ON THE DIAGRAM OF ONE TYPE MODAL OPERATORS ON INTUITIONISTIC FUZZY SETS: LAST EXPANDING WITH $Z^{\omega,\theta}_{\alpha,\beta}$

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Abstract. Intuitionistic Fuzzy Modal Operator was defined by Atanassov in [3] in 1999. In 2001, [4], he introduced the generalization of these modal operators. After this study, in 2004, Dencheva [14] defined second extension of these operators. In 2006, the third extension of these was defined in [6] by Atanassov. In 2007,[11], the author introduced a new operator over Intuitionistic Fuzzy Sets which is a generalization of Atanassov's and Dencheva's operators. At the same year, Atanassov defined an operator which is an extension of all the operators defined until 2007. The diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets was introduced first in 2007 by Atanassov [10]. In 2008, Atanassov defined the most general operator and in 2010 the author expanded the diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets with the operator $Z^{\omega}_{\alpha,\beta}$. Some relationships among these operators were studied by several researchers[5]-[8] [11], [13], [14]- [19]. The aim of this paper is to expand the diagram of one type modal operators over intuitionistic fuzzy sets . For this purpose, we defined a new modal oparator $Z^{\omega,\theta}_{\alpha,\beta}$ over intuitionistic fuzzy sets. It is shown that this oparator is the generalization of the operators $Z^{\omega}_{\alpha,\beta}, E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$.

1. Introduction

The theory of fuzzy sets (FSs), proposed by Zadeh[20], has gained successful applications in various fields. However, the membership function of the fuzzy set is a single value between zero and one, which combines the favoring evidence and the opposing evidence. According to Fuzzy Set Theory, if the membership degree of an element x is $\mu(x)$, the nonmembership degree is $1-\mu(x)$; thus, it is fixed.

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 [1] and formed an extension of fuzzy sets by enlarging the truth value set to the lattice $[0,1]\times[0,1]$ as defined below.

Definition 1.1. Let $L=[0,1]$ then $L^*=\{(x_1,x_2)\in[0,1]^2 : x_1+x_2\leq 1\}$ is a lattice with

$$(x_1,x_2)\leq(y_1,y_2) :\Longleftrightarrow \text{“}x_1\leq y_1 \textit{ and } x_2\geq y_2\text{”}$$

The units of this lattice are denoted by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. The lattice $(L^*,\leq)$ is a complete lattice: For each $A\subseteq L^*$,

$$supA=(sup\{x\in[0,1]:y\in[0,1],(x,y)\in A\}, inf\{y\in[0,1]:x\in[0,1],(x,y)\in A\})$$

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and

$infA = (inf\{x \in [0,1] : y \in [0,1], (x,y) \in A\}, sup\{y \in [0,1] : x \in [0,1], (x,y) \in A\})$

As it is well known, every lattice $(L^*, \leq)$ has an equivalent definition as an algebraic structure $(L, \wedge, \vee)$ where the meet operator "$\wedge$" and the join operator "$\vee$" are linked to the ordering " $\leq$ " as in the following equivalence, for $x, y \in L^*$,

$$x \leq y \Longleftrightarrow x \vee y = y \Longleftrightarrow x \wedge y = x$$

The operators $\wedge$ and $\vee$ (join and meets resp.) on $(L^*, \leq)$ are defined as follows, for $(x_1, y_1), (x_2, y_2) \in L^*$:

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2)$$

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$$

Definition 1.2. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 1.3. [1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.4. [1]Let $A \in IFS$ and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ then the set

$$A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$$

is called the complement of A.

The modal operators have been known to be important tools for IFSs where the operators are defined on the contrary to the FSs. Intuitionistic fuzzy operators and some properties of these operators were examined by several authors [6], [7], [13], [14]. In addition, the fuzzy closer operators, the intuitionistic fuzzy topological operators defined on the mathematical structure have also been studied [15], [16], [18], [19].

The notion of Intuitionistic Fuzzy Operators was firstly introduced by Atanassov [2]. The simplest one among them is presented as in the following definition.

Definition 1.5. [3] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X), \alpha, \beta \in [0,1]$.

(1) $\boxplus A = \{< x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} >: x \in X\}$
(2) $\boxtimes A = \{< x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2} >: x \in X\}$

After this definition, in 2001, Atanassov, in [4], defined the following extension of these operators:

Definition 1.6. [4] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$.

(1) $\boxplus_\alpha A = \{< x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha >: x \in X\}$
(2) $\boxtimes_\alpha A = \{< x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x) >: x \in X\}$

In these operators $\boxplus_\alpha$and $\boxtimes_\alpha$; If we choose $\alpha = \frac{1}{2}$, we get the operators $\boxplus$, $\boxtimes$, resp. Therefore, the operators $\boxplus_\alpha$and $\boxtimes_\alpha$ are the extensions of the operators $\boxplus$, $\boxtimes$, resp. Some relationships between these operators were studied by several authors ([14],[17]) In 2004, the second extension of these operators was introduced by Dencheva in [14].

Definition 1.7. [14] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$.

(1) $\boxplus_{\alpha,\beta} A = \{< x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta >: x \in X\}$ where $\alpha + \beta \in [0,1]$.
(2) $\boxtimes_{\alpha,\beta} A = \{< x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x) >: x \in X\}$where $\alpha + \beta \in [0,1]$.

