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# CATEGORICAL RELATIONS AMONG MATROIDS, FUZZY MATROIDS AND FUZZIFYING MATROIDS

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Abstract. The aim of this paper is to study the categorical relations between matroids, Goetschel-Voxman's fuzzy matroids and Shi's fuzzifying matroids. It is shown that the category of fuzzifying matroids is isomorphic to that of closed fuzzy matroids and the latter is concretely coreflective in the category of fuzzy matroids. The category of matroids can be embedded in that of fuzzifying matroids as a simultaneously concretely reflective and coreflective subcategory.

## 1. Introduction and Preliminaries

Matroids were introduced by Whitney in 1935 as a generalization of both graphs and metric spaces. It is well known that matroids play an important role in mathematics and especially in applied mathematics. Matroids are precisely the structures for which the very simple and efficient greedy algorithm works [4, 10]. In 1988, matroid theory was generalized to fuzzy fields by Goetschel and Voxman [7]. Their approach to the fuzzification of matroids preserves many basic properties of crisp matroids. Fuzzy circuits, fuzzy rank functions and fuzzy closure operators are widely studied [5, 6, 7, 8], while none of the corresponding axioms has been established. In [11], based on the idea of a fuzzifying topology [12], Shi introduced a fuzzifying matroid and established a one-to-one correspondence between all fuzzifying matroids and all fuzzy rank functions.

The aim of this paper is to study the relations among matroids, fuzzy matroids and fuzzifying matroids from a categorical viewpoint. The paper is organized as follows. In Section 2, we study the relations between the category of fuzzy matroids and that of closed fuzzy matroids and show that the category of closed fuzzy matroids **CFM** is a concretely coreflective full subcategory of the category of fuzzy matroids **FM**. In Section 3, the properties of fuzzifying matroids are studied. We show that **FYM**, the category of fuzzifying matroids, is isomorphic to **CFM**. In Section 4, we show that the category of matroids **M** can be embedded in **FYM** as a simultaneously concretely reflective and coreflective subcategory. In summary, we show that

$$\mathbf{M} \overset{\mathrm{r,c}}{\subseteq} \mathbf{FYM} \cong \mathbf{CFM} \overset{\mathrm{c}}{\subseteq} \mathbf{FM},$$

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where $r$ and $c$ mean reflective and coreflective, respectively. A conclusion is given in the last section.

In what follows, we recall some basic concepts and notions which will be used throughout this paper. $E$ is always assumed to be a finite set.

**Definition 1.1.** [10] Let $I$ be a nonempty family of subsets of $E$ satisfying:
(I1) If $A \in I$ and $B \subseteq A$, then $B \in I$;
(I2) If $A,\ B \in I$ and $|A| < |B|$, then there exists $e \in B - A$ such that $A \cup e \in I$.
The pair $(E,\ I)$ is called a (*crisp*) *matroid* on $E$ and the members of $I$ are called the *independent sets*.

Let $\mathcal{F}(E)$ denote the set of all fuzzy subsets on $E$, i.e.,

$$\mathcal{F}(E) = \{f : E \longrightarrow [0,1]|\ f \text{ is a mapping}\}.$$

For $\mu,\ \nu \in \mathcal{F}(E)$,
$\mu \leq \nu$ if and only if $\mu(e) \leq \nu(e)$ for all $e \in E$;
$\mu < \nu$ if and only if $\mu \leq \nu$ and $\mu(e) < \nu(e)$ for some $e \in E$;
$\text{supp}\mu = \{e \in E|\ \mu(e) > 0\}$;
$m(\mu) = \min\{\mu(e)|\ e \in \text{supp}\mu\}$ (Since $E$ is finite, $m(\mu)$ is well defined).
Let $r \in [0,1]$ and $\mu \in \mathcal{F}(E)$,
$C_r(\mu) = \{e \in E|\ \mu(e) \geq r\}$ is called the $r$-*cut* of $\mu$.
$\forall A \subseteq E$, $|A|$ is the cardinal number of $A$.

