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MINIMIZATION OF DETERMINISTIC FINITE AUTOMATA WITH VAGUE (FINAL) STATES AND INTUITIONISTIC FUZZY (FINAL) STATES

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Abstract. In this paper, relations among the membership values of generalized fuzzy languages such as intuitionistic fuzzy language, interval-valued fuzzy language and vague language are studied. It will aid in studying the properties of one language when the properties of another are known.

Further, existence of a minimized finite automaton with vague (final) states for any vague regular language recognized by a finite automaton with vague (final) states is shown in this paper. Finally, an efficient algorithm is given for minimizing the finite automaton with vague (final) states. Similarly, it can be shown for intuitionistic fuzzy regular language. These may contribute to a better understanding of the role of finite automaton with vague (final) states or the finite automaton with intuitionistic fuzzy (final) states while studying lexical analysis, decision making etc.

1. Introduction

Fuzziness reduces the gap between formal language and natural language in terms of precision, leading to describe fuzzy language. Fuzzy language and fuzzy grammars were formerly defined by Lee and Zadeh [12]. A fuzzy language $\tilde{L}$ in the set of finite alphabet Σ, is a class of strings $w \in \Sigma^*$ along with a grade of membership function $f_{\tilde{L}}(w)$. This membership function assigns to each string a grade of membership value in [0, 1]. Fuzzy language is further generalized as intuitionistic fuzzy language (IFL) [17], interval-valued fuzzy language (IVFL) [18] and vague language (VL) [6] using the notion of intuitionistic fuzzy sets [1], [2], interval-valued fuzzy sets [8] and vague sets [7] respectively. Our motive is to study the membership values of these languages in a generalized set up. Here, we have shown that there is a relation between the membership values of the strings in IFL, IVFL and VL respectively.

One of the most significant branches of the algebraic theory of languages and automata is Myhill-Nerode's theory [9], where recognizability of regular languages by finite automata is studied through right invariant equivalence classes. Also, it is a powerful tool for minimizing the number of redundant states in a finite automaton. Myhill-Nerode's theorem has been extended to fuzzy regular language and also an

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algorithm is given for minimizing the deterministic finite automaton with fuzzy (final) states in [13], [15]. Since finite automata constitutes a mathematical model on computation, fuzzy finite automata may be considered as an extended model which includes notions like vagueness and imprecision frequently encountered in the study of fuzzy language. The models of general fuzzy automata and general fuzzy recognizers are given in [10] and [11] respectively.

In this paper, we have shown that for any vague regular language and for any intuitionistic fuzzy regular language recognizable respectively by vague (final) states automaton and intuitionistic fuzzy (final) states automaton, there exists a minimal vague (final) states automaton and a minimal intuitionistic fuzzy (final) states automaton. These automata are unique up to an isomorphism. Furthermore, an efficient algorithm is given for minimizing vague (final) states automaton and intuitionistic fuzzy (final) states automaton. These may help out to a better understanding of the role of vague (final) states automaton or intuitionistic fuzzy (final) states automaton while studying lexical analysis, decision making, and some other areas involving the manipulation of imprecise data.

2. **Preliminaries**

Basic definitions on fuzzy sets and fuzzy finite automata can be found in [19] and [14]. In this section, we have given some definitions.

Definition 2.1. Let Σ be a finite alphabet set and $f_{\tilde{L}}(w) : \Sigma^* \to M$ a function, where M is a set of real numbers in $[0, 1]$. Then the set $\tilde{L} = \{(w, f_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$ is called a fuzzy language (FL) [15] over Σ and $f_{\tilde{L}}(w)$ the membership function of $\tilde{L}$.

Definition 2.2. Let '$\tilde{L}$' be a fuzzy language over Σ, the finite alphabet set with $f_{\tilde{L}}(w) : \Sigma^* \to M$ as its membership function. Then, '$\tilde{L}$' is called a fuzzy regular language (FRL) [15] if;

(i) the set $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ is finite, and

(ii) for each $m \in M$, the string $S_{\tilde{L}}(m)$ is regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = m\}$ [15] and a string is regular, if it is recognized by a finite automaton [9].

Definition 2.3. Let Σ be a set of finite alphabet and $f_{\tilde{L}}(w) : \Sigma^* \to M$, $g_{\tilde{L}}(w) : \Sigma^* \to N$ are the functions, where M and N are the finite set of real numbers in [0, 1]. Then we call the set, $\tilde{L} = \{(w, f_{\tilde{L}}(w), g_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$ an intuitionistic fuzzy language (IFL) [17] over Σ.
Here, $f_{\tilde{L}}(w), g_{\tilde{L}}(w)$ represents respectively, the membership and nonmembership functions of $\tilde{L}$ and for any $w \in \Sigma^*$, $0 \leq f_{\tilde{L}}(w) + g_{\tilde{L}}(w) \leq 1$.

Definition 2.4. Let '$\tilde{L}$' be an IFL over Σ, the finite alphabet set with $f_{\tilde{L}}(w)$ and $g_{\tilde{L}}(w)$ as its membership and nonmembership functions. We call '$\tilde{L}$' an intuitionistic fuzzy regular language (IFRL) [17] if,

(i) the sets $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ and $\{l \in N \mid S_{\tilde{L}}(l) \neq \emptyset\}$ are finite, and

(ii) for each $m \in M$ the string $S_{\tilde{L}}(m)$ and for each $l \in N$ the string $S_{\tilde{L}}(l)$ are regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = m\}$ and $S_{\tilde{L}}(l) = \{w \in \Sigma^* \mid g_{\tilde{L}}(w) = l\}$ [17].

