}_{\mathrm{cl}}(A))$. So $A \in (\mathcal{I}_{\mathrm{cl}})_{(a)} \Leftrightarrow A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset$.

Since $\mathrm{cl}|_{2^E \times \{a\}}$ satisfies (FC1)-(FC4), there is a crisp matroid $(E, \mathcal{I}_{\mathrm{cl}_a})$ with $\mathrm{cl}|_{2^E \times \{a\}}$ as its closure operator. Then $A \in \mathcal{I}_{\mathrm{cl}_a}$ if and only if $A \cap \left(\bigcap_{x \in A} \mathrm{cl}(A-x, a) \right) = \emptyset$. This implies that $\mathcal{I}_{\mathrm{cl}_a} = (\mathcal{I}_{\mathrm{cl}})_{(a)}$ and by Theorem 2.8, we conclude that $(E, \mathcal{I}_{\mathrm{cl}})$ is an L-fuzzifying matroid.

Now we check that $\mathrm{cl} = \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}$.

By (FC6) and (FC7), it follows that for any $A \in 2^E$, there exists $B \subseteq A$ such that $\mathrm{cl}(A, a) = \mathrm{cl}(B, a)$, $B \cap \left(\bigcap_{x \in B} \mathrm{cl}(B-x, a) \right) = \emptyset$, and for any $y \in \mathrm{cl}(B, a) - B$, $(B \cup \{y\}) \cap \left(\bigcap_{x \in B \cup \{y\}} \mathrm{cl}(B \cup \{y\} - x, a) \right) \neq \emptyset$. It is easy to see that $B \in (\mathcal{I}_{\mathrm{cl}})_{(a)}$ and $B \cup \{y\} \notin (\mathcal{I}_{\mathrm{cl}})_{(a)}$. Hence we have

$$|B| = (R_{\mathcal{I}_{\mathrm{cl}}})_{(a)}(B) \leqslant (R_{\mathcal{I}_{\mathrm{cl}}})_{(a)}(B \cup \{y\}) < |B \cup \{y\}| = |B| + 1,$$

i.e., $(R_{\mathcal{I}_{\mathrm{cl}}})_{(a)}(B) = (R_{\mathcal{I}_{\mathrm{cl}}})_{(a)}(B \cup \{y\})$. This shows $y \in \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(B, a)$. Therefore $\mathrm{cl}(A, a) = \mathrm{cl}(B, a) \subseteq \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(B, a) \subseteq \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(A, a)$.

Moreover, by the properties of closure operators of crisp matroids, it follows that there exists a base C of $(E, (\mathcal{I}_{\mathrm{cl}})_{(a)})$, let $B = A \cap C$, such that $\mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(A, a) = \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(B, a)$, $B \in (\mathcal{I}_{\mathrm{cl}})_{(a)}$ and $B \cup \{y\} \notin (\mathcal{I}_{\mathrm{cl}})_{(a)}$ for any $y \in \mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(B, a) - B$. In other words, $B\cap\left(\bigcap_{x\in B}\mathrm{cl}(B-x,a)\right)=\emptyset$ and $(B\cup\{y\})\cap\left(\bigcap_{x\in B\cup\{y\}}\mathrm{cl}(B\cup\{y\}-x,a)\right)\neq\emptyset$. Therefore $y\in\mathrm{cl}(B,a)$ by (FC7). Thus, $\mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(A,a)=\mathrm{cl}_{\mathcal{I}_{\mathrm{cl}}}(B,a)\subseteq\mathrm{cl}(B,a)\subseteq\mathrm{cl}(A,a)$. □

Theorem 4.6. *For an L-fuzzifying matroid* $(E,\mathcal{I})$, $\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}=\mathcal{I}$.

Proof. $\forall A\in 2^E$, by

$$\begin{aligned}\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}(A) &= \bigvee\left\{a\in\beta(\top):A\cap\left(\bigcap_{x\in A}\mathrm{cl}_{\mathcal{I}}(A-x,a)\right)=\emptyset\right\}\\ &= \bigvee\left\{a\in\beta(\top):A\in\mathcal{I}_{(a)}\right\}\\ &= \bigvee\left\{a\in\beta(\top):a\in\beta(\mathcal{I}(A))\right\}\\ &= \mathcal{I}(A).\end{aligned}$$