**Definition 1.2.** [7] Let $\Psi$ be a nonempty family of fuzzy subsets on $E$ satisfying:
(FI1) If $\mu \in \Psi,\ \nu \in \mathcal{F}(E)$ and $\nu \leq \mu$, then $\nu \in \Psi$;
(FI2) If $\mu,\ \nu \in \Psi$ and $|\text{supp}\mu| < |\text{supp}\nu|$, then there exists $\omega \in \Psi$ such that
(a) $\mu < \omega \leq \mu \vee \nu$;
(b) $m(\omega) \geq \min\{m(\mu), m(\nu)\}$.
Then the pair $(E, \Psi)$ is called a *fuzzy matroid* on $E$, and $\Psi$ is called the family of *independent* fuzzy sets.

**Example 1.3.** Let $E$ be a finite family of $n$-dimensional vectors over the real numbers $\mathbb{R}$. $A \in [0,1]^E$ is called independent if $\forall\{k_e|\ e \in E\} \subseteq \mathbb{R}$, $\sum_{e\in E} k_e A(e)e = \mathbf{0}$ always implies $\forall e \in E,\ k_e A(e) = 0$. It's easy to verify that $A \in [0,1]^E$ is independent iff $\text{supp}A$ is (crisp) independent. Then it is straightforward to show that the family of all independent fuzzy subsets of $E$ consists of a fuzzy matroid in the sense of Definition 1.2.

By [5, 7], we know that if $(E, \Psi)$ is a fuzzy matroid, then $(E, I_r)$ is a (crisp) matroid on $E$ for all $r \in (0,1]$, where $I_r = \{C_r(\mu)|\ \mu \in \Psi\}$, is the $r$-*level cut matroid* of $(E, \Psi)$. Sometimes we write $C_r(\Psi)$ instead of $I_r$.

**Proposition 1.4.** [7] *Let* $(E, \Psi)$ *be a fuzzy matroid. Then*

$$\Psi = \{\mu \in \mathcal{F}(E)|\ \forall r \in (0,1],\ C_r(\mu) \in I_r\}.$$

Since $E$ is finite, there are at most finite $r$-level cut matroids on $E$.

**Proposition 1.5.** [7] *Let* $(E, \Psi)$ *be a fuzzy matroid on* $E$ *and* $(E, I_r)$ *be the* $r$*-level cut matroid for any* $r \in (0, 1]$. *Then there is a finite sequence* $0 = r_0 < r_1 < \cdots < r_n = 1$ *such that*

(1) *If* $r_i < s, t < r_{i+1}$*, then* $I_s = I_t,\ 0 \le i \le n-1$*;*

(2) *If* $r_i < s < r_{i+1} < t < r_{i+2}$*, then* $I_s \supsetneqq I_t,\ 0 \le i \le n-2$.

*The sequence* $r_0,\ r_1, \cdots,\ r_n$ *is called the fundamental sequence of* $(E, \Psi)$.

For $1 \le i \le n$, let $\overline{r}_i = \frac{1}{2}(r_{i-1} + r_i)$. Then the decreasing sequence of crisp independent families $I_{\overline{r}_1} \supsetneqq I_{\overline{r}_2} \supsetneqq \cdots \supsetneqq I_{\overline{r}_n}$ is called the *induced matroids sequence* of $(E, \Psi)$.

**Remark 1.6.** (1) For a fuzzy matroid $(E, \Psi)$ with the fundamental sequence $0 = r_0 < r_1 < \cdots < r_n$, the $n$ numbers $r_1, \cdots, r_n$ separate the $r$-level cut matroids into $n$ classes $I_{\overline{r}_1} \supsetneqq I_{\overline{r}_2} \supsetneqq \cdots \supsetneqq I_{\overline{r}_n}$. However, for $r_i$ $(i = 1, 2, \cdots, n)$ we do not know whether $I_{r_i} = I_{\overline{r}_i}$ or $I_{r_i} = I_{\overline{r}_{i+1}}$. If for all $i \in \{1, 2, \cdots, n\}$, $I_{r_i} = I_{\overline{r}_i}$ holds, then the fuzzy matroid $(E, \Psi)$ is called *closed* [see Definition 2.1 below].

(2) Proposition 1.5 is partly cited from Observation 2.2 in [7], which says that for a fuzzy matroid $(E, \Psi)$, there is a finite sequence $r_0 < r_1 < \cdots < r_n$ such that

(i) $r_0 = 0,\ r_n \le 1$;

(ii) $I_s \ne \emptyset$ if $0 < s \le r_n$; $I_s = \emptyset$ if $s > r_n$;

(iii) If $r_i < s, t < r_{i+1}$, then $I_s = I_t,\ 0 \le i \le n-1$;

(iv) If $r_i < s < r_{i+1} < t < r_{i+2}$, then $I_s \supsetneqq I_t,\ 0 \le i \le n-2$.

