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FUZZY ACTS OVER FUZZY SEMIGROUPS AND SHEAVES

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Abstract. Although fuzzy set theory and sheaf theory have been developed and studied independently, Ulrich Hohle shows that a large part of fuzzy set theory is in fact a subfield of sheaf theory. Many authors have studied mathematical structures, in particular, algebraic structures, in both categories of these generalized (multi)sets.

Using Hohle's idea, we show that for a (universal) algebra A, the set of fuzzy algebras over A and the set of subalgebras of the constant sheaf of algebras over A are order isomorphic. Then, among other things, we study the category of fuzzy acts over a fuzzy semigroup, so to say, with its universal algebraic as well as classic algebraic definitions.

1. Introduction and Preliminaries

Although fuzzy set theory and sheaf theory have been developed and studied independently, Hohle [17, 18] gives the close relation between them. Both are in some sense multisets, and hence are generalizations of ordinary (single) sets. This property makes both theories strongly useful in mathematics and many of its applications.

Algebraic structures have specially been studied in both disciplines, see, for example, [3, 4, 5, 20, 8, 9, 13, 11, 10]. Using Hohle's idea, we show that for a (universal) algebra A, the poset of fuzzy algebras over A and the poset of subalgebras of the constant sheaf over A are order isomorphic.

Thus, one can use the power and the tools in both theories to study mathematics in these fields. The main purpose of this paper is to give briefly the above mentioned relation between fuzzy algebras and sheaves of algebras (Section 2) to open another door to study fuzzy algebras, and to study the category of fuzzy (so to say, universal algebraic) acts over a crisp semigroup and fuzzy (so to say, classic algebraic) acts over a fuzzy semigroup (Sections 3 and 4).

Let us recall the following definitions needed in the sequel, and introduce some notations used in this paper.

Definition 1.1. A set X together with a function $\mu : X \to [0, 1]$ is called a *fuzzy set* (over X) and is denoted by (X, μ) or $X^{(\mu)}$. We call X the *underlying set* and μ the *membership function* of the fuzzy set $X^{(\mu)}$, and $\mu(x) \in [0, 1]$ is the *grade of membership* of x in $X^{(\mu)}$.

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If μ is a constant function with value $a \in [0,1]$, $X^{(\mu)}$ is denoted by $X^{(a)}$. The fuzzy set $X^{(1)}$ is called a *crisp set* and may sometimes simply be denoted by X.

For a fuzzy set $X^{(\mu)}$ and $\alpha \in [0,1]$, $X^{(\mu)}_{\alpha} = X^{(\mu)}_{[\alpha} := \{x \in X \mid \mu(x) \geq \alpha\}$ is called the (*closed*) α-*cut* or the (*closed*) α-*level set* of the fuzzy set $X^{(\mu)}$, and $X^{(\mu)}_{(\alpha} := \{x \in X \mid \mu(x) > \alpha\}$ is called the *open* α-*cut* or the *open* α-*level set*. The cuts $S(\mu) = X^{(\mu)}_{(0} = \{x \mid \mu(x) > 0\}$ and $C(\mu) = X^{(\mu)}_{1} = \{x \mid \mu(x) = 1\}$ are called the *support* and the *core* (also called, the *kernel*, the *singular cut*) of μ, respectively.

A *fuzzy function* from $X^{(\mu)}$ to $Y^{(\eta)}$, written as $f : X^{(\mu)} \to Y^{(\eta)}$, is an ordinary function $f : X \to Y$ such that the following is a **fuzzy triangle**:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "f"'] & {[0,1]} \\ Y \arrow[ur, "\eta"'] & \end{tikzcd}$$

meaning that $\mu \leq \eta f$ (that is, $\mu(x) \leq \eta f(x)$ for all $x \in X$). The set of all fuzzy sets with a fixed underlying set X is called the *fuzzy power* or the set of *fuzzy subsets* of X and is denoted by $\mathbf{FSub}X$. Clearly fuzzy sets together with fuzzy functions between them form a category denoted by **FSet**.

Note that the category **FSet** is concrete over **Set**. Also, the left adjoint (free functor) F to the obvious forgetful functor $U : \mathbf{FSet} \to \mathbf{Set}$ takes a set X to the fuzzy set $X^{(0)}$, and a function $f : A \to B$ to itself.

Since the category **FSet** has the free object over a singleton set, a fuzzy function $f : X^{(\mu)} \to Y^{(\eta)}$ is monic if and only if $f : X \to Y$ is one to one and so, in this case, one may write $X^{(\mu)} \leq Y^{(\eta)}$. Also, if $f = id_X : X \to X$, then $X^{(\mu)} \leq X^{(\eta)}$ if $\mu \leq \eta$, that is, $\mu(x) \leq \eta(x)$, for all $x \in X$. In the case where $X \subseteq Y$ and f is the inclusion function, $X^{(\mu)}$ is called a *fuzzy subset* of $Y^{(\eta)}$, written as $X^{(\mu)} \subseteq Y^{(\eta)}$.