The concepts of the modal operators are introduced and studied by different researchers, [4, 5, 6, 7, 8], [14], [15], [17], etc.

In 2006, the third extension of the above operators was studied by Atanassov . He defined the following operators in [6]

Definition 1.8. [6]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$.

(1) $\boxplus_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma >: x \in X\}$ where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.
(2) $\boxtimes_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) >: x \in X\}$ where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.

If we choose $\alpha = \beta$ and $\gamma = \beta$ in the above operators, then we can see easily that $\boxplus_{\alpha,\alpha,\gamma} = \boxplus_{\alpha,\beta}$and $\boxtimes_{\alpha,\alpha,\gamma} = \boxtimes_{\alpha,\beta}$.Therefore, we can say that $\boxplus_{\alpha,\beta,\gamma}$and $\boxtimes_{\alpha,\beta,\gamma}$are the extensions of the operators $\boxplus_{\alpha,\beta}$, $\boxtimes_{\alpha,\beta}$,resp. From these extensions, we get the first diagram of one type modal operators over Intuitionistic Fuzzy Sets as displayed in Figure1.

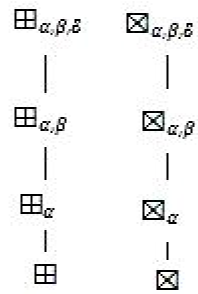

FIGURE 1

In 2007, after this diagram, the author [11] defined a new operator and studied some of its properties. This operator is named $E_{\alpha,\beta}$ and defined as follows:

Definition 1.9. [11]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta) >: x \in X\}$$

If we choose $\alpha = 1$ and subtitute α for β we get the operator $\boxplus_\alpha$. Similarly, if $\beta = 1$ is chosen and substituted for β, we get the operator $\boxtimes_\alpha$. In the wiev of this definition, the diagram of one type modal operators on IFSs is figured below:

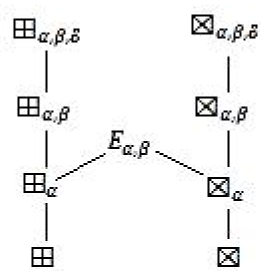

FIGURE 2

These extensions have been investigated by several authors [14], [7, 8, 9]. In particular, the authors have made significant contributions to these operators. In 2007, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [7].

Definition 1.10. [7] Let Xbe a set, A$\in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1$$

then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta >: x \in X\}$$

This operator changes the one type modal operators' diagram as in fig3,

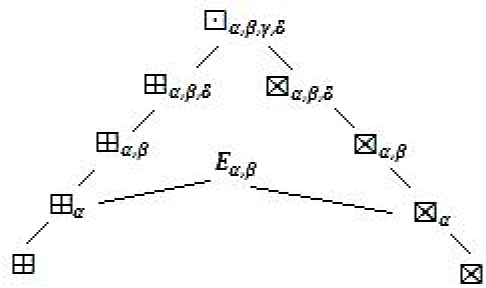

FIGURE 3

At the end of the these studies, Atanassov though that this diagram was completed. However, he realized that it wasn't totally true since there was an operator which was also an extension of two type modal oparators.

In 2008, he defined this most general operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$as following:

Definition 1.11. [8] Let X be a set,$A \in IFS(X)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$ such that

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1$$

and

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

then the operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined by

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \{< x, \alpha\mu_A(x) - \varepsilon\nu_A(x) + \gamma, \beta\nu_A(x) - \zeta\mu_A(x) + \delta >: x \in X\}$$

After this definition, the one type modal operators' diagram becomes as in Figure 4.

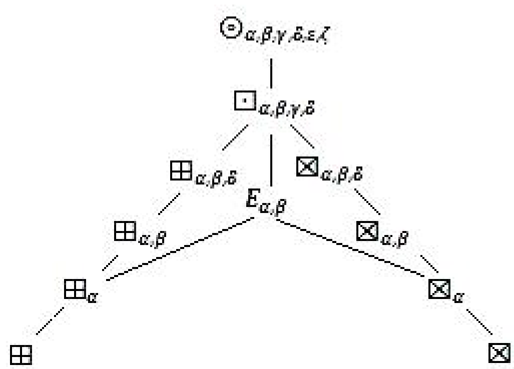

FIGURE 4

In 2010, the author [12] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 1.12. [12]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0,1]$. We define the following operator:

$$Z^{\omega}_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\}$$

Likewise the other one type modal operators, some properties of the operator $Z^{\omega}_{\alpha,\beta}$ were studied in [12] and [13].