The difference between this statement [7] and Proposition 1.5 is in $r_n$. In the sense of [7], there are two cases for $r_n$: is $r_n < 1$ and $r_n = 1$. If $r_n < 1$, by (ii), we have $I_s = \emptyset$ for all $s \in (r_n, 1]$, which is a contradiction, since $(E, I_s)$ is a matroid and $\emptyset \in I_s$. If $r_n = 1$, the statement (ii) is superfluous and Observation 2.2 is the same as Proposition 1.5. In fact, if $r_n < 1$, the status of $r_n$ in Observation 2.2 is the same as that of $r_{n-1}$ in Proposition 1.5.

For other concepts and notions, we refer the reader to [10] for matroids, [5, 7] for fuzzy matroids and [1] for category theory.

## 2. Categories of Fuzzy Matroids and Closed Fuzzy Matroids

**Definition 2.1.** [5] Let $(E, \Psi)$ be a fuzzy matroid with the fundamental sequence $0 = r_0 < r_1 < \cdots < r_n = 1$. $(E, \Psi)$ is called *closed* if $I_r = I_{\overline{r}_i}$ whenever $r_{i-1} < r \le r_i (1 \le i \le n)$.

From [5], we know that a closed fuzzy matroid has many properties that make it preferable to a general one (e.g. Theorem 1.9 and Theorem 1.10 in [5]).

**Proposition 2.2.** [5] *Let* $(E, \Psi)$ *be a fuzzy matroid with the fundamental sequence* $0 = r_0 < r_1 < \cdots < r_n = 1$ *and the induced matroids sequence* $I_{\overline{r}_1} \supsetneqq I_{\overline{r}_2} \supsetneqq \cdots \supsetneqq I_{\overline{r}_n}$. *For each* $r \in (r_{i-1}, r_i] (1 \le i \le n)$, *let* $\overline{I}_r = I_{\overline{r}_i}$. *Put*

$$\Psi^* = \{\mu \in \mathcal{F}(E) |\ C_r(\mu) \in \overline{I}_r,\ \forall r \in (0, 1]\}.$$

*Then* $(E, \Psi^*)$*, called the closure of* $(E, \Psi)$*, is the least closed fuzzy matroid containing* $(E, \Psi)$*(this means* $\Psi \subseteq \Psi^*$*). Obviously,* $(E, \Psi)$ *and* $(E, \Psi^*)$ *have the same fundamental sequence.*

**Definition 2.3.** Let $(E_i, I_i)(i = 1, 2)$ be two crisp matroids. A mapping $f : E_1 \longrightarrow E_2$ is called an *M-mapping* if $f^{-1}(A) \in I_1$ holds for all $A \in I_2$. We denote the concrete category of crisp matroids with $M$-mappings as morphisms by **M**.

**Definition 2.4.** Let $(E_i, \Psi_i)(i = 1, 2)$ be two fuzzy matroids. A mapping $f : E_1 \longrightarrow E_2$ is called an *FM-mapping* if $\mu \circ f \in \Psi_1$ holds for all $\mu \in \Psi_2$. We denote the concrete category of fuzzy matroids with $FM$-mappings as morphisms by **FM** and the full subcategory of **FM** formed by all closed fuzzy matroids by **CFM**.

**Proposition 2.5.** *Let $(E_i, \Psi_i)(i = 1, 2)$ be two fuzzy matroids and $f : E_1 \longrightarrow E_2$ be a mapping. $f$ is an FM-mapping from $(E_1, \Psi_1)$ to $(E_2, \Psi_2)$ if and only if $f$ is an $M$-mapping from $(E_1, C_r(\Psi_1))$ to $(E_2, C_r(\Psi_2))$ for all $r \in (0, 1]$.*