Note that one can easily see that the partially ordered set **FSub**X, with the above order, is a complete lattice with the lattice operations of membership functions as

$$(\bigwedge \mu_i)(x) := \bigwedge \mu_i(x)$$
$$(\bigvee \mu_i)(x) := \bigvee \mu_i(x)$$

Also, it easily follows that for all $X^{(\mu)}, X^{(\nu)} \in \mathbf{FSub}X$

- $X^{(\mu)} \leq X^{(\nu)}, \lambda \in [0,1] \Rightarrow X^{(\mu)}_{\lambda} \subseteq X^{(\nu)}_{\lambda}$.
- $\lambda \leq \gamma,\ \lambda, \gamma \in [0,1] \Rightarrow X^{(\mu)}_{\gamma} \leq X^{(\mu)}_{\lambda}$,
- $X^{(\mu)} = X^{(\nu)} \Longleftrightarrow X^{(\mu)}_{\lambda} = X^{(\nu)}_{\lambda}$, for all $\lambda \in [0,1]$.

2. Universal Algebra in the Categories Sh[0,1] and FSet

In this section, we first recall the general notion of a (universal) algebra in an ordinary category and then, using Hohle's idea [17, 18], we give the relation between the categories of algebras in the categories **FSet** of fuzzy sets and **Sh**[0,1] of sheaves on the frame (complete Heyting algebra) [0,1].

Definition 2.1. Let $\mathcal{E}$ be a finitely complete category (or at least with finite products). Then a (*universal*) *algebra* in $\mathcal{E}$ is an entity $(A, (\lambda_A)_{\lambda\in\Lambda})$, where A is an object of $\mathcal{E}$, Λ is a set and for each $\lambda \in \Lambda$, the λth operation $\lambda_A : A^{n_\lambda} \to A$ is a morphism in $\mathcal{E}$ where each n_λ is a finite cardinal number and A^{n_λ} is the n_λth power of A. The family $\tau = (n_\lambda)_{\lambda\in\Lambda}$ is called the *type* of this algebra. The algebra $(A, (\lambda_A)_{\lambda\in\Lambda})$ is simply denoted by A.

Also, a *homomorphism* $h : A \to B$ from an algebra $(A, (\lambda_A)_{\lambda\in\Lambda})$ to an algebra $(B, (\lambda_B)_{\lambda\in\Lambda})$, both of type τ in $\mathcal{E}$, is a morphism in $\mathcal{E}$ such that for each $\lambda \in \Lambda$, $\lambda_B \circ h^{n_\lambda} = h \circ \lambda_A$.

Algebras (of type τ) in $\mathcal{E}$ and homomorphisms between them form a category denoted by **Alg**$\mathcal{E}$. If $\mathcal{E}$ is the category **Set** of sets, **Alg**$\mathcal{E}$ is simply denoted by **Alg** (see [6]). Usually for the classical algebraic structures, such as groups, rings, and vector spaces, the operations satisfy some identities (such as associativity, commutativity, etc.). The full subcategory of **Alg**$\mathcal{E}$ whose objects are all algebras in $\mathcal{E}$ satisfying a set Σ of equations is denoted by **Mod**$(\Sigma, \mathcal{E})$.

No need to mention the importance of sheaves over topological spaces or frames and complete Heyting algebras, (that is, complete lattices in which meets distribute over arbitrary joins), see, for example, [22, 23]. Let us very briefly recall this notion and sheaves of algebras.

First note that every partially ordered set P can be considered as a category $\mathbf{P}$ whose objects are the elements of P and for every $a, b \in P$,

$$Hom(a,b) = \begin{cases} \{*\} & \text{if } a \leq b \\ \emptyset & \text{otherwise} \end{cases}$$

Thus, in particular, every frame is a category.

Now, recall that a *presheaf* (of sets) on a frame L is a functor $F : \mathbf{L^{op}} \to \mathbf{Set}$ in which for each $\beta \leq \alpha$, the corresponding function $F\alpha \to F\beta$ is denoted by ρ^α_β or $-|\beta$ and called a *restriction.* That is, for $\gamma \leq \beta \leq \alpha$,

$$\rho^\alpha_\alpha = id_{F\alpha} \quad \& \quad \rho^\alpha_\beta \circ \rho^\beta_\gamma = \rho^\alpha_\gamma.$$

Also, *morphisms* between presheaves are just natural transformations $\eta : F \to G$; that is, η is an L indexed family $(\eta_\alpha)_{\alpha\in L}$ of maps $F\alpha \xrightarrow{\eta_\alpha} G\alpha$ such that for each pair $\beta \leq \alpha$, the following diagram is commutative:

$$\begin{tikzcd} F\alpha \arrow[r, "\eta_\alpha"] \arrow[d, "\rho^\alpha_\beta"'] & G\alpha \arrow[d, "\rho^\alpha_\beta"] \\ F\beta \arrow[r, "\eta_\beta"'] & G\beta \end{tikzcd}$$

A presheaf F on L is called a *sheaf* if the following two conditions are satisfied:

- F is a *mono presheaf* (or *separated*), that is, if $\alpha = \bigvee_{i\in I} \alpha_i$ and $s, t \in F\alpha$, such that $s|\alpha_i = t|\alpha_i$, for all $i \in I$, then $s = t$.

- F satisfies the *patching* (or *gluing*) condition, that is, if $\alpha = \bigvee_{i\in I} \alpha_i$ and $(s_i)_{i\in I} \in \prod_{i\in I} F\alpha_i$ with $s_i|\alpha_i \wedge \alpha_j = s_j|\alpha_i \wedge \alpha_j$, for all $i, j \in I$, then there exists $s \in F\alpha$ (unique by separatedness) with $s|\alpha_i = s_i$, for all $i \in I$.