Proposition 1.13. [12]*Let $\alpha, \beta \in I$, $\alpha \geq \beta$ and let $A \in IFS(X)$ then*

(1) $Z^{1}_{\alpha,\beta}(A) = E_{\alpha,\beta}$
(2) $Z^{1}_{\frac{1}{2},1}(A) = \boxtimes A$
(3) $Z^{1}_{1,\frac{1}{2}}(A) = \boxplus A$
(4) $Z^{1}_{\alpha,1}(A) = \boxtimes_{\alpha} A$
(5) $Z^{1}_{1,\alpha}(A) = \boxplus_{\alpha} A$

Theorem 1.14. [12]*Let $\alpha, \beta, \omega \in I$, and let $A \in IFS(X)$.*

(1) *If* $\alpha \geq \beta$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\beta,\alpha}(A)$
(2) *If* $\lambda, \tau \in I$, $\tau \geq \beta, \alpha \geq \lambda, \alpha\beta = \lambda\tau$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\lambda,\tau}(A)$
(3) *If* $\theta \in I, \alpha\beta\theta = \omega$ *then* $Z^{\omega}_{\alpha,\beta}(Z^{\theta}_{\beta,\alpha}(A)) = Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A))$
(4) *If* $\theta \in I$, $\alpha \geq \beta, \alpha - \alpha\beta \geq \omega \geq \beta - \alpha\beta, \alpha - \alpha\beta \geq \theta \geq \beta - \alpha\beta$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta}_{\beta,\alpha}(A)$
(5) $Z^{\omega}_{\alpha,\beta}(A) = (Z^{\omega}_{\beta,\alpha}(A^c))^c$
(6) *If* $\alpha \neq 0$ *and* $\beta \leqslant \omega$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta}_{\alpha,\omega}(A)$ *and* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(7) *If* $\beta \neq 0$ *and* $\omega \leqslant \alpha$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\alpha}_{\omega,\beta}(A)$ *and* $Z^{\beta}_{\alpha,\omega}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$
(8) *If* $\omega \neq 0$ *and* $\beta \leqslant \alpha$ *then* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$ *and* $Z^{\beta}_{\alpha,\omega}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(9) *If* $\beta \leqslant \alpha \leqslant \omega$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$ *and* $Z^{\alpha}_{\omega,\beta}(A) \sqsubseteq Z^{\beta}_{\alpha,\omega}(A)$
(10) *If* $\alpha \leqslant \beta \leqslant \omega$ *then* $Z^{\beta}_{\omega,\alpha}(A) \sqsubseteq Z^{\alpha}_{\beta,\omega}(A)$
(11) *If* $\beta \leqslant \omega \leqslant \alpha$ *then* $Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta}_{\omega,\alpha}(A)$
(12) *If* $\theta \in I, \beta \neq 1, \alpha \geq \beta, \theta > \omega, \alpha^2 \geq \frac{1-\alpha}{1-\beta}$ $A \in IFS(X)$ *then* $Z^{\omega}_{\beta,\alpha}(Z^{\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega}_{\alpha,\beta}(A))$
(13) *If* $\alpha \neq 1$, $0 \leqslant \alpha + \beta \leqslant 1$ *then* $Z^{\frac{\beta}{1-\alpha}}_{1,\alpha}(A) = \boxplus_{\alpha,\beta}(A)$ *and* $Z^{\frac{\beta}{1-\alpha}}_{\alpha,1}(A) = \boxtimes_{\alpha,\beta}(A)$
(14) *If* $\alpha + \beta \leqslant 1$ *then*$Z^{\omega}_{\alpha,\beta}$*represents* $\boxplus_{\alpha,\beta}$ *and* $\boxtimes_{\alpha,\beta}$
(15) *If we write* $\alpha\beta, \beta\omega(1-\alpha), \alpha\beta, \alpha\omega(1-\beta)$ *instead of* $\alpha, \beta, \gamma, \delta$ *resp. then* $\boxdot_{\alpha,\beta,\gamma,\delta}$ *represents* $Z^{\omega}_{\alpha,\beta}$.

The opposite of the Theorem1.14.(15) is not valid. Regarding the properties represented above, the diagram of one type modal operators over IFSs is displayed in Figure 5.

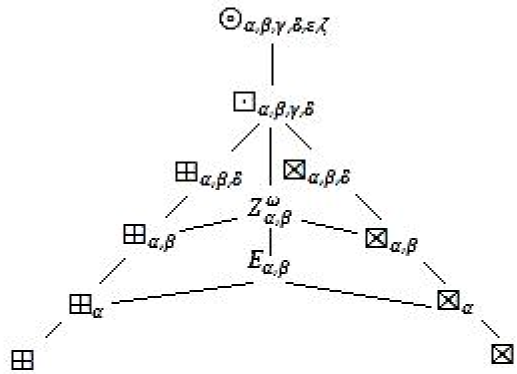

FIGURE 5

2. Some Relationships of One Type Modal Operators Over Intuitionistic Fuzzy Sets

In [13], the relationships between some One Type Modal Operators over Intuitionistic Fuzzy Sets have been studied. Some important relations are as follows:

Theorem 2.1. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I$ *and* $A \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A))$
(2) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$, *for* $\theta_1 \leq \theta_2$

Corollary 2.2. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \max\{\theta_1, \theta_2\} + \theta_1 \in I, A \in IFS(X)$

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$
(2) *If* $\theta_1 \leq \theta_2$ *then* $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_1}(A))$
(3) *If* $\theta_2 \leq \theta_1$ *then* $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_2,\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_2}(A))$

Theorem 2.3. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I, A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$$

Theorem 2.4. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2, \theta_3 \in I, \max\{\theta_1, \theta_2\} + \theta_3 \in I$, $A \in IFS(X)$.