*Proof.* Let $f$ be an $FM$-mapping from $(E_1, \Psi_1)$ to $(E_2, \Psi_2)$. We must prove that $f^{-1}(C_r(\mu)) \in C_r(\Psi_1)$ for all $\mu \in \Psi_2$ and all $r \in (0, 1]$. Clearly, it is sufficient to find a $\nu \in \Psi_2$ satisfying $f^{-1}(C_r(\mu)) = C_r(\nu)$ and we see that $\nu = \mu \circ f \in \Psi_2$ satisfies the equation. Conversely, if $\mu \in \Psi_2$, we have $C_r(\mu) \in C_r(\Psi_2)$ and $f^{-1}(C_r(\mu)) \in C_r(\Psi_1)$. We need only to prove that $C_r(\mu \circ f) = f^{-1}(C_r(\mu))$ which can be verified by Proposition 1.4. □

**Proposition 2.6.** *Let $(E_i, \Psi_i)(i = 1, 2)$ be two fuzzy matroids and $(E_1, \Psi_1)$ be closed. If $f : (E_1, \Psi_1) \longrightarrow (E_2, \Psi_2)$ is an FM-mapping, then so is $f : (E_1, \Psi_1) \longrightarrow (E_2, \Psi_2^*)$.*

*Proof.* By Proposition 2.5, we need only to prove that Small $f : (E_1, C_r(\Psi_1)) \longrightarrow (E_2, C_r(\Psi_2^*))$ is an $M$-mapping for all $r \in (0, 1]$. Let $0 = s_0 < s_1 < \cdots < s_n = 1$ and $0 = r_0 < r_1 < \cdots < r_m = 1$ be the fundamental sequences of $(E_1, \Psi_1)$ and $(E_2, \Psi_2)$ respectively. $\forall r \in (0, 1]$, there exist $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots m\}$ such that $r \in (s_{i-1}, s_i] \cap (r_{j-1}, r_j]$. Let $\varepsilon = \frac{1}{2}\min\{r - s_{i-1}, r - r_{j-1}\}$. Then $r - \varepsilon \in (s_{i-1}, s_i] \cap (r_{j-1}, r_j]$. It is easy to verify that $C_r(\Psi_1) = C_{r-\varepsilon}(\Psi_1)$ and $C_r(\Psi_2^*) = C_{r-\varepsilon}(\Psi_2)$ and, since $f : (E_1, \Psi_1) \longrightarrow (E_2, \Psi_2)$ is an $FM$-mapping, this implies that $f : (E_1, C_r(\Psi_1)) \longrightarrow (E_2, C_r(\Psi^*))$ is an $M$-mapping. □

**Corollary 2.7.** *Let $(E_i, \Psi_i)(i = 1, 2)$ be two fuzzy matroids. If $f : (E_1, \Psi_1) \longrightarrow (E_2, \Psi_2)$ is an $FM$-mapping, then so is $f : (E_1, \Psi_1^*) \longrightarrow (E_2, \Psi_2^*)$.*

Therefore, there is a functor $cl$, called the closure functor, from **FM** to **CFM** which sends every fuzzy matroid $(E, \Psi)$ to its closure $(E, \Psi^*)$.

**Corollary 2.8.** *Let $i$ : **CFM** $\longrightarrow$ **FM** be the inclusion functor. Then*
(1) $\forall (E, \Psi) \in |\mathbf{FM}|$, $i \circ cl(\Psi) \supseteq \Psi$;
(2) $\forall (E, \Psi) \in |\mathbf{CFM}|$, $cl \circ i(\Psi) = \Psi$.

*Proof.* It is trivial. □

**Theorem 2.9.** **CFM** *is a concretely coreflective full subcategory of* **FM**.

## 3. The Category of Fuzzifying Matroids as a Subcategory of FM

In [11], Professor Fu-Gui Shi introduced a new fuzzification of matroids, called fuzzifying matroids, from another point of view and defined and studied their fuzzy rank functions. He also established a one-to-one correspondence between the sets of all fuzzifying matroids and all fuzzy rank functions.

**Definition 3.1.** [11] Let $\mathcal{I} : 2^E \longrightarrow [0,1]$ be a mapping satisfying:
(FYI1) $\mathcal{I}(\emptyset) = 1$;
(FYI2) If $A \subseteq B$, then $\mathcal{I}(A) \geq \mathcal{I}(B)$;
(FYI3) If $|A| < |B|$, then there exists $e \in B\text{-}A$ such that $\mathcal{I}(A \cup e) \geq \min\{\mathcal{I}(A), \mathcal{I}(B)\}$.
The pair $(E, \mathcal{I})$ is called a *fuzzifying matroid* on $E$.