In the following, the frame L is considered to be the unit interval $[0, 1]$. So every presheaf F on $[0, 1]$ is an indexed family $(F\alpha)_{\alpha\in[0,1]}$ with the restriction functions $(\rho^\alpha_\beta : F\alpha \to F\beta)_{\alpha\geq\beta}$. Specially, for an arbitrary set A, the corresponding *constant sheaf* $\tilde{A}$ is defined by $\tilde{A}\alpha = A$, if $\alpha \neq 0$, and $\tilde{A}0 = \{*\}$.

One can easily see that a subsheaf G of a sheaf F is a sheaf with $G\alpha \subseteq F\alpha$, for each $\alpha \in L$, and each $\rho^\alpha_\beta : G\alpha \to G\beta$ is the restriction of each $\rho^\alpha_\beta : F\alpha \to F\beta$. The following lemma is a characterization of the subsheaves of a constant sheaf on $L = [0, 1]$.

Lemma 2.2. *F is a subsheaf of a constant sheaf on $[0, 1]$ if and only if $(F\alpha)_{\alpha\in[0,1]}$ fulfills any one of the following equivalent conditions:*
(i) $F0 = \{*\},\ F\alpha \subseteq F\beta,$ where $0 < \beta \leq \alpha$.
(ii) $\bigcap_{0<\beta<\alpha} F\beta = F\alpha$.

Proof. It is clear that every subsheaf of a constant sheaf $\tilde{A}$ is of the form (i). So we just show that (i) and (ii) are equivalent. Let a sheaf F satisfy condition (i). Thus, $F\alpha \subseteq F\beta$, for every $0 < \beta \leq \alpha$. Hence $F\alpha \subseteq \bigcap_{0<\beta\leq\alpha} F\beta$. On the other hand, let $a \in \bigcap_{0<\beta<\alpha} F\beta$. Then consider $a_\beta = a \in F\beta$, for every $\beta < \alpha$, with the property that $\rho^\beta_{\beta'}(a_\beta) = a_{\beta'}$, for every $\beta' \leq \beta < \alpha$. Since $\alpha = \bigvee_{\beta<\alpha} \beta$, (by the patching property of sheaves) there is $a_\alpha \in F\alpha$ such that $\rho^\alpha_\beta(a_\alpha) = a_\beta = a$. That is, $a \in F\alpha$ and hence $\bigcap_{0<\beta<\alpha} F\beta = F\alpha$. Now let F be a sheaf satisfying (ii). Then F, being a sheaf, gives that $F0 = \{*\}$ and for $0 < \beta \leq \alpha$, $F\alpha \subseteq F\beta$, since $\bigcap_{0<\beta<\alpha} F\beta = F\alpha$. □

It is easy to see that finite limits in the category $\mathbf{Sh}L$ of sheaves on L are computed componentwise. In particular, finite product of sheaves $F_1, \ldots, F_n$, is computed as $(\prod_{i=1}^n F_i)\alpha = \prod_{i=1}^n F_i\alpha$, for every $\alpha \in L$.

Since $\mathbf{Sh}L$ is finitely complete (in fact a topos), algebras of type τ can be defined in $\mathbf{Sh}L$ as in Definition 2.1. That is to say, a *sheaf of algebras* of type τ on L is a sheaf $A \in \mathbf{Sh}L$ if its n-ary operations is a natural transformation $\lambda_A : A^n \to A$.

It can be proved that A is an algebra of type τ in $\mathbf{Sh}L$ if and only if each $A\alpha$ is an ordinary algebra of the same type. Also A satisfies a set Σ of equations if and only if each $A\alpha$ satisfies Σ.

Now by Lemma 2.2 we have the following.

Corollary 2.3. *For every $A \in$ **Alg**, the corresponding constant sheaf $\tilde{A}$ is an algebra of the same type in $\mathbf{Sh}L$. Also, every subalgebra of $\tilde{A}$ is represented by the chain $(F(\alpha))_{\alpha\in(0,1]}$ of subalgebras of A, where $F(\alpha) \subseteq F(\beta)$ for $0 < \beta \leq \alpha$, and $F(0) = \{*\}$.*

The set of all subalgebras of the constant sheaf $\tilde{A}$ (of algebras) is denoted by $\mathbf{ShSub_a}\tilde{A}$.

Finally in this section, we introduce the category of (universal) algebras of type τ in the category **FSet** of fuzzy sets. But first we note that the category **FSet** is finitely complete, in particular the product of fuzzy subsets $A^{(\mu)}$ and $B^{(\eta)}$ is $(A \times B)^{(\mu \wedge \eta)}$, and thus one can define (universal) algebras in this category as follows.

Definition 2.4. (i) By an *n-array fuzzy operation* λ on a fuzzy set $A^{(\mu)}$ in the category **FSet** we mean a fuzzy function from A^n to A. More explicitly, an n-array operation λ on a fuzzy set $A^{(\mu)}$ gives the following fuzzy triangle.

$$\begin{tikzcd} A^n \arrow[r, "\bigwedge_{i=1}^{n} \mu"] \arrow[d, "\lambda"'] & {[0,1]} \\ A \arrow[ur, "\mu"'] & \end{tikzcd}$$

meaning that for every $a_1, ..., a_n \in A$,

$$\bigwedge_{i=1,...,n} \mu(a_i) \leq \mu(\lambda_A(a_1, ..., a_n)).$$

(ii) A (universal) algebra of type τ in the category **FSet**, called a *fuzzy (universal) algebra*, is a fuzzy set $A^{(\mu)}$ together with a family $\{\lambda_i : A^n \to A\}_{i \in I}$ of fuzzy operations.