(1) *If* $\theta_2 \leq \min\{\theta_1, \theta_3\}$ *then* $Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2,\theta_3}(A))$
(2) *If* $\theta_1 \leq \theta_2$ *then* $Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1\theta_2\theta_3}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$

Corollary 2.5. [13]*Let* $\alpha, \beta, \omega, \theta, \tau \in I, \theta \leq \tau$ *and let* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta,\theta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta,\theta,\tau}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta}(A))$$

Theorem 2.6. [13]*Let* $A, B \in IFS(X)$ *then* $A \sqsubseteq B \Rightarrow Z^{\omega}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega}_{\alpha,\beta}(B)$

Theorem 2.7. [13]*Let* $\alpha, \beta, \omega \in I$ *and* $A, B \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(A \cap B) = Z^{\omega}_{\alpha,\beta}(A) \cap Z^{\omega}_{\alpha,\beta}(B)$
(2) $Z^{\omega}_{\alpha,\beta}(A \cup B) = Z^{\omega}_{\alpha,\beta}(A) \cup Z^{\omega}_{\alpha,\beta}(B)$

Theorem 2.8. [13]*Let* $\alpha, \beta, \omega \in I, \beta \leq \alpha$ *and* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega}_{\beta,\alpha}(A)) \sqsubseteq Z^{\omega}_{\beta,\alpha}(Z^{\omega}_{\alpha,\beta}(A))$$

Corollary 2.9. [13]*Let* $\alpha, \beta, \omega \in I, \alpha\beta \leq \omega, \alpha \leq \beta$ *and* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(E_{\beta,\alpha}(A)) \sqsubseteq Z^{\omega}_{\beta,\alpha}(E_{\alpha,\beta}(A))$$

Theorem 2.10. [13]*Let* $\alpha, \beta, \omega, \theta \in I,\ A \in IFS(X)$ *then*

(1) $Z^{\omega}_{\alpha,\beta}(\boxplus(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes(A))$
(2) $Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta}(A))$
(3) $\theta_1 + \theta_2 \leq 1 \Rightarrow Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2}(A))$
(4) $\max\{\theta_1 + \theta_2\} + \theta_3 \leq 1 \Rightarrow Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_3}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1,\theta_2,\theta_3}(A))$

Corollary 2.11. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I, \theta_1 + \theta_2 \in I$ *and let* $A \in IFS(X)$ *then*

$$Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1\theta_2}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\boxtimes_{\theta_1}(A))$$

Theorem 2.12. [13]*Let* $\alpha, \beta, \omega, \theta \in I,\ A \in IFS(X)$ *then*

(1) $\boxplus(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega}_{\alpha,\beta}(A))$
(2) $\boxplus_{\theta}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta}(Z^{\omega}_{\alpha,\beta}(A))$
(3) $\theta_1 + \theta_2 \in I \Rightarrow \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A))$
(4) $\max\{\theta_1 + \theta_2\} + \theta_3 \in I \Rightarrow \boxplus_{\theta_1,\theta_2,\theta_3}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2,\theta_3}(Z^{\omega}_{\alpha,\beta}(A))$

Theorem 2.13. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2, \theta_1 + \theta_2 \in I$ *and let* $A \in IFS(X)$ *then*

(1) $\boxplus_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A))$
(2) $\boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A))$

Corollary 2.14. [13]*Let* $\alpha, \beta, \omega, \theta_1, \theta_2 \in I,\ \theta_1 + \theta_2 \in I,\ A \in IFS(X)$.

(1) $\boxplus_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1,\theta_2}(Z^{\omega}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\theta_1}(Z^{\omega}_{\alpha,\beta}(A))$
(2) $E_{\alpha,\beta}(\boxplus_{\theta_1}(A)) \sqsubseteq E_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2}(A)) \sqsubseteq E_{\alpha,\beta}(\boxplus_{\theta_1,\theta_2,\theta_1}(A))$
(3) $\alpha, \beta, \theta, \tau \in I, \theta \leq \tau$ *and let* $A \in IFS(X)$ *then*

$$E_{\alpha,\beta}(\boxtimes_{\theta,\theta}(A)) \sqsubseteq E_{\alpha,\beta}(\boxtimes_{\theta,\theta,\tau}(A)) \sqsubseteq E_{\alpha,\beta}(\boxtimes_{\theta}(A))$$

Theorem 2.15. [13]*Let* $\alpha, \beta, \omega \in I, A \in IFS(X)$ *then*

(1) $I_{\mu}(Z^{\omega}_{\alpha,\beta}(A)) = Z^{\omega}_{\alpha,\beta}(I_{\mu}(A))$
(2) $Z^{\omega}_{\alpha,\beta}(I_{\nu}(A)) \sqsubseteq I_{\nu}(Z^{\omega}_{\alpha,\beta}(A))$

3. The One Type Modal Operator $Z_{\alpha\beta}^{\omega\theta}$

It can be asked that whether or not there is any one type modal operator on IFs different from the others and is it possible to extend the last one type modal operators diagram. The answer is "yes". In this study, we have defined a new one type modal operator on IFS, that is generalization of the some one type modal operators. The new operator defined as follows:

Definition 3.1. Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0, 1]$. We define the following operator:

$$Z_{\alpha,\beta}^{\omega,\theta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\}$$