In Definition 3.1, if we replace $[0,1]$ by $\{0,1\}$, then a fuzzifying matroid is just a crisp matroid. A fuzzy matroid in the sense of Goetschel and Voxman is the fuzzification of a crisp matroid by fuzzifying the domain, while a fuzzifying matroid is the fuzzification of a crisp matroids by fuzzifying the codomain.

Let $(E, \mathcal{I})$ be a fuzzifying matroid. $\forall r \in (0,\ 1]$, define $\mathcal{I}_{[r]} = \{A \subseteq E |\ \mathcal{I}(A) \geq r\}$. Then $(E, \mathcal{I}_{[r]})$ is a (crisp) matroid, called the *r-cut matroid* of $(E, \mathcal{I})$. $\forall s \in [0,\ 1)$, define $\mathcal{I}_{(s)} = \{A \subseteq E |\ \mathcal{I}(A) > s\}$. Then $(E, \mathcal{I}_{(s)})$ is a (crisp) matroid, called the *s-strong cut matroid* of $(E, \mathcal{I})$.

**Proposition 3.2.** [11] *Let* $\mathcal{I} : 2^E \longrightarrow [0,1]$ *be a map. Then the following are equivalent:*
(1) $(E, \mathcal{I})$ *is a fuzzifying matroid.*
(2) $\forall r \in (0,1]$, $(E, \mathcal{I}_{[r]})$ *is a crisp matroid.*
(3) $\forall s \in [0,1)$, $(E, \mathcal{I}_{(s)})$ *is a crisp matroid.*

**Example 3.3.** Let $(E, I)$ be a weighted matroid [3] with $w : E \longrightarrow [0,1]$ as the weighting function. Define $\mathcal{I}_I : 2^E \longrightarrow [0,1]$ by

$$\forall A \subseteq E, \mathcal{I}_I(A) = \begin{cases} \min\limits_{e \in A} w(e) & A \in I; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $(E, \mathcal{I}_I)$ is a fuzzifying matroid. Furthermore, if there is no loop in $(E, I)$, i.e., $\forall e \in E,\ \{e\} \in I$, then $\forall e \in E,\ w(e) = \mathcal{I}_I(\{e\})$. Thus, each loop-free weighted matroid can be viewed as a fuzzifying matroid.

**Remark 3.4.** A weighted matroid [2, 3] can be constructed from a weighted graph. In many practical examples the loop in a weighted graph may have no use. For example, the cities in a map and the roads between them consist of a weighted graph with the length of roads as the weight. When one want to travel from one city to another, the loops will never be walked through. Thus loops in these cases have no use.

**Theorem 3.5.** *Let* $(E, \mathcal{I})$ *be a fuzzifying matroid and* $(E, \mathcal{I}_{[r]})$ *be defined as above. Then there is a finite sequence* $0 = r_0 < r_1 < \cdots < r_n = 1$ *such that*
(1) *If* $r_i < s,\ t \leq r_{i+1}$, *then* $\mathcal{I}_{[s]} = \mathcal{I}_{[t]},\ 0 \leq i \leq n-1$;
(2) *If* $r_i < s \leq r_{i+1} < t \leq r_{i+2}$, *then* $\mathcal{I}_{[s]} \supsetneq \mathcal{I}_{[t]},\ 0 \leq i \leq n-2$.
*The sequence* $r_0, r_1, \cdots, r_n$ *is called the fundamental sequence of* $(E, \mathcal{I})$.

In order to prove Theorem 3.5, we first need some lemmas.

We shall define an equivalence relation $\sim$ on (0, 1] by $a \sim b \Leftrightarrow I_{[a]} = I_{[b]}$. Since $E$ is a finite set, the number of crisp matroids on $E$ is finite. Thus there exist at most a finite number of equivalence classes, which we denote by $A_1, A_2, \cdots, A_n$.