(iii) A *fuzzy homomorphism* $f : A^{(\mu)} \to B^{(\eta)}$ between fuzzy algebras of the same type is a fuzzy map as well as an algebra homomorphism (preserving the operations) from A to B. The set of all fuzzy algebras over a fixed algebra A is denoted by $\mathbf{FSub_a}A$, and the category of all fuzzy algebras of type τ and the fuzzy homomorphisms between them is denoted by $\mathbf{FAlg}(\tau)$, or simply by **FAlg**.

Lemma 2.5. *Let $A^{(\mu)} \in \mathbf{FAlg}$. Then every α-cut $A^{(\mu)}_\alpha$ is an (ordinary) algebra.*

Proof. Suppose that $a_1, \ldots, a_n$ belong to $A^{(\mu)}_\alpha$. Then we have $\alpha \leq \bigwedge_{i=1,...,n} \mu(a_i) \leq \mu(\lambda_A(a_1, ..., a_n))$, which proves the result. $\square$

Note that, for an ordinary (universal) algebra $A \in \mathbf{Alg}$, the sets $\mathbf{ShSub_a}\tilde{A}$ and $\mathbf{FSub_a A}$ are partially ordered, with the obvious orders. In the next theorem we show that they are order isomorphic.

Theorem 2.6. *Let $A \in \mathbf{Alg}$. Then, the partially ordered sets $\mathbf{FSub_a}A$ and $\mathbf{ShSub_a}\tilde{A}$ are order isomorphic.*

Proof. Define the map $G : \mathbf{FSub_a}A \to \mathbf{ShSub_a}\tilde{A}$ by $G(A^{(\mu)}) := F_\mu$ in which $F_\mu(\alpha) := A^{(\mu)}_\alpha$, if $\alpha \neq 0$, and $F_\mu(0) := \{*\}$. Corollary 2.3 and Lemma 2.5 ensure that F_μ is a subalgebra of the constant sheaf $\tilde{A}$. Note that G is order-preserving, because if $A^{(\nu)} \leq A^{(\mu)}$ then, for every $\lambda \in [0,1]$, $A^{(\nu)}_\lambda \subseteq A^{(\mu)}_\lambda$. Hence $F_\nu \leq F_\mu$.

Now we define $H : \mathbf{ShSub_a}\tilde{A} \to \mathbf{Fsub_a}A$ by $H(F) := A^{(\mu_F)}$ in which $\mu_F(a) := \bigvee_{a \in F(\lambda)} \lambda$, for every $a \in A$. We note that H is order-preserving. Because, if $F \leq F'$, then $F(\alpha) \subseteq F'(\alpha)$ for every $\alpha \in [0,1]$, and hence $\bigvee_{a \in F(\lambda)} \lambda \leq \bigvee_{a \in F'(\lambda)} \lambda$.

Also $A^{(\mu_F)}$ is a fuzzy algebra of type τ, because for every n-ary operation λ_A, and $a_1, \ldots, a_n \in A$, if $\mu_F(a_i) \neq 0$, for every $i = 1, \ldots, n$, then $a_j \in F(\bigwedge_{i=1}^{n} \mu_F(a_i))$, for every $j = 1, \ldots, n$. So $\lambda_A(a_1, \ldots, a_n) \in F(\bigwedge_{i=1}^{n} \mu_F(a_i))$ and hence $\bigwedge_{i=1}^{n} \mu_F(a_i) \leq \mu_F(\lambda_B(a_1, \ldots, a_n))$. Also if there exists $1 \leq j \leq n$ such that $\mu_F(a_j) = 0$, then obviously, $0 = \bigwedge_{i=1}^{n} \mu_F(a_i) \leq \mu_F(\lambda_B(a_1, \ldots, a_n))$.

Now we show that H and G are inverses of each other. For each fuzzy algebra $A^{(\mu)}$, we have:

$$\begin{aligned} \mu_{F_\mu}(a) &= \bigvee_{a \in F_\mu(\lambda)} \lambda \\ &= \bigvee_{\mu(a) \geq \lambda} \lambda \\ &= \mu(a) \end{aligned}$$

and also

$$\begin{aligned} F_{\mu_F}(\alpha) &= \{a \in A \mid \mu_F(a) \geq \alpha\} \\ &= \{a \in A \mid \bigvee_{a \in F(\lambda)} \lambda \geq \alpha\} \\ &= \begin{cases} F(\alpha) & \text{if } \alpha \neq 0 \\ \{*\} & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

The last equality is because $F(\alpha) \subseteq \{a \in A \mid \bigvee_{a \in F(\lambda)} \lambda \geq \alpha\}$, for α is one of λ,s in (1), and since $\bigvee_{a \in F(\lambda)} \lambda \geq \alpha$, there exists λ_0 with $a \in F(\lambda_0)$ and $\alpha \leq \lambda_0$. Hence for $F(\lambda_0) \subseteq F(\alpha)$, $a \in F(\alpha)$. $\square$

3. Fuzzy Acts over Fuzzy Semigroups

Universal algebras having a semigroup (monoid or group) S of unary operations have always been of interest to mathematicians, specially to computer scientists and logicians. The algebraic structures so obtained are called S-acts (or S-sets, S-automata, S-operands, S-polygons, S-systems, see [12, 19, 2, 7]).

As a very interesting example, used in computer sciences as a convenient means of algebraic specification of process algebras (see [14, 15]), consider the monoid $(\mathbb{N}^\infty, \cdot)$ acting on sets, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $n < \infty, \forall n \in \mathbb{N}$ and $m \cdot n = min\{m, n\}$ for $m, n \in \mathbb{N}^\infty$ (see, for example, [14, 16]).