If A$\in IFS(X)$ then $\mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$. For $\alpha, \beta \in I$ if we use $\alpha\beta - \beta \leq 0, \alpha\theta - 1 \leq 0$ and $1 - \omega \geq 0, 1 - \beta \geq 0$ then

$$(\alpha\beta - \beta)(1 - \omega) + (\alpha\theta - 1)(1 - \beta) \leq 0.$$

Thus,

$$\begin{aligned} &\alpha\beta(\mu_A(x) + \nu_A(x)) + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta \\ &\leq \alpha\beta + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta \\ &= \alpha\beta + \beta\omega - \alpha\beta\omega + \alpha\theta - \alpha\beta\theta + \beta - \beta + 1 - 1 \\ &= \alpha\beta(1 - \omega) - \beta(1 - \omega) + \alpha\theta(1 - \beta) - (1 - \beta) + 1 \\ &= (\alpha\beta - \beta)(1 - \omega) + (\alpha\theta - 1)(1 - \beta) + 1 \\ &\leq 1 \end{aligned}$$

therefore if $A \in IFS(X)$ then $Z_{\alpha,\beta}^{\omega,\theta}(A) \in IFS(X)$. It is clear that

$$Z_{\alpha,\beta}^{\omega,\omega}(A) = Z_{\alpha,\beta}^{\omega}(A)$$

Similar to other operators, this operator has some properties and relationships by itself. In this section, we study these properties.

Theorem 3.2. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and let $A \in IFS(X)$ then*

$$Z_{\alpha,\beta}^{\omega,\theta}(A) \sqsubseteq Z_{\alpha,\beta}^{\omega}(A)$$

Proof. If we use $\omega \leq \theta$ then $\alpha\omega \leq \alpha\theta$.

$$\begin{aligned} Z_{\alpha,\beta}^{\omega,\theta}(A) &= \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\} \\ &\sqsubseteq \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\} \\ &= Z_{\alpha,\beta}^{\omega}(A) \end{aligned}$$

□

Theorem 3.3. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and let $A \in IFS(X)$ then*

$$Z_{\alpha,\beta}^{\omega,\theta}(A) \sqsubseteq Z_{\alpha,\beta}^{\theta,\omega}(A)$$

Proof. If we use $\omega \leq \theta$ and $\alpha\beta \leq \beta$ then $\alpha\beta\left(\theta-\omega\right) \leq \beta\left(\theta-\omega\right).$

$$\begin{aligned}\beta\omega-\alpha\beta\omega &\leq \beta\theta-\alpha\beta\theta\\ &\Rightarrow \alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega \leq \alpha\beta\mu_A(x)+\beta\theta-\alpha\beta\theta\end{aligned}$$

now if we use $\alpha\beta \leq \alpha$ then

$$\begin{aligned}\alpha\beta\left(\theta-\omega\right) &\leq \alpha\left(\theta-\omega\right).\\ &\Rightarrow \alpha\omega-\alpha\beta\omega \leq \alpha\theta-\alpha\beta\theta\\ &\Rightarrow \alpha\beta\nu_A(x)+\alpha\omega-\alpha\beta\omega \leq \alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta\end{aligned}$$

So,

$$\begin{aligned}Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{<x,\beta(\alpha\mu_A(x)+\omega-\omega.\alpha),\alpha(\beta\nu_A(x)+\theta-\theta.\beta)>: x\in X\}\\ &\sqsubseteq \{<x,\beta(\alpha\mu_A(x)+\theta-\theta.\alpha),\alpha(\beta\nu_A(x)+\omega-\omega.\beta)>: x\in X\}\\ &= Z^{\theta,\omega}_{\alpha,\beta}(A)\end{aligned}$$

□

Theorem 3.4. *Let $\alpha,\beta,\omega,\theta \in I$, $\alpha \geq \beta$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(A)$$

Proof. If we use $\beta \leq \alpha$ then $\beta\omega \leq \alpha\omega$ and $\alpha\theta \geq \beta\theta$.

$$\begin{aligned}Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{<x,\beta\alpha\mu_A(x)+\beta\omega-\omega\beta\alpha,\alpha\beta\nu_A(x)+\theta\alpha-\theta\alpha\beta>: x\in X\}\\ &\sqsubseteq \{<x,\beta\alpha\mu_A(x)+\alpha\omega-\beta\alpha\omega,\alpha\beta\nu_A(x)+\theta\beta-\theta\alpha\beta>: x\in X\}\\ &= Z^{\omega,\theta}_{\beta,\alpha}(A)\end{aligned}$$

Hence,

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(A)$$

□

Theorem 3.5. *Let $\alpha,\beta,\omega,\theta \in I$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A^c) = Z^{\theta,\omega}_{\beta,\alpha}(A)^c$$

Proof. If we use definition of A^c it is clear that

$$\begin{aligned}Z^{\omega,\theta}_{\alpha,\beta}(A^c) &= \{<x,\beta\alpha\nu_A(x)+\beta\omega-\omega\beta\alpha,\alpha\beta\mu_A(x)+\theta\alpha-\theta\alpha\beta>: x\in X\}\\ &= Z^{\theta,\omega}_{\beta,\alpha}(A)^c\end{aligned}$$