Definition 2.5. Let Σ be a finite alphabet set. Then we call the set, $\tilde{L} = \{(w, [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]) \mid w \in \Sigma^*\}$ an interval-valued fuzzy language (IVFL) [18], where $f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w) : \Sigma^* \to [0,1]$ represents the lower and the upper membership functions of $\tilde{L}$ respectively.
Here, for any $w \in \Sigma^*, 0 \leq f^L_{\tilde{L}}(w) \leq f^U_{\tilde{L}}(w) \leq 1$ and $0 \leq f^L_{\tilde{L}}(w) + (1 - f^U_{\tilde{L}}(w)) \leq 1$.

In short, $\tilde{L} = \{(w, f_{\tilde{L}}(w)) \mid w \in \Sigma^*\}$, where $f_{\tilde{L}}(w) = [f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]$ $\forall w \in \Sigma^*$.

Definition 2.6. Let '$\tilde{L}$' be an IVFL over Σ, the finite alphabet set, and $f_{\tilde{L}}(w)$ the membership function of '$\tilde{L}$'. Then we call, '$\tilde{L}$' an interval-valued fuzzy regular language (IVFRL) [18] if;

(i) the set $\{[m,n] \in I[0,1] \mid S_{\tilde{L}}[m,n] \neq \emptyset\}$ is finite, and

(ii) for each $[m,n] \in I[0,1]$ the string $S_{\tilde{L}}[m,n]$ is regular,

where $S_{\tilde{L}}[m,n] = \{w \in \Sigma^* \mid f_{\tilde{L}}(w) = [m,n]\}$ [18].

Definition 2.7. Let Σ be a finite alphabet set. Then we call the set, $\tilde{L} = \{(w, [t_{\tilde{L}}(w), 1 - f_{\tilde{L}}(w)]) \mid w \in \Sigma^*\}$ a vague language (VL) [6] over Σ. Here, $t_{\tilde{L}}(w), f_{\tilde{L}}(w) : \Sigma^* \to [0,1]$ represents respectively, the truth membership and the false membership functions of $\tilde{L}$, such that $0 \leq t_{\tilde{L}}(w) \leq 1 - f_{\tilde{L}}(w) \leq 1$ or
$0 \leq t_{\tilde{L}}(w) + f_{\tilde{L}}(w) \leq 1$.

Definition 2.8. Let '$\tilde{L}$' be a VL over Σ, the finite alphabet set with $t_{\tilde{L}}(w) : \Sigma^* \to M$, $f_{\tilde{L}}(w) : \Sigma^* \to N$ as its truth and false membership functions respectively. Then we call, '$\tilde{L}$' a vague regular language (VRL) [6] if,

(i) the sets $\{m \in M \mid S_{\tilde{L}}(m) \neq \emptyset\}$ and $\{n \in N \mid S_{\tilde{L}}(1-n) \neq \emptyset\}$ are finite, and

(ii) for each $m \in M$ and $n \in N$ the strings $S_{\tilde{L}}(m)$ and $S_{\tilde{L}}(1-n)$ are regular,

where $S_{\tilde{L}}(m) = \{w \in \Sigma^* \mid t_{\tilde{L}}(w) = m\}$ and $S_{\tilde{L}}(1-n) = \{w \in \Sigma^* \mid 1 - f_{\tilde{L}}(w) = 1 - n\}$ [6].

Definition 2.9. A nondeterministic finite automaton with vague (final) states (NDFA-VS) [6] '$\tilde{A}$' is a 7-tuple $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$, where Q is the finite set of states, Σ is the finite set of input alphabets, δ, γ: $Q \times \Sigma :\to 2^Q$ are the state transition functions i.e., $\delta(p,a) = q$ and $\gamma(p,a) = q$ for $p \in Q, q \in 2^Q$ and $a \in \Sigma$, q_0 is the vague starting state and $\tilde{T}_{F_{\tilde{A}}}$, $\tilde{F}_{F_{\tilde{A}}}$: $Q \to [0,1]$ are respectively, the truth and false membership functions of vague (final) state set.

Define,

$td_{\tilde{A}}(x) = \max\{\tilde{T}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \delta^*\}$ and

$fd_{\tilde{A}}(x) = \min\{\tilde{F}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \gamma^*\}$ or $fd_{\tilde{A}}(x) = \max\{1 - \tilde{F}_{F_{\tilde{A}}}(q) \mid (q_0, x, q) \in \gamma^*\}$, where $\delta^*, \gamma^* : Q \times \Sigma^* \to 2^Q$ are respectively, the reflexive and transitive closure of δ and γ.
The string 'x' is accepted by '$\tilde{A}$' with the truth degree $td_{\tilde{A}}(x)$ and the false degree $fd_{\tilde{A}}(x)$ with the condition $0 \leq td_{\tilde{A}}(x) + fd_{\tilde{A}}(x) \leq 1$.

The vague regular language accepted by '$\tilde{A}$' is denoted by $\tilde{L}(\tilde{A})$ and is given by the set, $\tilde{L}(\tilde{A}) = \{(x, [td_{\tilde{A}}(x), 1 - fd_{\tilde{A}}(x)]) \mid x \in \Sigma^*\}$.

Definition 2.10. A deterministic finite automaton with vague (final) states (DFA-VS) [6] $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ is a NDFA-VS with δ, γ: $Q \times \Sigma \to Q$ being functions instead of a relation.