In this section we are going to define the notion of the actions of a (fuzzy) semigroup on a fuzzy set, which will be studied in this and the last section of this paper.

First note that, for a semigroup S and $s \in S$, we get the left and the right translations $\lambda_s, \rho_s : S \to S$ defined by $\lambda_s(x) = sx,\ \rho_s(x) = xs$.

Now, according to the definition of a fuzzy (universal) algebra given in Section 2, we have the following definition of a fuzzy semigroup.

Definition 3.1. A semigroup S together with a function $\nu : S \to [0,1]$ is called a *fuzzy semigroup* if its multiplication is a fuzzy function: for every $s, r \in S$, $\nu(s) \wedge \nu(r) \leq \nu(sr)$; that is, the following is a fuzzy triangle:

$$\begin{tikzcd} S \times S \arrow[r, "\nu\wedge\nu"] \arrow[d, "\lambda_S"'] & {[0,1]} \\ S \arrow[ur, "\nu"'] & \end{tikzcd}$$

in which $\lambda_S : S \times S \to S$ is the multiplicatin of the semigroup S. If S has an identity 1, one usually add the condition $\nu(1) = 1$.

Now, note that, for a (crisp) semigroup S, a (crisp) set A can be made into an (ordinary) S-act in the following two equivalent ways:

(*Universal algebraic*) The set A together with a family $(\lambda_s : A \to A)_{s\in S}$ of unary operations satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda_s(a)$.

(*Classic algebraic*) The set A together with a function $\lambda : S \times A \to A$ satisfying $(st)a = s(ta)$ (and $1a = a$, if S has an identity) where $sa = \lambda(s, a)$.

Now, having these two, so called, universal and classic actions of S on A, we get the following two, not necessarily equivalent, definitions for a fuzzy act over a fuzzy monoid.

Definition 3.2. Let $S^{(\nu)}$ be a fuzzy semigroup and $A^{(\mu)}$ be a fuzzy set such that A is an S-act, as defined above. Then, $A^{(\mu)}$ is called:

(*Universal*) A *fuzzy S-act* (or fuzzy $S^{(1)}$-act, where $S^{(1)}$ is the crisp semigroup S to emphasize fuzziness) if each λ_s is a fuzzy function; that is $\mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$ (with no mention of ν). That is, for every $s \in S$, the following triangle is fuzzy:

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\lambda_s"'] & {[0,1]} \\ A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(*Classic*) A *fuzzy $S^{(\nu)}$-act* if $\lambda : S \times A \to A$ is a fuzzy function; that is, $\nu(s) \wedge \mu(a) \leq \mu(sa)$, for every $s \in S$ and $a \in A$. That is, the following triangle is fuzzy:

$$\begin{tikzcd} S \times A \arrow[r, "\nu\wedge\mu"] \arrow[d, "\lambda"'] & {[0,1]} \\ S \arrow[ur, "\mu"'] & \end{tikzcd}$$

Note that Universal implies Classic, and if $S^{(1)} = S$ is a (crisp) semigroup, then $\nu(s) \wedge \mu(a) = 1 \wedge \mu(a) = \mu(a)$, and so the above two definitions are equivalent.

Also, every fuzzy semigroup $S^{(\nu)}$ is naturally a fuzzy $S^{(\nu)}$-act (classically) and $S^{(1)} = S$ is a fuzzy $S^{(1)} = S$-act (universally, and hence classically). Also, if $S^{(\nu)}$ is a fuzzy left ideal, then it is a fuzzy S-act (universally, and hence classically).

A morphism between fuzzy $S^{(\nu)}$-acts (universal or classic), also called an $S^{(\nu)}$-map is simply an S-map (that is, a map together with action-preserving property) as well as a fuzzy function. The set of all (fuzzy) $S^{(\nu)}$-acts with a fixed A is denoted by $S^{(\nu)}$-$\mathbf{FSub}A$, and the category of all fuzzy $S^{(\nu)}$-acts with morphisms between them is denoted by $S^{(\nu)}$**-FAct**.

Also note that, since an S-act A is naturally a (unary) universal algebra, the universal definition of fuzzy acts, being compatible with Definition 2.4, may be considered to be more algebraic than the classic one. Thus, by Theorem 2.6, the partially ordered set $S^{(1)}$-$\mathbf{FSub}A$ of (universal) fuzzy S-acts is order isomorphic to the partially ordered set $\mathbf{ShSub_a}\tilde{A}$ of subsheaves of the constant sheaf of S-acts $\tilde{A}$. In particular, we have the counterpart of Lemma 2.5 as follows:

Theorem 3.3. *An S-act A with $\mu : A \to [0,1]$ is a fuzzy $S = S^{(1)}$-act if and only if for every $\alpha \in [0,1]$, $A^{(\mu)}_\alpha$ is an ordinary S-subact of A.*

Proof. One way is clear by Lemma 2.5. Now let for every $\alpha \in [0,1]$, $A^{(\mu)}_\alpha$ be an ordinary subact of A, $s \in S$, and $a \in A$. Then, since $a \in A^{(\mu)}_{\mu(a)}$ and $A^{(\mu)}_{\mu(a)}$ is a subact of A, $sa \in A^{\mu}_{\mu(a)}$. That is $\mu(a) \le \mu(sa)$, for every $s \in S$ and $a \in A$. □

As we have seen, Lemma 2.5 and Theorem 2.6 are true for the universal definition of fuzzy acts. In the following we are going to give the counterparts of these results for the classic definition of fuzzy acts.