**Lemma 3.6.** (1) $\forall a, b \in (0,1]$, *if* $a \leq b$, *then* $I_{[a]} \supseteq I_{[b]}$.
(2) *Each* $A_i (i = 1, 2, \cdots, n)$ *is an interval.*

*Proof.* (1) is trivial. To show (2), we only need to show that $\forall i \in I$, $\forall a, b \in A_i$ with $a \leq b$, if $c \in [a, b]$, then $c \in A_i$. Suppose that $(E, I)$ is the cut matroid of the elements in $A_i$. Then $\mathcal{I}_{[a]} = \mathcal{I}_{[b]} = I$. Since $a \leq c \leq b$, we know by (1) that $I = \mathcal{I}_{[b]} \subseteq \mathcal{I}_{[c]} \subseteq \mathcal{I}_{[a]} = I$ and $\mathcal{I}_{[c]} = I$. Thus $c \in A_i$ by the definition of $A_i$ and so $A_i$ is an interval. □

**Lemma 3.7.** $\forall i \in \{1, 2, \cdots, n\}$, $\sup A_i \in A_i$.

*Proof.* Suppose $r_i = \sup A_i$ and $I$ is the corresponding independent family with respect to $A_i$. Then, obviously, $I \supseteq I_{[r_i]}$. $\forall A \in I$, $A \in \mathcal{I}_{[r]}$ holds for all $r \in A_i$. Hence $\mathcal{I}(A) \geq r$ for all $r \in A_i$, which implies $\mathcal{I}(A) \geq r_i$. Thus $A \in I_{[r_i]}$, which implies that $I \subseteq I_{[r_i]}$. Hence $I_{[r_i]} = I$ and $r_i \in A_i$. □

By Lemma 3.6 and Lemma 3.7, we conclude that there exist $r_1, r_2, \cdots, r_n \in (0, 1]$ which separate different cut matroids from each other. Suppose that $r_1 < r_2 < \cdots < r_n$. Then $r_n$ must be equal to 1 and Theorem 3.5 follows.

**Remark 3.8.** The corresponding independent family with respect to $A_1$ may equal $\mathcal{I}_{[0]} = 2^E$.

In what follows we shall study the relations between a fuzzifying matroid and its fundamental sequence.

**Proposition 3.9.** *Let* $0 = r_0 < r_1 < \cdots < r_n = 1$ *be a finite sequence,and* $I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_n$ *be a finite sequence of crisp independent families on* $E$. $\forall r \in (0, 1]$, *if* $r \in (r_{i-1}, r_i] (i \in \{1, \cdots, n\})$, *put* $I_{[r]} = I_{r_i}$ *and define* $\mathcal{I} : 2^E \longrightarrow [0, 1]$ *by*

$$\forall A \subseteq E, \mathcal{I}(A) = \max\{r \in (0, 1] |\ A \in I_{[r]}\}.$$

*Then* $(E, \mathcal{I})$ *is a fuzzifying matroid on* $E$ *and* $\mathcal{I}_{[r]} = I_r$ *for all* $r \in (0, 1]$.

*Proof.* Obviously, $\mathcal{I}$ is well-defined and (FYI1), (FYI2) hold. Suppose that $|A| < |B|$ and $r = \min\{\mathcal{I}(A), \mathcal{I}(B)\}$. If $r \neq 0$, then $r \in (r_{i-1}, r_i]$ for an $i \in \{1, 2, \cdots, n\}$. Hence $A, B \in I_{[r]} = I_{r_i}$, which implies that there exists $e \in B$-$A$ such that $A \cup e \in I_{r_i}$. Thus, $\mathcal{I}(A \cup e) \geq r_i \geq r$. Also, it is easy to show that $\mathcal{I}_{[r]} = I_r$ for all $r \in (0, 1]$. □

**Corollary 3.10.** *Let* $(E, \mathcal{I})$ *be a fuzzifying matroid with the fundamental sequence* $0 = r_0 < r_1 < \cdots < r_n = 1$. *Then*

$$\mathcal{I}(A) = \max\{r_i |\ A \in \mathcal{I}_{[r_i]},\ i \in \{1, 2, \cdots, n\}\}.$$

*Thus the codomain of* $\mathcal{I} : 2^E \longrightarrow [0, 1]$ *is contained in* $\{r_0, r_1, \cdots, r_n\}$.

