Iranian Journal of Fuzzy Systems **Vol. 7, No. 1,** (2010) pp. 59-64

FUZZY QUASI-METRIC VERSIONS OF A THEOREM OF GREGORI AND SAPENA

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Abstract. We provide fuzzy quasi-metric versions of a fixed point theorem of Gregori and Sapena for fuzzy contractive mappings in G-complete fuzzy metric spaces and apply the results to obtain fixed points for contractive mappings in the domain of words.

1. **Introduction and Preliminaries**

Fixed point theories in fuzzy metric spaces and probabilistic metric spaces are closely related. The fixed point theory of the fuzzy metric spaces was introduced by Grabiec [2], where a fuzzy metric version of the Banach contraction principle was proved. In order to obtain his theorem, Grabiec considered a notion of completeness, now called G-completeness, cf. [5]. Subsequently, Gregori and Sapena [5] introduced a new class of contractive mappings in G- complete fuzzy metric spaces. Recent results related to the paper of Gregori and Sapena [5] may be found in [9], [10], [11], [12], [13], [18].

Unfortunately, G-completeness is a very restricting notion, and as is shown in [17], even the induced fuzzy metric space $(\mathbb{R}, M, Min)$, where

$$M(x,y,t) = \frac{t}{t + \mid x - y \mid}$$

is not G-complete. This fact motivated the alternative notion of M-completeness [4], borrowed from probabilistic metric space theory [16]. On the other hand, it is shown by Romaguera et. al [14] that G-completeness provides an efficient tool for obtaining fixed points for fuzzy contraction mappings on complete Non-archimedean fuzzy quasi-metric spaces, and thus it can be successfully applied to obtain fixed points for contractive mappings in the domain of words.

The aim of this paper is to provide fuzzy quasi-metric versions of the fixed point theorem of Gregori and Sapena [5]. The existence of a solution for a recurrence equation, which appears in the average case analysis of Quicksort algorithms is obtained as an application. Our basic references are [3], Chapter X, [5], [6] and [14].

A *fuzzy quasi-metric* on a nonempty set X is a pair $(M, *)$, where $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

Received: August 2008; Accepted: April 2009

Key words and phrases: Fuzzy metric space, Non-archimedean fuzzy quasi-metric, G-bicomplete, Domain of words.

$(i) M(x,y,0)=0$;
$(ii) x=y$ if and only if $M(x,y,t)=M(y,x,t)=1$ for all $t>0$;
$(iii) M(x,z,t+s)\geqslant M(x,y,t)*M(y,z,s)$ for all $t,s\geqslant 0$;
$(iv) M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous.

If the triangle inequality (iii) is replaced by:

$$M(x,z,t)\geq M(x,y,t)*M(y,z,t) \text{ for all } x,y,z\in X, t>0$$

then $(M,*)$ is called a *Non-archimedean fuzzy quasi-metric.*

A fuzzy quasi-metric $(M,*)$ satisfying the symmetry axiom $M(x,y,t)=M(y,x,t)$ for all $x,y\in X$ and $t>0$ is a fuzzy metric in the sense of Kramosil and Michalek [8].

Definition 1.1. A triple $(X,M,*)$, where $(M,*)$ is a (Non-archimedean) fuzzy quasi-metric on X is said to be a *(Non-archimedean) fuzzy quasi-metric space.*

If $(M,*)$ is a fuzzy quasi-metric on X, then $(M^{-1},*)$ is also a fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X\times X\times[0,\infty)$ defined by $M^{-1}(x,y,t)$ $=M(y,x,t)$. Moreover, if we denote by M^i the fuzzy set in $X\times X\times[0,\infty)$ given by $M^i(x,y,t)=min\{M(x,y,t),M^{-1}(x,y,t)\}$, then $(M^i,*)$ is a fuzzy metric on X [6].