□

Theorem 3.6. *Let $\alpha,\beta,\omega,\theta \in I$, $\omega=\theta,\omega.\theta=\alpha.\beta$ and let $A \in IFS(X)$ then*

$$Z^{\beta,\alpha}_{\omega,\theta}(A) = Z^{\omega,\theta}_{\alpha,\beta}(A)$$

Proof. The proof is clear. □

Theorem 3.7. *Let $\alpha,\beta,\omega,\theta,\sigma \in I$, $\omega \leq \sigma$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\sigma,\theta}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \sigma$ then $\omega\beta(1-\alpha) \leq \sigma\beta(1-\alpha)$. With this inequality

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(A) &= \{< x, \beta\alpha\mu_A(x) + \beta\omega - \omega\beta\alpha, \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\mu_A(x) + \beta\sigma - \sigma\beta\alpha, \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &= Z^{\sigma,\theta}_{\alpha,\beta}(A) \end{aligned}$$

□

Theorem 3.8. *Let $\alpha, \beta, \omega, \theta, \sigma \in I$, $\max\{\omega, \sigma\} \leq \theta$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta,\sigma}_{\alpha,\beta}(A)$$

Proof. If we use $\omega \leq \theta$ then

$$\begin{aligned} \omega - \omega\alpha &\leq \theta - \alpha\theta \\ &\Rightarrow \beta\alpha\mu_A(x) + \beta\omega - \alpha\beta\omega \leq \beta\alpha\mu_A(x) + \beta\theta - \alpha\beta\theta \end{aligned}$$

and if we use $\sigma \leq \theta$ then

$$\begin{aligned} \theta - \beta\theta &\geq \sigma - \beta\sigma \\ &\Rightarrow \alpha\beta\nu_A(x) + \theta\alpha - \theta\alpha\beta \geq \alpha\beta\nu_A(x) + \alpha\sigma - \alpha\beta\sigma \end{aligned}$$

Thus,

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta,\sigma}_{\alpha,\beta}(A)$$

□

Corollary 3.9. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \leq \theta$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\theta}_{\alpha,\beta}(A)$$

Theorem 3.10. *Let $\alpha, \beta \in I$, $\alpha \leq \beta$ and $A \in IFS(X)$ then*

$$Z^{\alpha,\beta}_{\alpha,\beta}(A) \sqsubseteq Z^{\beta,\alpha}_{\alpha,\beta}(A)$$

Proof. If we use $\alpha \leq \beta$ then $\alpha\beta \leq \beta^2$ and $\alpha\beta \geq \alpha^2$. So,

$$\begin{aligned} Z^{\alpha,\beta}_{\alpha,\beta}(A) &= \{< x, \beta\alpha\mu_A(x) + \beta\alpha - \beta\alpha^2, \alpha\beta\nu_A(x) + \alpha\beta - \alpha\beta^2 >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\mu_A(x) + \beta^2 - \beta^2\alpha, \alpha\beta\nu_A(x) + \alpha^2 - \alpha^2\beta >: x \in X\} \\ &= Z^{\beta,\alpha}_{\alpha,\beta}(A) \end{aligned}$$

□

Theorem 3.11. *Let $\alpha, \beta, \lambda, \tau, \theta, \omega \in I$, $\tau \geq \beta, \alpha \geq \lambda, \omega \leq \sigma, \theta \geq \varepsilon, \alpha\beta = \lambda\tau$, and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) \sqsubseteq Z^{\sigma,\varepsilon}_{\lambda,\tau}(A)$$

Proof. If we use the above inequalities

$$\begin{aligned} \beta\omega(1-\alpha) &\leq \tau\sigma(1-\lambda) \\ &\Rightarrow \beta\omega \leq \alpha\beta\omega + \tau\sigma - \tau\sigma\lambda \\ &\Rightarrow \beta\alpha\mu_A(x) + \beta\omega - \alpha\beta\omega \leq \lambda\tau\mu_A(x) + \tau\sigma - \tau\sigma\lambda \end{aligned}$$

and

$$\begin{aligned} \alpha\theta\left(1-\beta\right) &\geq \lambda\varepsilon\left(1-\tau\right) \\ &\Rightarrow \alpha\beta\nu_A(x)+\theta\alpha-\theta\alpha\beta \geq \lambda\tau\nu_A(x)+\lambda\varepsilon-\tau\lambda\varepsilon \end{aligned}$$

Therefore,

$$Z_{\alpha,\beta}^{\omega,\theta}(A) \sqsubseteq Z_{\lambda,\tau}^{\sigma,\varepsilon}(A)$$

□

Corollary 3.12. *Let $\alpha,\beta,\lambda,\tau,\theta,\omega \in I$, $\tau \geq \beta, \alpha \geq \lambda, \alpha\beta = \lambda\tau$ and let $A \in IFS(X)$ then*

$$Z_{\alpha,\beta}^{\omega,\theta}(A) \sqsubseteq Z_{\lambda,\tau}^{\omega,\theta}(A)$$

Theorem 3.13. *Let $\alpha,\beta,\omega,\theta \in I$ and $A, B \in IFS(X)$ then*