First recall from [11] that, for a semigroup S in $\mathbf{Sh}[0,1]$ (sheaf of semigroups), an *S-sheaf* is a sheaf A together with a natural transformation $\eta : S \times A \to A$ such that $(A\alpha, \eta_\alpha)$ is an $S\alpha$-act, for each $\alpha \in [0,1]$ and $\rho^\alpha_\beta : A\alpha \to A\beta$ is equivariant over $\rho'^\alpha_\beta : S\alpha \to S\beta$ for every $\beta \le \alpha$; that is, $\rho^\alpha_\beta(sa) = \rho'^\alpha_\beta(s)\rho^\alpha_\beta(a)$ $(sa|\beta = (s|\beta)(a|\beta))$, for every $s \in S\alpha$, $a \in A\alpha$.

Since, in view of Theorem 2.6, every fuzzy semigroup $S^{(\nu)}$ corresponds to a sheaf F_ν of semigroups, we can consider F_ν-sheaves. Now we claim that the partially ordered set $S^{(\nu)}$-$\mathbf{FSub}A$ of fuzzy $S^{(\nu)}$-subacts of A is order isomorphic to the partially ordered set F_ν-$\mathbf{ShSub}\tilde{A}$ of subalgebra of the constant F_ν-sheaf $\tilde{A}$ of algebras. To prove the claim, first we note the following lemma.

Lemma 3.4. *Let A be an S-act. Then, for every fuzzy semigroup $S^{(\nu)}$, the constant sheaf $\tilde{A}$ is an F_ν-sheaf.*

Proof. Since A is an S-act, we can define the natural transformation $\eta : F_\nu \times \tilde{A} \to \tilde{A}$ to be the family $(\eta_\alpha)_{\alpha\in[0,1]}$ in which η_α maps every $(s,a) \in (F_\nu \times A)\alpha = F_\nu\alpha \times A\alpha$ to sa, for every $\alpha \in [0,1]$. We note that η is a natural transformation because if $\beta \le \alpha$, then $\rho^\alpha_\beta : (F_\nu \times \tilde{A})\alpha \to (F_\nu \times \tilde{A})\beta$ is nothing but the product of the inclusion map

$S_{\alpha}^{(\nu)} \hookrightarrow S_{\beta}^{(\nu)}$ and $id : A\alpha \to A\beta$. So we have the following commutative diagram

$$\begin{tikzcd} (s,a) \arrow[r, squiggly] \arrow[d, "\rho_{\beta}^{\alpha}"'] & sa \arrow[d, "\rho_{\beta}^{\alpha}"] \\ (s,a) \arrow[r, squiggly] & sa \end{tikzcd}$$

Also, in view of the above diagram, it is clear that $\rho_{\beta}^{\alpha} : A\alpha \to A\beta$ is equivariant over $\rho'^{\alpha}_{\beta} : S_{\alpha}^{(\nu)} \to S_{\beta}^{(\nu)}$ for every $\beta \leq \alpha$. □

The following is the counterpart of Theorem 3.3 which leads us to our claim.

Theorem 3.5. *$A^{(\mu)}$ is a fuzzy $S^{(\nu)}$-act if and only if $A_{\lambda}^{(\mu)}$ is an $S_{\lambda}^{(\nu)}$-act for every $\lambda \in [0, 1]$. That is F_{μ} is an F_{ν}-sheaf.*

Proof. First suppose that $A^{(\mu)}$ is a fuzzy $S^{(\nu)}$-act of A, $\lambda \in [0, 1]$, $s \in S_{\lambda}^{(\nu)}$, and $a \in A_{\lambda}^{(\mu)}$. Then since $\lambda \leq \nu(s) \wedge \mu(a) \leq \mu(sa)$, $sa \in A_{\lambda}^{(\mu)}$. Now let $S_{\lambda}^{(\nu)}$ be $A_{\lambda}^{(\nu)}$-act, for every $\lambda \in [0, 1]$. Then for every $a \in A$ and $s \in S$, take $\lambda = \nu(s) \wedge \mu(a)$. Since $a \in A_{\lambda}^{(\mu)}$ and $s \in S_{\lambda}^{(\nu)}$, $sa \in A_{\lambda}^{(\mu)}$. Hence $\nu(s) \wedge \mu(a) \leq \mu(sa)$. □

Theorem 3.6. *The partially ordered set $S^{(\nu)}$-**FSub**A is order isomorphic to the partially ordered set of subsheaf acts over $[0, 1]$ of the constant F_{ν}-sheaf $\tilde{A}$ which is denoted by F_{ν}-**ShSub**$\tilde{A}$.*

Proof. Define the order preservig map $G : S^{(\nu)}\text{-}\mathbf{FSub}A \to F_{\nu}\text{-}\mathbf{ShSub}\tilde{A}$ to be $G(A^{(\mu)}) := F_{\mu}$. We have already seen, in Theorems 2.6 and Lemma 3.4, that F_{μ} is a subact of the constant sheaf $\tilde{A}$ and $\tilde{A}$ is an F_{ν}-sheaf and in Theorem 3.5 we saw that F_{μ} is an F_{ν}-sheaf. It is easily seen that if $A^{(\mu)} \leq A^{(\gamma)}$, then $F_{\mu} \leq F_{\gamma}$.