*Proof.* We need only prove that $\mathcal{I}(A) = \max\{r \in (0,1]|\ A \in \mathcal{I}_{[r]}\}$. If $A \in \mathcal{I}_{[r]}$, then $\mathcal{I}(A) \geq r$. Hence $\mathcal{I}(A) \geq \max\{r \in (0,1]|\ A \in \mathcal{I}_{[r]}\}$. Now let $\mathcal{I}(A) = a$. Then $A \in \mathcal{I}_{[a]}$ and so $\max\{r \in (0,1]|\ A \in \mathcal{I}_{[r]}\} \geq a = \mathcal{I}(A)$. □

**Definition 3.11.** Let $(E_i, \mathcal{I}_i)(i = 1,2)$ be two fuzzifying matroids. A mapping $f : E_1 \longrightarrow E_2$ is called an *FYM-mapping* if $\mathcal{I}_1(f^{-1}(A)) \geq \mathcal{I}_2(A)$ holds for all $A \subseteq E$. We denote the category of fuzzifying matroids with $FYM$-mappings as morphisms by **FYM**.

**Proposition 3.12.** *Let $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ be two fuzzifying matroids and $f : E_1 \longrightarrow E_2$ be a mapping. Then $f$ is an FYM-mapping from $(E_1, \mathcal{I}_1)$ to $(E_2, \mathcal{I}_2)$ if and only if $f$ is an M-mapping from $(E_1, \mathcal{I}_{1[r]})$ to $(E_2, \mathcal{I}_{2[r]})$ for all $r \in (0,1]$.*

*Proof.* Let $f$ be an $FYM$-mapping from $(E_1, \mathcal{I}_1)$ to $(E_2, \mathcal{I}_2)$ and $r \in (0,1]$. $\forall A \in \mathcal{I}_{2[r]}$, we have $\mathcal{I}_1(f^{-1}(A)) \geq \mathcal{I}_2(A) \geq r$ and $f^{-1}(A) \in \mathcal{I}_{1[r]}$. In other words, $f$ is an $M$-mapping from $(E_1, \mathcal{I}_{1[r]})$ to $(E_2, \mathcal{I}_{2[r]})$. For the converse, we must prove that $\mathcal{I}_1(f^{-1}(A)) \geq \mathcal{I}_2(A)$ for any $A \subseteq E_2$. By Corollary 3.10, we need only show that $\{r \in (0,1]|\ f^{-1}(A) \in \mathcal{I}_{1[r]}\} \supseteq \{r \in (0,1]|\ A \in \mathcal{I}_{2[r]}\}$. This is equivalent to the statement that $f$ is an $M$-mapping from $(E_1, \mathcal{I}_{1[r]})$ to $(E_2, \mathcal{I}_{2[r]})$ for all $r \in (0,1]$. □

In the following discussion, we will study the relations between closed fuzzy matroids and fuzzifying matroids from the viewpoint of category theory.

Let $(E, \Psi)$ be a closed fuzzy matroid with the fundamental sequence $0 = r_0 < r_1 < \cdots < r_n = 1$ and the induced matroids sequence $I_{\overline{r}_1} \supsetneqq I_{\overline{r}_2} \supsetneqq \cdots \supsetneqq I_{\overline{r}_n}$. We define a mapping $\mathcal{I}_\Psi : 2^E \longrightarrow [0,1]$ by

$$\forall A \in 2^E, \mathcal{I}_\Psi(A) = \max\{s \in (0,1]|\ A \in I_s\}.$$

By Definition 3.1 and Proposition 3.9, $(E,\ \mathcal{I}_\Psi)$ is a fuzzifying matroid, called the fuzzifying matroid induced by $(E, \Psi)$. Obviously, every closed matroid and its induced fuzzifying matroid have the same fundamental sequence. Then, by Proposition 2.5 and Proposition 3.12, we easily obtain the following corollary.

**Corollary 3.13.** *Let $(E_i, \Psi_i)(i = 1,2)$ be two closed fuzzy matroids. If $f : (E_1, \Psi_1) \rightarrow (E_1, \Psi_2)$ is an FM-mapping, then $f$ is an FYM-mapping from $(E_1, \mathcal{I}_{\Psi_1})$ to $(E_2, \mathcal{I}_{\Psi_2})$.*

Let $(E, \mathcal{I})$ be a fuzzifying matroid with the fundamental sequence $0 = s_0 < s_1 < \cdots < s_n = 1$. $\forall s \in (s_{i-1},\ s_i](i \in \{1,2,\cdots,m)$, let $I_s = \mathcal{I}_{[s_i]}$ and

$$\Psi_\mathcal{I} = \{\mu \in \mathcal{F}(E)|\ \forall s \in (0,1],\ C_s(\mu) \in I_s\}.$$

By Proposition 1.4 and Proposition 2.2, $(E,\ \Psi_\mathcal{I})$ is a closed fuzzy matroid, called the fuzzy matroid induced by the fuzzifying matroid $(E, \mathcal{I})$. Obviously, every fuzzifying matroid and its induced fuzzy matroid have the same fundamental sequence. Corollary 3.14 follows easily from Proposition 2.5 and Proposition 3.12.