(1) $Z_{\alpha,\beta}^{\omega,\theta}(A\cap B) = Z_{\alpha,\beta}^{\omega,\theta}(A)\cap Z_{\alpha,\beta}^{\omega,\theta}(B)$
(2) $Z_{\alpha,\beta}^{\omega,\theta}(A\cup B) = Z_{\alpha,\beta}^{\omega,\theta}(A)\cup Z_{\alpha,\beta}^{\omega,\theta}(B)$

Proof. We know that for every $x, y, c \in I$ $\min\{x+c, y+c\} = \min\{x, y\}+c$ and $\max\{x+c, y+c\} = \max\{x, y\}+c$. (1)If we use the above equalities then we get,

$$\begin{gathered} Z_{\alpha,\beta}^{\omega,\theta}(A)\cap Z_{\alpha,\beta}^{\omega,\theta}(B) \\ = \{< x, \min(\alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega, \alpha\beta\mu_B(x)+\beta\omega-\alpha\beta\omega), \\ \max\{\alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta, \alpha\beta\nu_B(x)+\alpha\theta-\alpha\beta\theta\} >: x \in X\} \\ = \{< x, \min(\alpha\beta\mu_A(x), \alpha\beta\mu_B(x))+\beta\omega-\alpha\beta\omega, \\ \max(\alpha\beta\nu_A(x), \alpha\beta\nu_B(x))+\alpha\theta-\alpha\beta\theta >: x \in X\} \\ = Z_{\alpha,\beta}^{\omega}(A\cap B) \\ Z_{\alpha,\beta}^{\omega,\theta}(A\cap B) = Z_{\alpha,\beta}^{\omega,\theta}(A)\cap Z_{\alpha,\beta}^{\omega,\theta}(B) \end{gathered}$$

(2) If we use the same equalities as above then we get,

$$\begin{gathered} Z_{\alpha,\beta}^{\omega,\theta}(A)\cup Z_{\alpha,\beta}^{\omega,\theta}(B) \\ = \{< x, \max(\alpha\beta\mu_A(x)+\beta\omega-\alpha\beta\omega, \alpha\beta\mu_B(x)+\beta\omega-\alpha\beta\omega), \\ \min\{\alpha\beta\nu_A(x)+\alpha\theta-\alpha\beta\theta, \alpha\beta\nu_B(x)+\alpha\theta-\alpha\beta\theta\} >: x \in X\} \\ = \{< x, \max(\alpha\beta\mu_A(x), \alpha\beta\mu_B(x))+\beta\omega-\alpha\beta\omega, \\ \min(\alpha\beta\nu_A(x), \alpha\beta\nu_B(x))+\alpha\theta-\alpha\beta\theta >: x \in X\} \\ = Z_{\alpha,\beta}^{\omega,\theta}(A\cup B) \end{gathered}$$

So, we get

$$Z_{\alpha,\beta}^{\omega}(A\cup B) = Z_{\alpha,\beta}^{\omega}(A)\cup Z_{\alpha,\beta}^{\omega}(B)$$

□

Theorem 3.14. *Let $\alpha,\beta,\omega,\theta \in I$, $\beta \leq \alpha$ and $A \in IFS(X)$ then*

$$Z_{\alpha,\beta}^{\omega,\theta}(Z_{\alpha,\beta}^{\omega,\theta}(A)) \sqsubseteq Z_{\alpha,\beta}^{\omega,\theta}(Z_{\beta,\alpha}^{\omega,\theta}(A))$$

Proof. If we use $\beta \leq \alpha$ then $\beta^2 \leq \alpha\beta$ and $\alpha^2 \geq \alpha\beta$ So

$$\begin{aligned} \alpha\beta^2 + \beta^2 &\leq \alpha\beta + \alpha^2\beta \\ &\Rightarrow \alpha\beta^2 - \alpha\beta \leq \alpha^2\beta - \beta^2 \\ &\Rightarrow \alpha^2\beta^2\mu_A(x) + \alpha\beta^2\omega - \alpha^2\beta^2\omega + \beta\omega - \alpha\beta\omega \end{aligned}$$

$$\leq \alpha^2\beta^2\mu_A(x) + \alpha^2\beta\omega - \alpha^2\beta^2\omega + \beta\omega - \beta^2\omega$$

on the other hand

$$\begin{aligned} \alpha^2\beta + \alpha^2 &\geq \alpha\beta^2 + \alpha\beta \\ &\Rightarrow \alpha^2\beta\theta - \alpha\beta\theta \geq \alpha\beta^2\theta - \alpha^2\theta \\ &\Rightarrow \alpha^2\beta^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha\beta\theta \end{aligned}$$

$$\geq \alpha^2\beta^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha^2\theta$$

Hence,

$$Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

□

Theorem 3.15. *Let $\alpha, \beta, \omega, \theta \in I$, $\omega \geq \theta$, $\theta(1-\theta) \geq \omega(1-\beta)$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(Z^{\alpha,\beta}_{\omega,\theta}(A)) \sqsubseteq Z^{\alpha,\beta}_{\omega,\theta}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. The proof is clear. □