Also define the functor $H : F_{\nu}\text{-}\mathbf{ShSub}\tilde{A} \to S^{(\nu)}\text{-}\mathbf{FSub}A$ to be $H(F) := \mu_F$ (as we defined in Theorem 2.6). Now we just show that μ_F is a fuzzy $S^{(\nu)}$-subact of A. So consider $\mu_F(sa)$, if $sa \in F\alpha$ for some $\alpha \neq 0$, then $\mu_F(sa) = \bigvee_{sa \in F\lambda} \lambda$. Specially for $\lambda_0 = \nu(s) \wedge \mu_F(a)$, $sa \in F\lambda_0$. Hence $\nu(s) \wedge \mu_F(a) \leq \mu_F(sa)$. Also if $sa \notin F\alpha$, for every $\alpha \in [0, 1]$, then $a \notin F\alpha$ or $s \notin F\alpha$, for every $\alpha \in [0, 1]$, because otherwise, if $a \in F\alpha$ and $s \in F\beta$, then $sa \in F(\alpha \wedge \beta)$. So if $\mu_F(sa) = 0$ then $\mu_F(a) \wedge \nu(s) = 0$. Now in view of Theorem 2.6 there is nothing to prove. □

4. The Category of Fuzzy Acts

In this section we give some categorical properties of fuzzy $S^{(\nu)}$-acts (see [1, 21] for categorical notions.)

Theorem 4.1. *The* **product** *of two fuzzy $S^{(\nu)}$-acts $A^{(\mu)}$ and $B^{(\eta)}$ in the category $S^{(\nu)}$-**FAct** is $(A \times B)^{(\mu\wedge\eta)}$.*

Proof. Since $(\mu \wedge \eta)(a, b) \wedge \nu(s) = \mu(a) \wedge \eta(b) \wedge \nu(s) \wedge \nu(s) \leq \mu(sa) \wedge \eta(sb) = (\mu \wedge \eta)(s(a, b))$, for every $(a, b) \in A \times B$ and $s \in S$, $(A \times B)^{(\mu\wedge\eta)}$ is an $S^{(\nu)}$-act in which the action of S over $A \times B$ is defined component wise, that is $(s(a, b)) = (sa, sb)$.

Also, since the following triangles are clearly fuzzy triangles, the projections are fuzzy maps:

$$\begin{tikzcd}
B \arrow[dr, "\eta"'] & A\times B \arrow[l, "\pi_B"'] \arrow[r, "\pi_A"] \arrow[d, "\mu\wedge\eta"] & A \arrow[dl, "\mu"] \\
 & {[0,1]} &
\end{tikzcd}$$

Now let $C^{(\gamma)}$ be an $S^{(\nu)}$-act with fuzzy $S^{(\nu)}$-maps f and g in the following diagram of fuzzy triangles:

$$\begin{tikzcd}
B \arrow[dr, "\eta"'] & C \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\gamma"] & A \arrow[dl, "\mu"] \\
 & {[0,1]} &
\end{tikzcd}$$

Then, since $\gamma \leq \mu g$ and $\gamma \leq \eta f$, $\gamma \leq \mu g \wedge \eta f$, the $S^{(\nu)}$-map $(g, f) : C \to A \times B$ with $(g, f)(c) = (g(c), f(c))$ is a fuzzy map, that is, makes the diagram

$$\begin{tikzcd}
C \arrow[r, "\gamma"] \arrow[d, "(f,g)"'] & {[0,1]} \\
A\times B \arrow[ur, "\mu\wedge\eta"'] &
\end{tikzcd}$$

a fuzzy triangle. It is easy to verify that $(g, f) : C \to A\times B$ is the unique fuzzy $S^{(\nu)}$-map which can commute the above diagram. This clearly proves the universality of $(A \times B)^{(\mu\wedge\eta)}$, and hence we get the result. $\square$

Also, note that the **terminal object** in $S^{(\nu)}$**-FSet**, which is the product of the empty family, is $T : \{*\} \to [0, 1]$ which maps $*$ to 1 with the constant action. The following is easily proved.

Theorem 4.2. *The* **coproduct** *of two fuzzy $S^{(\nu)}$-acts $A^{(\mu)}$ and $B^{(\eta)}$ is the fuzzy S-act $(A\dot{\cup}B)^{(\mu\dot{\cup}\eta)}$, where $\mu\dot{\cup}\nu|_A = \mu$ and $\mu\dot{\cup}\eta|_B = \eta$.*

Theorem 4.3. *The* **equalizer** *of two fuzzy $S^{(\nu)}$-act maps $f, g : A^{(\mu)} \to B^{(\eta)}$ is the fuzzy $S^{(\nu)}$-act $E^{(e)}$, where $E = \{a \in A \mid f(a) = g(a)\}$ and $e := \mu|_E : E \to [0, 1]$.*

Proof. First note that $E^{(e)}$ is clearly a fuzzy $S^{(\nu)}$-subact of $A^{(\mu)}$ equalizing f and g.

Now suppose $C^{(\gamma)}$ is a fuzzy $S^{(\nu)}$-act and $h : C^{(\gamma)} \to A^{(\mu)}$ is a fuzzy $S^{(\nu)}$-map ($\gamma \leq \mu h$) such that $fh = gh$. Then, one can easily see that the map $\overline{h} : C^{(\gamma)} \to E^{(e)}$, taking every $c \in C$ to $h(c) \in E$, is the unique fuzzy $S^{(\nu)}$-map which gives the universality of $E^{(e)}$, and hence we get the result. $\square$

Theorem 4.4. *The* **pullback** *of two fuzzy $S^{(\nu)}$-maps $f : A^{(\mu)} \to C^{(\gamma)}$ and $g : B^{(\eta)} \to C^{(\gamma)}$ is the fuzzy equalizer of the two fuzzy $S^{(\nu)}$-maps $f \circ \pi_A,\ g \circ \pi_B : (A \times B)^{(\mu\wedge\eta)} \to C^{(\gamma)}$*

Now using the above theorem about pullbacks, we easily get the following facts.