**Corollary 3.14.** *Let $(E_i, \mathcal{I}_i)(i = 1,2)$ be two fuzzifying matroids and $(E_i, \Psi_{\mathcal{I}_i})(i = 1,2)$ be the closed fuzzy matroid induced by $\mathcal{I}_i(i = 1,2)$. If $f : (E_1, \mathcal{I}_1) \longrightarrow (E_2, \mathcal{I}_2)$ is an FYM-mapping, then $f(E_1, \Psi_{\mathcal{I}_1}) \longrightarrow (E_2, \Psi_{\mathcal{I}_2})$ is an FM-mapping.*

Corollaries 3.13 and 3.14 indicate that there exists a pair of functors between the categories **CFM** and **FYM**.

**Corollary 3.15.** *Let* $(E, \Psi)$ *and* $(E, \mathcal{I})$ *respectively be a closed fuzzy matroid and a fuzzifying matroid. Then*
(1) $\Psi_{\mathcal{I}_\Psi} = \Psi$;
(2) $\mathcal{I}_{\Psi_\mathcal{I}} = \mathcal{I}$.

*Proof.* (1) and (2) follow straightforwardly from Proposition 1.4 and Corollary 3.10, respectively. □

Now, by Corollaries 3.13-3.15, we have

**Theorem 3.16.** **CFM** *is isomorphic to* **FYM**.

## 4. The Embedding of M in FYM

Let $(E, I)$ be a crisp matroid. Define $\mathcal{I} : 2^E \longrightarrow [0, 1]$ by $\mathcal{I} = \chi_I$, the characteristic function of $I$. Then $(E, \mathcal{I})$ is a fuzzifying matroid. Let $i : \mathbf{M} \longrightarrow \mathbf{FYM}$, $(E,\ I) \mapsto (E, \chi_I)$ be the inclusion functor.

In the next proposition, we will define two functors from **FYM** to **M**.

**Proposition 4.1.** (1) $F : \mathbf{FYM} \longrightarrow \mathbf{M}$, $(E, \mathcal{I}) \mapsto (E, \mathcal{I}_{[1]})$ *is a functor.*
(2) $G : \mathbf{FYM} \longrightarrow \mathbf{M}$, $(E, \mathcal{I}) \mapsto (E, \mathcal{I}_{(0)})$ *is a functor.*

*Proof.* The proof is trivial. □

**Proposition 4.2.** (1) *For any matroid* $(E, I)$, $F \circ i(I) = G \circ i(I) = I$.
(2) *For any fuzzifying matroid* $(E, \mathcal{I})$, $i \circ F(\mathcal{I}) \leq \mathcal{I}$, $i \circ G(\mathcal{I}) \geq \mathcal{I}$.

*Proof.* The proof is again trivial. □

Since both **M** and **FYM** are concrete categories, we have

**Theorem 4.3.** *Both* $(F, i)$ *and* $(i, G)$ *are Galois correspondences between* **M** *and* **FYM**, *and* $i$ *is the right inverse of both* $F$ *and* $G$. *Hence* **M** *can be embedded in* **FYM** *as a simultaneously reflective and coreflective subcategory.*

## 5. Conclusion

In this paper, it is shown that the category of fuzzifying matroids is isomorphic to that of closed fuzzy matroids, which is concretely coreflective in the category of fuzzy matroids. The category of matroids can be embedded in that of fuzzifying matroids as a simultaneously concretely reflective and coreflective subcategory. As has been stated in Section 1, none of the corresponding axioms related to circuits, bases, closure operators and rank functions has been established. Since closed fuzzy matroids have many more nice properties than fuzzy matroids (see Goetschel and Voxman's previous works [2-5]) and the recent paper on fuzzy matroids [9] is also mainly based on closed fuzzy matroids, the isomorphism between the category of fuzzifying matroids and that of closed fuzzy matroids indicates that the corresponding axioms may firstly be established on closed fuzzy matroids. We should also try to find a more reasonable definition of fuzzy matroids.

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