For each $x \in \Sigma^*$, $td_{\tilde{A}}(x) = \tilde{T}_{F_{\tilde{A}}}(q)$, where $q = \delta^*(q_0, x)$ and
$fd_{\tilde{A}}(x) = \tilde{F}_{F_{\tilde{A}}}(q)$, where $q = \gamma^*(q_0, x)$.
Define, $td_{\tilde{A}}(x) = 0$ and $fd_{\tilde{A}}(x) = 1$ if $\delta^*(q_0, x)$ and $\gamma^*(q_0, x)$ are not defined (i.e., there is no transition for a string x from state q_0).

Note: Deterministic and nondeterministic finite automata with vague (final) states are called vague (final) states automaton.

Definition 2.11. A nondeterministic finite automaton with intuitionistic fuzzy (final) states (NDFA-IFS) [5] '$\tilde{A}$' is a 7-tuple $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$, where Q is the finite set of states, Σ is the finite set of input alphabets, δ, γ: $Q \times \Sigma :\to 2^Q$ are the state transition functions i.e., $\delta(p, a) = q$ and $\gamma(p, a) = q$ for $p \in Q, q \in 2^Q$ and $a \in \Sigma$, q_0 is the intuitionistic fuzzy starting state and $\tilde{F}_{1\tilde{A}}$, $\tilde{F}_{2\tilde{A}}$: $Q \to [0, 1]$ represents the membership and the nonmembership functions of intuitionistic fuzzy (final) state set respectively.

Define,
$d_{\tilde{A}}(x) = \max\{\tilde{F}_{1\tilde{A}}(q) \mid (q_0, x, q) \in \delta^*\}$ and
$n_{\tilde{A}}(x) = \min\{\tilde{F}_{2\tilde{A}}(q) \mid (q_0, x, q) \in \gamma^*\}$, where $\delta^*, \gamma^* : Q \times \Sigma^* \to 2^Q$ are the reflexive and transitive closure of δ and γ respectively.
The string 'x' is accepted by '$\tilde{A}$' with the degree $d_{\tilde{A}}(x)$ and the nondegree $n_{\tilde{A}}(x)$ such that $0 \le d_{\tilde{A}}(x) + n_{\tilde{A}}(x) \le 1$.
The intuitionistic fuzzy regular language accepted by '$\tilde{A}$' is denoted by $\tilde{L}(\tilde{A})$ and is given by the set, $\tilde{L}(\tilde{A}) = \{(x, d_{\tilde{A}}(x), n_{\tilde{A}}(x)) \mid x \in \Sigma^*\}$.

Definition 2.12. A deterministic finite automaton with intuitionistic fuzzy (final) states (DFA-IFS) [5] $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ is a NDFA-IFS with δ, γ: $Q \times \Sigma \to Q$ being functions instead of a relation.

For each $x \in \Sigma^*$, $d_{\tilde{A}}(x) = \tilde{F}_{1\tilde{A}}(q)$, where $q = \delta^*(q_0, x)$ and
$n_{\tilde{A}}(x) = \tilde{F}_{2\tilde{A}}(q)$, where $q = \gamma^*(q_0, x)$.
Define, $d_{\tilde{A}}(x) = 0$ and $n_{\tilde{A}}(x) = 1$ if $\delta^*(q_0, x)$ and $\gamma^*(q_0, x)$ are not defined (i.e., there is no transition for a string x from state q_0).

Note: Deterministic and nondeterministic finite automata with intuitionistic fuzzy (final) states are called intuitionistic fuzzy (final) states automaton.

3. Relation Between IFL, IVFL and VL

Intuitionistic fuzzy sets, interval-valued fuzzy sets and vague sets are the extensions of fuzzy set. Atanassov and Gargov [3] shown that, interval-valued fuzzy sets can be express in the form of intuitionistic fuzzy sets and Bustince and Burillo [4] shown that vague sets are equivalent to intuitionistic fuzzy sets. In this section, an attempt has been made to compare three models that extend fuzzy language theory: intuitionistic fuzzy language, interval-valued fuzzy language and vague language theory. Our exposition recalls the concept of their membership values resulting in some relations among them. With the help of this, the property of one language can be used to study the property of another.

The difference between IFL and IVFL is due to the definition of their membership values. In IFL, we have $(f_{\tilde{L}}(w), g_{\tilde{L}}(w))$ as the membership value of a string 'w', where each of $f_{\tilde{L}}(w)$ and $g_{\tilde{L}}(w)$ represents a value of 'w' in [0, 1]. These are respectively, the membership and the nonmembership values of 'w', with the condition that $0 \leq f_{\tilde{L}}(w) + g_{\tilde{L}}(w) \leq 1$. Whereas, in IVFL the membership value of a string 'w' is given by $[f^L_{\tilde{L}}(w), f^U_{\tilde{L}}(w)]$, where each of $f^L_{\tilde{L}}(w)$ and $f^U_{\tilde{L}}(w)$ represents a value in [0, 1]. These represents respectively, the lower and the upper membership values of 'w', with the condition that $0 \leq f^L_{\tilde{L}}(w) + (1 - f^U_{\tilde{L}}(w)) \leq 1$. It has been observed that the semantics of the membership value $(f_{\tilde{L}}(w))$ of a string 'w' in IFL is the same as lower membership value $(f^L_{\tilde{L}}(w))$ of the string 'w' in IVFL and the nonmembership value $(g_{\tilde{L}}(w))$ of the string 'w' in IFL is the same as $(1 - f^U_{\tilde{L}}(w))$ (i.e., complement of the upper membership value) of the string 'w' in IVFL.