Hence $\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}=\mathcal{I}$. □

Example 4.7. Let $(E,\mathcal{I})$ be the I-fuzzifying matroid in Example 3.2, where $I=[0,1]$. It is easy to check that

$$R_{(a)}(\emptyset)=0\quad(a\in[0,1)),\qquad R_{(a)}(\{x\})=\begin{cases}0, & a\in[0.7,1);\\ 1, & a\in[0,0.7),\end{cases}$$

$$R_{(a)}(\{y\})=\begin{cases}0, & a\in[0.5,1);\\ 1, & a\in[0,0.5),\end{cases}\qquad R_{(a)}(\{z\})=\begin{cases}0, & a\in[0.3,1);\\ 1, & a\in[0,0.3),\end{cases}$$

$$R_{(a)}(\{x,y\})=\begin{cases}0, & a\in[0.7,1);\\ 1, & a\in[0.5,0.7);\\ 2, & a\in[0,0.5),\end{cases}\qquad R_{(a)}(\{x,z\})=\begin{cases}0, & a\in[0.7,1);\\ 1, & a\in[0.3,0.7);\\ 2, & a\in[0,0.3),\end{cases}$$

$$R_{(a)}(\{y,z\})=\begin{cases}0, & a\in[0.5,1);\\ 1, & a\in[0,0.5),\end{cases}\qquad R_{(a)}(E)=\begin{cases}0, & a\in[0.7,1);\\ 1, & a\in[0.5,0.7);\\ 2, & a\in[0,0.5).\end{cases}$$

By

$$\mathrm{cl}_{\mathcal{I}}(A,a)=\{x\in E:R_{(a)}(A)=R_{(a)}(A\cup x)\},$$

we obtain

$$\mathrm{cl}_{\mathcal{I}}(\emptyset,a)=\begin{cases}E, & a\in[0.7,1);\\ \{y,z\}, & a\in[0.5,0.7);\\ \{z\}, & a\in[0.3,0.5);\\ \emptyset, & a\in[0,0.3),\end{cases}\qquad \mathrm{cl}_{\mathcal{I}}(\{x\},a)=\begin{cases}E, & a\in[0.5,1);\\ \{x,z\}, & a\in[0.3,0.5);\\ \{x\}, & a\in[0,0.3),\end{cases}$$

$$\mathrm{cl}_{\mathcal{I}}(\{y\},a)=\begin{cases}E, & a\in[0.7,1);\\ \{y,z\}, & a\in[0,0.7),\end{cases}\qquad \mathrm{cl}_{\mathcal{I}}(\{z\},a)=\begin{cases}E, & a\in[0.7,1);\\ \{y,z\}, & a\in[0.5,0.7);\\ \{z\}, & a\in[0,0.5),\end{cases}$$

$$\mathrm{cl}_{\mathcal{I}}(\{x,z\},a)=\begin{cases}E, & a\in[0.5,1);\\ \{x,z\}, & a\in[0,0.5),\end{cases}\qquad \mathrm{cl}_{\mathcal{I}}(\{y,z\},a)=\begin{cases}E, & a\in[0.7,1);\\ \{y,z\}, & a\in[0,0.7),\end{cases}$$

$$\mathrm{cl}_{\mathcal{I}}(\{x,y\},a)=\mathrm{cl}_{\mathcal{I}}(E,a)=E\ \ (a\in[0,1)).$$

It is easy to check that the map $\mathrm{cl}_{\mathcal{I}}:2^E\times[0,1)\to 2^E$ is an I-fuzzifying closure operator satisfying Theorem 4.3, and $\mathrm{cl}_{\mathcal{I}}$ is precisely the I-fuzzifying closure operator cl in Example 4.2. Now according to Theorem 4.5 we can obtain the map $\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}:2^E\to[0,1]$ as follows:

$$\begin{aligned}\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}(A) &= \bigvee\left\{a\in[0,1):A\cap\left(\bigcap_{x\in A}\mathrm{cl}_{\mathcal{I}}(A-x,a)\right)=\emptyset\right\}\\ &= \begin{cases}0, & A\in\{E,\{y,z\}\};\\ 0.3, & A\in\{\{z\},\{x,z\}\};\\ 0.5, & A\in\{\{y\},\{x,y\}\};\\ 0.7, & A=\{x\};\\ 1, & A=\emptyset.\end{cases}\end{aligned}$$

It is easy to see that $\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}=\mathcal{I}$ and $\mathrm{cl}_{\mathcal{I}_{\mathrm{cl}_{\mathcal{I}}}}=\mathrm{cl}_{\mathcal{I}}$.

5. Conclusion

In this paper, properties of fuzzy rank functions are studied. The notion of L-fuzzifying closure operators is presented and used to characterize L-fuzzifying matroids. It is shown that there is a one-to-one correspondence between an L-fuzzifying matroid and their L-fuzzifying closure operators.

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