Corollary 4.5. *The* **inverse image** *of* $B^{(\eta)} \leq B^{(\gamma)}$ *along a fuzzy* $S^{(\nu)}$*-map* $f : A^{(\mu)} \to B^{(\gamma)}$ *is* $f^{-1}(\eta) : E \to [0,1]$ *with* $E = \{(a,b) \in A \times B \mid f(a) = b\}$ *and*

$$f^{-1}(\eta)(a,b) = \mu(a) \wedge \eta(b).$$

So we can consider $f^{-1}(\eta) : A \to [0,1]$ to be $f^{-1}(\eta)(a) = \mu(a) \wedge \eta(f(a))$ which is a fuzzy $S^{(\nu)}$-subact of μ. Because $f^{-1}(\eta)(a) = \mu(a) \wedge \eta(f(a)) \leq \mu(a) \wedge \gamma(f(a)) = \mu(a)$, for every $a \in A$.

In the rest of this section we are going to construct a sort of image and inverse image of an $S^{(\nu)}$-map between fuzzy $S^{(\nu)}$-acts in the category $S^{(\nu)}$-**FSub**A. To do this, first we give the following definition.

Definition 4.6. Let $A^{(\mu)}$ be a fuzzy subact of A and B be a set, and $f : A \to B$ be a map. Then we define the fuzzy subset $f(\mu) : B \to [0,1]$ to be:

$$f(\mu)(b) := \bigvee \{\mu(a) \mid f(a) = b\}, \quad \text{for every } b \in B.$$

Theorem 4.7. *Suppose that* $A^{(\mu)}$ *is a fuzzy* $S^{(\nu)}$*-subact of* A*,* B *is an* S*-act, and* $f : A \to B$ *is an* S*-map, then* $f(\mu)$ *is a fuzzy* $S^{(\nu)}$*- subact of* B*.*

Proof. To prove, consider $f(\mu)(sb) = \bigvee\{\mu(a) \mid f(a) = sb\}$ and $f(\mu)(b) := \bigvee\{\mu(a) \mid f(a) = b\}$. But if $f(a) = b$, then $sf(a) = f(sa) = sb$. Now since $\nu(s) \wedge \mu(a) \leq \mu(sa)$, $\nu(s) \wedge \bigvee\{\mu(a) \mid f(a) = b\} \leq \bigvee\{\mu(a) \mid f(a) = sb\}$ and hence $f(\mu)(b) \leq f(\mu)(sb)$. □

Theorem 4.8. *Let* $f : A \to B$ *be an* S*-map between* S*-acts* A *and* B*, and* $B^{(\mu)}$ *be a fuzzy* $S^{(\nu)}$*-subact of* B*. Then* $\mu(f) : A \to [0,1]$*, defined by* $\mu(f)(a) := \mu(f(a))$*, is a fuzzy* $S^{(\nu)}$*-subact of* A*.*

Proof. $\nu(s) \wedge \mu(f)(a) = \nu(s) \wedge \mu(f(a)) \leq \mu(sf(a)) = \mu(f(sa)) = \mu(f)(sa)$, for every $a \in A$ and $s \in S$. □

In view of Theorems 4.7 and 4.8 we can consider the functors $\mathbf{f^{-1}} : S^{(\nu)}$-**FSub**$B \to S^{(\nu)}$-**FSub**$A$ to be $\mathbf{f^{-1}}(B^{(\mu)}) = A^{(\mu(f))}$ and call it the *inverse image* of f, and $\mathbf{f} : S^{(\nu)}$-**FSub**$A \to S^{(\nu)}$-**FSub**B to be $\mathbf{f}(A^{(\eta)}) = B^{(f(\mu))}$ and call it the *image* of f, where $f : A \to B$ is an S-map between S-acts A and B. The terms used for the above functors are justified, since $\mathbf{f}$ is a left adjoint of $\mathbf{f}^{-1}$. More explicitly:

Proposition 4.9. *If* $f : A \to B$ *is an* S*-map between* S*-acts* A *and* B*, and* $B^{(\mu)}$ *and* $A^{(\nu)}$ *are respectively fuzzy* $S^{(\nu)}$*-subacts of* B *and* A*, then* $f(\mu(f)) \leq \mu$ *and* $\eta \leq (f(\eta))(f)$*.*

Proof. For every $b \in B$ we have:

$$\begin{aligned} f(\mu(f))(b) &= \bigvee\{\mu(f)(a) \mid f(a) = b\} \\ &= \bigvee\{\mu(f(a)) \mid f(a) = b\} \\ &= \begin{cases} \mu(b) & \text{if } f^{-1}(b) \neq \emptyset \\ 0 & \text{other wise} \end{cases} \\ &\leq \mu(b) \end{aligned}$$

Also we have:

$$\begin{aligned} ((f(\eta))(f))(a) &= (f(\eta))(f(a)) \\ &= \bigvee\{\eta(x) \mid f(x) = f(a)\} \\ &\geq \eta(a) \end{aligned}$$

for every $a \in A$. So $\eta \leq (f(\eta))(f)$. $\square$

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