Also, the membership value plays an important role while differentiating IFL and VL. The membership value of a string 'w' in IFL is explained above. The membership value of a string 'w' in VL is given by $[t_{\tilde{L}}(w), 1 - f_{\tilde{L}}(w)]$, where each of $t_{\tilde{L}}(w)$ and $f_{\tilde{L}}(w)$ represents a value in [0, 1]. These are respectively, the truth membership and the false membership values of 'w', with the condition that $0 \leq t_{\tilde{L}}(w) + f_{\tilde{L}}(w) \leq 1$. It has been observed that the semantics of the membership value $(f_{\tilde{L}}(w))$ of 'w' in IFL is the same as that of the truth membership value $(t_{\tilde{L}}(w))$ of 'w' in VL and the nonmembership value $(g_{\tilde{L}}(w))$ of 'w' in IFL is the same as that of the false membership value $(f_{\tilde{L}}(w))$ of 'w' in VL.

From the above discussion, one can obtain an equality relation between the membership values of the string 'w' in aforementioned languages (in sequence IFL, IVFL and VL) as follows:

(i) $f_{\tilde{L}}(w) = f^L_{\tilde{L}}(w) = t_{\tilde{L}}(w)$ and

(ii) $g_{\tilde{L}}(w) = 1 - f^U_{\tilde{L}}(w) = f_{\tilde{L}}(w)$.

4. Myhill-Nerode Theorem for Vague Regular Language

One classical problem in the theory of automata is equivalence, reduction and minimization of finite state automata. Myhill-Nerode theory is a branch of the algebraic theory of languages and automata in which formal languages and deterministic automata are studied through right invariant equivalence classes. This theorem has also been extended to FRL. In our study we have further extended Myhill-Nerode theory to VRL and IFRL and have provided an efficient algorithm for minimizing DFA-VS and DFA-IFS. It provides necessary and sufficient conditions for VL and IFL to be regular in terms of equivalence classes. In particular, right invariant equivalence classes have shown oneself to be very useful in the proof of existence and construction of the minimal DFA-VS and DFA-IFS recognizing VRL and IFRL, respectively.

Theorem 4.1. *The following three statements are equivalent to one another:*

(i) Some finite automaton with vague (final) states can accept a vague regular language $\tilde{L}$ over Σ.

(ii) $\tilde{L}$ is the union of some equivalence classes of a right invariant equivalence relation of finite index.

(iii) Let the relation $R_{\tilde{L}} \subseteq \Sigma^* \times \Sigma^*$ be defined as $xR_{\tilde{L}}y$ iff $\forall\, z \in \Sigma^*$, $t_{\tilde{L}}(xz) = t_{\tilde{L}}(yz)$ and $f_{\tilde{L}}(xz) = f_{\tilde{L}}(yz)$, then $R_{\tilde{L}}$ is an equivalence relation of finite index.

Proof. $(i) \Rightarrow (ii)$ Let $\tilde{L}$ be a vague regular language over Σ the finite alphabet set. Assume that $\tilde{L}$ is accepted by some DFA-VS $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$. Let $R_{\tilde{A}}$ be the equivalence relation $xR_{\tilde{A}}y$ iff $\delta(q_0, x) = \delta(q_0, y)$ and $\gamma(q_0, x) = \gamma(q_0, y)$. $R_{\tilde{A}}$ is right invariant since, for any z, $\delta(q_0, xz) = \delta(q_0, yz)$, if $\delta(q_0, x) = \delta(q_0, y)$ and $\gamma(q_0, xz) = \gamma(q_0, yz)$, if $\gamma(q_0, x) = \gamma(q_0, y)$. Then the index of $R_{\tilde{A}}$ is finite, since the index is at most the number of states in Q. Furthermore, $\tilde{L}$ is the union of those equivalence classes having a string 'x' such that $\delta(q_0, x)$ and $\gamma(q_0, x)$ is in $\tilde{T}_{F_{\tilde{A}}}$ and $\tilde{F}_{F_{\tilde{A}}}$ respectively (*i.e.,* the equivalence classes corresponding to the final states).

$(ii) \Rightarrow (iii)$ We show that any equivalence relation 'E' satisfying (ii) is a refinement of $R_{\tilde{L}}$; *i.e.,* some equivalence class of $R_{\tilde{L}}$ will be the superset of every equivalence class 'E'. Thus, the index of $R_{\tilde{L}}$ cannot be greater than the index of 'E' and so is finite. Assume that 'xEy'. For each $z \in \Sigma^*$, '$xzEyz$' and thus $\tilde{L}(xz) = \tilde{L}(yz)$ (since 'E' is right invariant). Hence, $xR_{\tilde{L}}y$. We conclude that each equivalence class of 'E' is the subset of some equivalence class of $R_{\tilde{L}}$.

$(iii) \Rightarrow (i)$ To show that $R_{\tilde{L}}$ is right invariant, suppose $xR_{\tilde{L}}y$, and let $w \in \Sigma^*$, we must prove that $xwR_{\tilde{L}}yw$; *i.e.,* for any z, $\tilde{L}(xwz) = \tilde{L}(ywz)$. Since $xR_{\tilde{L}}y$, for any v, $\tilde{L}(xv) = \tilde{L}(yv)$ (by the definition of $R_{\tilde{L}}$). Consider '$v = wz$' to prove $R_{\tilde{L}}$ is right invariant.