Definition 1.2. Let $(X,M,*)$ be a fuzzy metric space. A sequence $(x_n)_{n\in N}$ in X is said to be *M-convergent* if there exists $x\in X$ such that

$$\lim_{n\to\infty} M(x,x_n,t)=1\ \forall t>0.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,M,*)$ is called *Cauchy* if for each $\varepsilon\in(0,1)$ and $t>0$ there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for all $m,n\geq n_0$. The space $(X,M,*)$ is called *complete* if every Cauchy sequence is convergent.

Definition 1.3. [6] A sequence $\{x_n\}$ in a fuzzy quasi-metric space $(X,M,*)$ is called *G-Cauchy* if it is a G- Cauchy sequence in the fuzzy metric space $(X,M^i,*)$. A fuzzy quasi-metric space $(X,M,*)$ is called *G- bicomplete* if the fuzzy metric space $(X,M^i,*)$ is G- complete.

Each G-(bi)complete fuzzy quasi- metric space is (bi) complete, but the converse is not true ([17]).

Definition 1.4. Let $(X,M,*)$ be a fuzzy quasi- metric space. A sequence $\{x_n\}$ in X is called *left G-Cauchy* if

$$\lim_{n\to\infty} M(x_n,x_{n+1},t)=1$$

for all $t>0$. The space $(X,M,*)$ is called *G-complete* if every left G-Cauchy sequence is M^{-1}-convergent.

2. Fuzzy Quasi-metric Versions of the Theorem of Gregori and Sapena

We begin this section by recalling the celebrated fixed point theorem of Gregori and Sapena [5].

Definition 2.1. [5] A *fuzzy contractive mapping* on a fuzzy metric space $(X, M, *)$ is a self-mapping f of X satisfying the following condition for all $x, y \in X$, $t > 0$ and fixed $k \in (0, 1)$.

$$(c) \qquad M(x,y,t) > 0 \Longrightarrow \frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$$

Theorem 2.2. *(*[5]*, Theorem 5.2) Let $(X, M, *)$ be a G -complete fuzzy metric space and $f : X \to X$ be a fuzzy contractive mapping such that $M(x, f(x), t) > 0$, $\forall t > 0$ for some $x \in X$. Then f has a fixed point.*

The following two theorems are fuzzy quasi-metric versions of the theorem of Gregori and Sapena.

Theorem 2.3. *Let $(X, M, *)$ be a G-bicomplete fuzzy quasi-metric space and f be a fuzzy contractive mapping on X. If there exists x in X such that $M(x, f(x), t) > 0$, $\forall t > 0$ and $M(f(x), x, t) > 0$, $\forall t > 0$, then f has a fixed point.*

Proof. Let x, y, t be such that $M^i(x, y, t) > 0$. Then $M(x, y, t) > 0$ and $M(y, x, t) > 0$, hence the following two relations hold.

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$$

and

$$\frac{1}{M(f(y), f(x), t)} - 1 \leq k(\frac{1}{M(y,x,t)} - 1)$$

It follows that

$$k(\frac{1}{M^i(x,y,t)} - 1) \geq \frac{1}{M(f(x), f(y), t)} - 1$$

and

$$k(\frac{1}{M^i(x,y,t)} - 1) \geq \frac{1}{M(f(y), f(x), t)} - 1.$$

Hence

$$\begin{aligned} k(\frac{1}{M^i(x,y,t)} - 1) &\geq \frac{1}{min\{M(f(x), f(y), t), M(f(y), f(x), t)\}} - 1 \\ &= \frac{1}{M^i(f(x), f(y), t)} - 1. \end{aligned}$$

Thus, f is a fuzzy contractive mapping on the complete fuzzy metric space $(X, M^i, *)$ and so, by Theorem 2.2, f has a fixed point. □

Theorem 2.4. *Let $(X, M, *)$ be a G-complete fuzzy quasi-metric space and f be a fuzzy contractive mapping on X. If every M^{-1}-convergent sequence has a unique limit and there exists x_0 in X such that $M(x_0, f(x_0), t) > 0$, $\forall t > 0$, then f has a fixed point.*