Now we present the minimized DFA-VS by constructing equivalence classes of $R_{\tilde{L}}$: Let Q' be the finite set of equivalence classes of $R_{\tilde{L}}$ and $[x] \in Q'$ containing x. Define, $\delta'([x], a) = [xa]$ and $\gamma'([x], a) = [xa]$. This definition is consistent as $R_{\tilde{L}}$ is right invariant. If we choose 'y' instead of 'x' from [x], we will have $\delta'([x], a) = [ya]$ and $\gamma'([x], a) = [ya]$. But $xR_{\tilde{L}}y$, so $\tilde{L}(xz) = \tilde{L}(yz)$. In particular, if $z = az'$, $\tilde{L}(xaz') = \tilde{L}(yaz')$, so $xaR_{\tilde{L}}ya$ and [xa]=[ya].

Let $q_0' = [\epsilon]$, $\tilde{T}'_{F_{\tilde{A}}} = \{[x] \mid x \in \tilde{L}\}$ and $\tilde{F}'_{F_{\tilde{A}}} = \{[x] \mid x \in \tilde{L}\}$. The finite automaton $\tilde{A}' = (Q', \Sigma, \delta', \gamma', q_0', \tilde{T}'_{F_{\tilde{A}'}}, \tilde{F}'_{F_{\tilde{A}'}})$ accepts $\tilde{L}$, since $\delta'(q_0', x) = \delta'([\epsilon], x) = [x]$ and $\gamma'(q_0', x) = \gamma'([\epsilon], x) = [x]$. Thus $\tilde{L}(\tilde{A}') = \tilde{L}(\tilde{A})$. $\square$

Algorithm 4.2. *Algorithm for minimizing deterministic finite automata with vague (final) states (DFA-VS)*

Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ be a DFA-VS. Assume that $Q = \{q_0, q_1, ..., q_n\}$, $\;n \geq 0$ and let $P = \{(q_i, q_j) \mid q_i, q_j \in Q \;\; and \;\; 0 \leq i < j \leq n\}$.

begin

Step 1: **for** each pair $(q_i, q_j) \in P$, and $\tilde{T}_{F_{\tilde{A}}}(q_i) \neq \tilde{T}_{F_{\tilde{A}}}(q_j)$ or $\tilde{F}_{F_{\tilde{A}}}(q_i) \neq \tilde{F}_{F_{\tilde{A}}}(q_j)$ do mark (q_i, q_j);

Step 2: **for** each unmarked pair $(q_i, q_j) \in P$ do

if for some $x \in \Sigma$, $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ is marked then

Step 2.1: mark (q_i, q_j);

Step 2.2: recursively mark all unmarked pairs on the list of (q_i, q_j) and on the list of other pairs that are marked at this step.

else

Step 2.3: **for** all input symbols 'x' do put (q_i, q_j) on the list for $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ unless $\delta(q_i, x) = \delta(q_j, x)$ and $\gamma(q_i, x) = \gamma(q_j, x)$.

Step 3: Equivalence classes of Q are constructed as follows;

For i = 0 to n - 1 do

For j = i + 1 to n do

if (q_i, q_j) is unmarked, q_j is in $[q_i]$, the equivalence class containing q_i.

Step 4: Define a minimum DFA-VS $\tilde{A}' = (Q', \Sigma, \delta', \gamma', q_0', \tilde{T}'_{F_{\tilde{A}'}}, \tilde{F}'_{F_{\tilde{A}'}})$ as follows;

$Q' = \{[q_i] \mid q_i \in Q\}$, $\delta'([q_i], a) = [\delta(q_i, a)]$, $\gamma'([q_i], a) = [\gamma(q_i, a)]$, $q_0' = [q_0]$, $\tilde{T}'_{F_{\tilde{A}'}}([q_i]) = \tilde{T}_{F_{\tilde{A}}}(q_i)$ and $\tilde{F}'_{F_{\tilde{A}'}}([q_i]) = \tilde{F}_{F_{\tilde{A}}}(q_i)$.

end.

Example 4.3. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{T}_{F_{\tilde{A}}}, \tilde{F}_{F_{\tilde{A}}})$ be a DFA-VS (Figure 1). Here, $Q = \{a, b, c, d, e, f\}$, $\Sigma = \{0, 1\}$, $q_0 = \{a\}$ the vague starting state with truth membership value $\tilde{T}_{F_{\tilde{A}}}(a) = 0.4$ and false membership value $\tilde{F}_{F_{\tilde{A}}}(a) = 0.5$. δ, $\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(a, 0) = \gamma(a, 0) = c$, $\delta(a, 1) = \gamma(a, 1) = b$, $\delta(b, 0) = \gamma(b, 0) = d$, $\delta(b, 1) = \gamma(b, 1) = a$, $\delta(c, 0) = \gamma(c, 0) = f$, $\delta(c, 1) = \gamma(c, 1) = e$, $\delta(d, 0) = \gamma(d, 0) = f$, $\delta(d, 1) = \gamma(d, 1) = e$, $\delta(e, 0) = \gamma(e, 0) = f$, $\delta(e, 1) = \gamma(e, 1) = e$, $\delta(f, 0) = \gamma(f, 0) = f$, $\delta(f, 1) = \gamma(f, 1) = f$, and $\tilde{T}_{F_{\tilde{A}}}(b) = 0.4$, $\tilde{F}_{F_{\tilde{A}}}(b) = 0.6$, $\tilde{T}_{F_{\tilde{A}}}(c) = 0.6$, $\tilde{F}_{F_{\tilde{A}}}(c) = 0.7$, $\tilde{T}_{F_{\tilde{A}}}(d) = 0.6$, $\tilde{F}_{F_{\tilde{A}}}(d) = 0.7$, $\tilde{T}_{F_{\tilde{A}}}(e) = 0.7$, $\tilde{F}_{F_{\tilde{A}}}(e) = 0.9$, and $\tilde{T}_{F_{\tilde{A}}}(f) = 0.2$, $\tilde{F}_{F_{\tilde{A}}}(f) = 0.3$ shows the truth and false membership values of the states {b}, {c}, {d}, {e}, and {f} respectively.

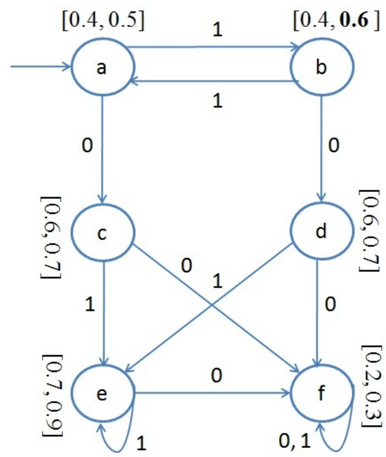

FIGURE 1. DFA-VS

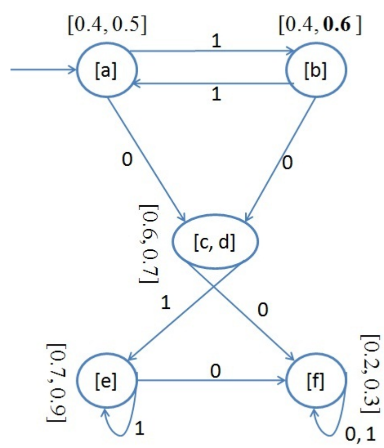

FIGURE 2. Minimized DFA-VS of Figure 1

Above DFA-VS (Figure 1) and its minimized DFA-VS (Figure 2) will accept the vague regular language;

$\tilde{L} = \{\mathbf{1(11)}^*/[\mathbf{0.4}, \mathbf{0.6}],\ (11)^*/[0.4, 0.5],\ 1^*0/[0.6, 0.7],\ 1^*01^+/[0.7, 0.9],\ 1^*00(0 + 1)^*/[0.2, 0.3],\ 1^*01^+0(0+1)^*/[0.2, 0.3]\}$.

Similarly, we can prove the Myhill-Nerode theorem for intuitionistic fuzzy regular language. An algorithm for minimizing DFA-IFS is given below.

Algorithm 4.4. *Algorithm for Minimizing Deterministic Finite Automata with Intuitionistic fuzzy (final) States (DFA-IFS)*

Let $\tilde{B} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{B}}, \tilde{F}_{2\tilde{B}})$ be a DFA-IFS [5]. Assume that $Q = \{q_0, q_1, ..., q_n\}$, $n \geq 0$ and let $P = \{(q_i, q_j) \mid q_i, q_j \in Q \ \ and \ \ 0 \leq i < j \leq n\}$.

begin

Step 1: **for** each pair $(q_i, q_j) \in P$, and $\tilde{F}_{1\tilde{B}}(q_i) \neq \tilde{F}_{1\tilde{B}}(q_j)$ or $\tilde{F}_{2\tilde{B}}(q_i) \neq \tilde{F}_{2\tilde{B}}(q_j)$ do mark (q_i, q_j);

Step 2: **for** each unmarked pair $(q_i, q_j) \in P$ do
  if for some $x \in \Sigma$, $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ is marked then
    Step 2.1: mark (q_i, q_j);
    Step 2.2: recursively mark all unmarked pairs on the list of (q_i, q_j) and on the list of other pairs that are marked at this step.
  else
    Step 2.3: **for** all input symbols 'x' do
    put (q_i, q_j) on the list for $(\delta(q_i, x), \delta(q_j, x))$ and $(\gamma(q_i, x), \gamma(q_j, x))$ unless $\delta(q_i, x) = \delta(q_j, x)$ and $\gamma(q_i, x) = \gamma(q_j, x)$.

Step 3: Equivalence classes of Q are constructed as follows;
  For i = 0 to n - 1 do
    For j = i + 1 to n do
      if (q_i, q_j) is unmarked, q_j is in $[q_i]$, the equivalence class containing q_i.

Step 4: Define a minimum DFA-IFS $\tilde{B}' = (Q', \Sigma, \delta', \gamma', q'_0, \tilde{F}'_{1\tilde{B}'}, \tilde{F}'_{2\tilde{B}'})$ as follows;
  $Q' = \{[q_i] \mid q_i \in Q\}$, $\delta'([q_i], a) = [\delta(q_i, a)]$, $\gamma'([q_i], a) = [\gamma(q_i, a)]$,
  $q'_0 = [q_0]$, $\tilde{F}'_{1\tilde{B}'}([q_i]) = \tilde{F}_{1\tilde{B}}(q_i)$ and $\tilde{F}'_{2\tilde{B}'}([q_i]) = \tilde{F}_{2\tilde{B}}(q_i)$.

end.

If intuitionistic fuzzy sets are reduced to fuzzy sets, we will consider only membership value. In that case DFA-IFS becomes deterministic finite automaton with fuzzy (final) states (DFA-FS). This DFA-FS may reduce further depending on the state transition and membership of each states (where strings with membership 0 in DFA-FS will not be considered as explained in example 4.2). For reducing DFA-FS we apply the algorithm given in [15]. Again, if fuzzy set is reduced to crisp set, then we will consider state with membership value zero in DFA-FS as non-final state in deterministic finite automaton (DFA) and all other states as final states in DFA. Thus DFA-FS becomes DFA. Depending on the state transition, DFA may get further reduced. For reducing automata, we will apply algorithm given in [9].