Proof. Let $x_n := f^n(x_0) \quad (n \in \mathbb{N})$. From $M(x_0, f(x_0), t) > 0,\ \forall t > 0$ and the contractive condition (c), it immediately follows that

$$M(x_n, x_{n+1}, t) \geq k^n M(x_0, x_1, t) \quad (n \in \mathbb{N}, t > 0),$$

which implies that $\{x_n\}$ is a left G-Cauchy sequence. Since $(X, M, *)$ is G-complete, there is $x \in X$ such that $\{x_n\}$ is M^{-1}-convergent to x. As f is continuous, x_{n+1} is M^{-1}-convergent to $f(x)$. From the uniqueness of the limit we conclude that $f(x) = x$. □

The following proposition states sufficient conditions for the fixed point to be unique.

Proposition 2.5. *Let f be a fuzzy contractive mapping on a fuzzy quasi metric space $(X, M, *)$. If x, y are fixed points of f such that $M(x, y, t) > 0$ and $M(y, x, t) > 0\ \forall t > 0$, then $x = y$.*

Proof. Since $M(f(x), f(y), t) = M(x, y, t) > 0$, by induction on n it can be proved that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k^n \left(\frac{1}{M(x, y, t)} - 1\right)$$

for every n. Therefore $M(f(x), f(y), t) = 1$ for all $t > 0$. Similarly, $M(f(y), f(x), t) = 1$ for all $t > 0$, implying $f(x) = f(y)$, that is, $x = y$. □

3. Application to the Domain of Words

Although the condition of G-completeness is quite restrictive, in the next proposition, which slightly improves Theorem 3 from [14], we show that every complete Non-archimedean fuzzy metric space under a t-norm of Hadžić type is G-complete. Recall that a t-norm T is said to be of Hadžić-type if the family of its iterates is equicontinuous at the point $x = 1$ (see [7], Chapter 1).

Proposition 3.1. *Every bicomplete Non- archimedean fuzzy quasi-metric space (X, M, T) with T of Hadžić-type is G-bicomplete.*

Proof. Since T is of Hadžić type, for given $\varepsilon \in (0, 1)$ there is λ in $(0, 1)$ such that

$$T^{m-1}(1 - \lambda, ..., 1 - \lambda) > 1 - \varepsilon\ \forall m \in \mathbb{N}.$$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a G-Cauchy sequence. Fix $\varepsilon \in (0, 1)$ and $t > 0$ and consider n_0 such that $M^i(x_n, x_{n+1}, t) > 1 - \lambda$ for all $n \geq n_0$. Then, for all $n \geq n_0$ and $j > 0$ we have:

$$M^i(x_n, x_{n+j}, t) \geq T(M^i(x_n, x_{n+1}, t), ..., M^i(x_{n+j-1}, x_{n+j}, t)) > 1 - \varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, M, T). Therefore, there is $x \in X$ such that $lim_{n \to \infty} M^i(x_n, x, t) = 1$ for all $t > 0$. It follows that (X, M^i, T) is G-complete, that is, (X, M, T) is G-bicomplete. □

Let Σ^∞ be the set of all (finite and infinite) sequences over a nonempty alphabet Σ. Denote by $l(x)$ the length of x and by $\sqsubset$ the prefix order on Σ^∞, i.e., $x \sqsubset y \Leftrightarrow x$ is a prefix of y. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y.

With the convention $2^{-\infty} = 0$, let M be defined as $M(x, y, 0) = 0$ for all $x, y \in \Sigma^\infty$, $M(x, y, t) = 1 \quad (t > 0)$ if x is a prefix of y and

$$M(x, y, t) = \begin{cases} 1 - 2^{-l(x \sqcap y)}, & \text{if } x \text{ is not a prefix of } y \text{ and } t \in (0, 1]; \\ 1, & \text{if } x \text{ is not a prefix of } y \text{ and } t > 1. \end{cases}$$

Lemma 3.2. [14] *$l(\Phi(x)) = l(x)+1$ for all $x \in \Sigma^\infty$ and $l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y))$ for all $x, y \in \Sigma^\infty$.*

By Proposition 3.1 with $T = \wedge, \wedge(a, b) = Min\{a, b\}$, we may prove the following proposition. [14]:

Proposition 3.3. *([14], Proposition 4) $(\Sigma^\infty, M, \wedge)$ is a bicomplete Non-archimedean fuzzy quasi-metric space.*

Proposition 3.3 allows us to apply Theorem 2.3 to show, in a direct way, the existence and uniqueness of solution for the following recurrence equation: $T(1) = 0$ and

$$T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1), n \geqslant 2.$$

This equation appears in the average case analysis of Quicksort algorithms, see [1], [14].