Further if intuitionistic fuzzy sets reduced to fuzzy sets, intuitionistic fuzzy regular language reduces to fuzzy refular language, where each string contains some membership value in [0, 1] and it can be recognized by DFA-FS and NDFA-FS [15]. The state reduction of DFA-FS will depend on membrship value of the state and state transition. Again, if fuzzy sets are reduced to crisp sets, fuzzy regular language becomes regular language. This regular language is accepted by a finite automaton and same can be reduced to its minimized form if possible [9].

Example 4.5. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a DFA-IFS (Figure 3). Here, $Q = \{p, q, r, s, t, u\}$, $\Sigma = \{a, b\}$, $q_0 = \{p\}$ the intuitionistic fuzzy starting state with membership value $\tilde{F}_{1\tilde{A}}(p) = 0.2$ and nonmembership value $\tilde{F}_{2\tilde{A}}(p) = 0.4$. δ,

$\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(p,a) = \gamma(p,a) = q$, $\delta(p,b) = \gamma(p,b) = t$, $\delta(q,a) = \gamma(q,a) = s$, $\delta(q,b) = \gamma(q,b) = r$, $\delta(r,a) = \gamma(r,a) = t$, $\delta(r,b) = \gamma(r,b) = u$, $\delta(s,a) = \gamma(s,a) = u$, $\delta(s,b) = \gamma(s,b) = r$, $\delta(t,a) = \gamma(t,a) = t$, $\delta(t,b) = \gamma(t,b) = t$, $\delta(u,a) = \gamma(u,a) = u$, $\delta(u,b) = \gamma(u,b) = u$, and $\tilde{F}_{1\tilde{A}}(q) = 0.3$, $\tilde{F}_{2\tilde{A}}(q) = 0.5$, $\tilde{F}_{1\tilde{A}}(r) = 0.2$, $\tilde{F}_{2\tilde{A}}(r) = 0.4$, $\tilde{F}_{1\tilde{A}}(s) = 0.3$, $\tilde{F}_{2\tilde{A}}(s) = 0.6$, $\tilde{F}_{1\tilde{A}}(t) = 0$, $\tilde{F}_{2\tilde{A}}(t) = 1$, and $\tilde{F}_{1\tilde{A}}(u) = 0$, $\tilde{F}_{2\tilde{A}}(u) = 1$ shows the membership and nonmembership value of the states {q}, {r}, {s}, {t}, and {u} respectively.

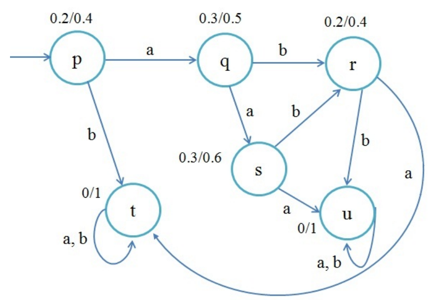

Figure 3. DFA-IFS

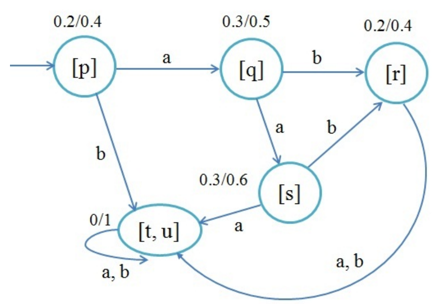

Figure 4. Minimized DFA-IFS of Figure 3

Above DFA-IFS (Figure 3) and its minimized DFA-IFS (Figure 4) will accept the intuitionistic fuzzy regular language;

$\tilde{L} = \{\epsilon/0.2/0.4, a/0.3/0.5, aa/0.3/0.6, ab, aab/0.2/0.4\}$.

If we change this DFA-IFS (Figure 3) to DFA-FS, the number of states of reduced DFA-FS [15] is the same as the number of states of reduced DFA-IFS. Again, if we change DFA-FS to DFA, the number of states of reduced DFA [9] will be the same as the number of states of reduced DFA-FS (here, (in DFA) we consider states with membership zero in DFA-FS as non final states).

Example 4.6. Let $\tilde{A} = (Q, \Sigma, \delta, \gamma, q_0, \tilde{F}_{1\tilde{A}}, \tilde{F}_{2\tilde{A}})$ be a DFA-IFS (Figure 5). Here, $Q = \{a, b, c, d, e\}$, $\Sigma = \{0, 1\}$, $q_0 = \{a\}$ the intuitionistic fuzzy starting state with membership value $\tilde{F}_{1\tilde{A}}(a) = 0.5$ and nonmembership value $\tilde{F}_{2\tilde{A}}(a) = 0.5$. δ, $\gamma : Q \times \Sigma \to Q$ are the transition functions given as $\delta(a, 0) = \gamma(a, 0) = b$, $\delta(a, 1) = \gamma(a, 1) = d$, $\delta(b, 0) = \gamma(b, 0) = c$, $\delta(b, 1) = \gamma(b, 1) = e$, $\delta(c, 0) = \gamma(c, 0) = b$, $\delta(c, 1) = \gamma(c, 1) = e$, $\delta(d, 0) = \gamma(d, 0) = c$, $\delta(d, 1) = \gamma(d, 1) = e$, $\delta(e, 0) = \gamma(e, 0) = e$, $\delta(e, 1) = \gamma(e, 1) = e$, and $\tilde{F}_{1\tilde{A}}(b) = 0.6$, $\tilde{F}_{2\tilde{A}}(b) = 0.3$, $\tilde{F}_{1\tilde{A}}(c) = 0.6$, $\tilde{F}_{2\tilde{A}}(c) = 0.2$, $\tilde{F}_{1\tilde{A}}(d) = 0.6$, $\tilde{F}_{2\tilde{A}}(d) = 0.3$, and $\tilde{F}_{1\tilde{A}}(e) = 0$, $\tilde{F}_{2\tilde{A}}(e) = 1$ shows the membership and nonmembership value of the states {b}, {c}, {d}, and {e} respectively.

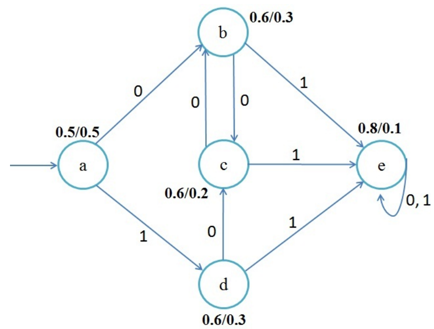

Figure 5. DFA-IFS

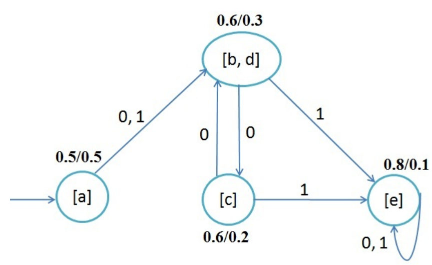

Figure 6. Minimized DFA-IFS of Figure 5

Above DFA-IFS (Figure 5) and its minimized DFA-IFS (Figure 6) will accept the intuitionistic fuzzy regular language;

$\tilde{L} = \{\epsilon/0.5/0.5,\ 0(00)^*/0.6/0.3,\ 0(00)^*0/0.6/0.2,\ 1(00)^*/0.6/0.3,\ 1(00)^*0/0.6/0.2,\ 0(00)^*1(0+1)^*/0.8/0.1,\ 1(00)^*1(0+1)^*/0.8/0.1,\ 0(00)^*01(0+1)^*/0.8/0.1,\ 1(00)^*01(0+1)^*/0.8/0.1\}$.

DFA-IFS (Figure 5) is changed to DFA-FS (Figure 7) and its minimized DFA-FS is given in Figure 8 accepting the fuzzy regular language

$\tilde{L} = \{\epsilon/0.5, 0^+, 10^*/0.6, 0^+1(0+1)^*, 10^*1(0+1)^*/0.8\}$ [13].

DFA-FS (Figure 7) is changed to DFA (Figure 9). Its minimized DFA is given in Figure 10 and both of these will accept the regular language; $\tilde{L} = \{0+1\}^*$ [9].

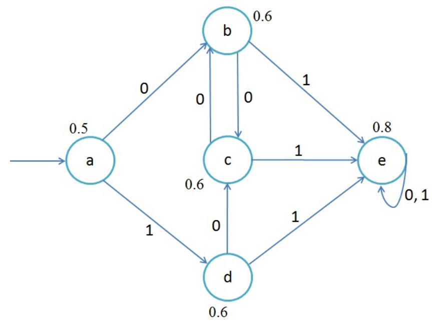

Figure 7. DFA-FS

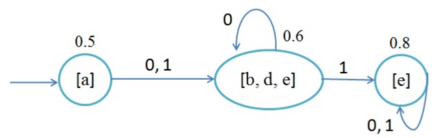

Figure 8. Minimized DFA-FS of Figure 7

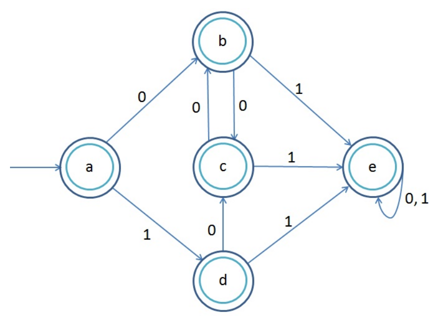

Figure 9. DFA

5. **Conclusion**

The extensive research being done on intuitionistic fuzzy sets, interval-valued fuzzy sets and vague sets (a survey [16] lists over 400 publications in the domain of intuitionistic fuzzy set theory alone, and the number is still growing fast) shows a mounting interest in these models. This paper has attempted to mend the situation by obtaining a relation between the membership values of IFL, IVFL and VL. It discusses the extended Myhill-Nerode theorem in the framework of VRL and IFRL also, it explains the method of minimizing DFA-VS and DFA-IFS through an algorithm. The theory of VL and IFL may prove to be of relevance in the construction of better models for natural languages. These may contribute to a better

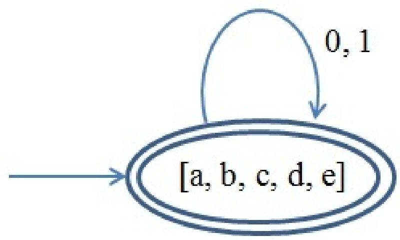

FIGURE 10. Minimized DFA of Figure 9

understanding of the role of vague (final) states automaton or intuitionistic fuzzy (final) states automaton in lexical analysis, decision making, pattern recognition, learning systems and other processes involving the manipulation of imprecise data.

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