To this end, we associate with T, the functional $\Phi : \Sigma^\infty \to \Sigma^\infty$ as follows: we write $x = x_1 x_2 ... x_n$, if $x \in \Sigma^\infty$ has length $n < \infty$ and $x = x_1 x_2 ...$, if x is an infinite word and define $(\Phi(x))_n$ by $(\Phi(x))_1 = T(1)$ and $\Phi(x)_n = \frac{2(n-1)}{n} + \frac{n+1}{n} x_{n-1}$, for all $n \geqslant 2$. We claim that Φ is a fuzzy contractive mapping on $(\Sigma^\infty, M, \wedge)$ with $k = 1/2$. Indeed, if x is a prefix of y, then $M(\Phi(x), \Phi(y), t) = M(x, y, t) = 1$. If x is not a prefix of y, then $M(\Phi(x), \Phi(y), t) = 1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}$. Therefore,

$$\frac{1}{M(\Phi(x), \Phi(y), t)} - 1 = \frac{1 - M(\Phi(x), \Phi(y), t)}{M(\Phi(x), \Phi(y), t)} = \frac{2^{-l(\Phi(x) \sqcap \Phi(y))}}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}}$$
$$\leq \frac{2^{-l(\Phi(x \sqcap y)}}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}}$$

and

$$\frac{1}{2}\left(\frac{1}{M(x, y, t)} - 1\right) = \frac{2^{-l(x \sqcap y) - 1}}{1 - 2^{-l(x \sqcap y)}} = \frac{2^{-l(\Phi(x \sqcap y)}}{1 - 2^{-l(x \sqcap y)}}.$$

Since $l(x \sqcap y) \leq l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y))$, it follows that

$$\frac{1}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}} \leq \frac{1}{1 - 2^{-l(x \sqcap y)}}$$

and thus

$$\frac{1}{M(\Phi(x), \Phi(y), t)} - 1 \leq \frac{1}{2}\left(\frac{1}{M(x, y, t)} - 1\right).$$

Next, since $(\Phi(x))_1 = 0$, it follows that $M^i(x, \Phi(x), t) > 0 \quad (t > 0)$ for every $x = 0 x_2 x_3 ... \in \Sigma^\infty$ and hence, by Theorem 3.2, Φ has a fixed point $z = z_1 z_2, ...$ Next, since every fixed point $y = y_1 ...$ of Φ, has $y_1 = 0$, it follows that if y, z are fixed points of Φ then $M^i(z, y, t) > 0$ for all $t > 0$. Therefore, by Proposition 3.1, the fixed point z is unique and it is the unique solution to the recurrence equation T, i.e. $z_1 = 0$ and $z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$ for all $n \geqslant 2$.

Remark 3.4. A similar approach can be found in [14], where it is shown that the mapping Φ is a probabilistic B contraction on the bicomplete fuzzy metric space $(\Sigma^{\infty}, M_1, \wedge)$, where $M_1(x, y, t) = 1$, if x is a prefix of y and $M_1(x, y, t) = \frac{t}{t+2^{-l(x \sqcap y)}}$ otherwise is the fuzzy quasi-metric induced by the Baire quasi-metric $d_{\sqsubseteq}$ defined through $d_{\sqsubseteq}(x, y) = 0$ if $x \sqsubset y$ and $d_{\sqsubseteq}(x, y) = 2^{-l(x \sqcap y)}$ otherwise. We note that the mapping Φ is not a probabilistic B-contraction on $(\Sigma^{\infty}, M, \wedge)$.

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