Theorem 3.16. *Let $\alpha, \beta, \omega, \theta \in I$, $\beta \leq \alpha, \alpha.\beta.\omega \leq \theta$ and $A \in IFS(X)$ then*

$$Z^{\theta,\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A)) \sqsubseteq Z^{\theta,\omega}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\beta \leq \alpha$ and $\alpha.\beta.\omega \leq \theta$ we get

$$\alpha\beta\omega(\alpha - \beta) \leq \theta(\alpha - \beta)$$

with this inequality

$$Z^{\theta,\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

$$= \{< x, \beta^2\alpha^2\mu_A(x) + \alpha^2\beta\omega - \beta^2\alpha^2\omega + \beta\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\}$$

$$\sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \alpha\theta - \alpha\beta\theta,$$

$$\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \beta\omega - \alpha\beta\omega >: x \in X\}$$

$$= Z^{\theta,\omega}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

In wiev of this properties, we can say that the operator $Z^{\omega,\theta}_{\alpha,\beta}$ is a generalization of $Z^{\omega}_{\alpha,\beta}$,and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$. If we use this result, we get the new diagram of one type modal operators as in Figure 6:

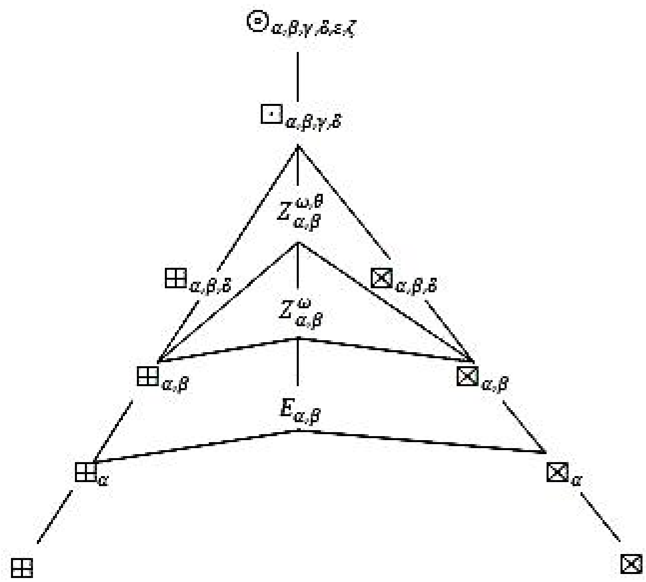

FIGURE 6

This is the last diagram of one type modal operators on IFSs. If we use the above properties, we can say that there is no one type modal operator on IFSs so that it is written inside of the last diagram. On the other hand, in the future, we may say that there are some one type modal operators that can be written outside of the last diagram.

4. Some Relationships Among One Type Modal Operators on Intuitionistic Fuzzy Sets

In several papers, some authors have discussed the relationships among the one type modal operators, some of which are given in the above section. First and foremost, we want to study standard relationships between the operator $Z^{\omega,\theta}_{\alpha,\beta}$ and the others. Afterwards, we will show the different relationships between them.

Theorem 4.1. *Let $\alpha, \beta, \omega, \theta \in I, \alpha \geq \beta$ and $A \in IFS(X)$ then*

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

Proof. If we use $\beta \leq \alpha$ then $\alpha\beta^2 \leq \alpha^2\beta$

$$\begin{aligned} &Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \\ &= \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \beta\omega - \alpha\beta\omega, \\ &\beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\} \\ &\sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha^2\beta\omega - \beta^2\alpha^2\omega + \beta\omega - \alpha\beta\omega, \\ &\beta^2\alpha^2\nu_A(x) + \alpha\beta^2\theta - \alpha^2\beta^2\theta + \alpha\omega - \alpha\beta\omega >: x \in X\} \end{aligned}$$

$$= Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

Thus,

$$Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega}_{\alpha,\beta}(Z^{\omega,\theta}_{\beta,\alpha}(A))$$

□

Theorem 4.2. *Let $\alpha, \beta, \omega, \theta \in I, \alpha \geq \beta$ and $A \in IFS(X)$ then*

$$Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\alpha \geq \beta$ then $\alpha\theta \geq \beta\theta$

$$\begin{gathered} Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \\ = \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \beta\theta - \alpha\beta\theta, \\ \beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \alpha\theta - \alpha\beta\theta >: x \in X\} \\ \sqsubseteq \{< x, \beta^2\alpha^2\mu_A(x) + \alpha\beta^2\omega - \beta^2\alpha^2\omega + \alpha\theta - \alpha\beta\theta, \\ \beta^2\alpha^2\nu_A(x) + \alpha^2\beta\theta - \alpha^2\beta^2\theta + \beta\theta - \alpha\beta\theta >: x \in X\} \\ = Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \end{gathered}$$

Therefore,

$$Z^{\theta}_{\alpha,\beta}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq Z^{\theta}_{\beta,\alpha}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.3. *Let $\alpha, \beta, \omega, \theta, \tau \in I$, $A \in IFS(X)$ then*

(1) $Z^{\omega,\theta}_{\alpha,\beta}(\boxplus(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes(A))$
(2) $Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$
(3) $\tau_1 + \tau_2 \leq 1 \Rightarrow Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1,\tau_2}(A))$
(4) $\max\{\tau_1, \tau_2\} + \tau_3 \leq 1 \Rightarrow Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_3}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau_1,\tau_2,\tau_3}(A))$

Proof. (1)If we use definitions of $\boxplus(A)$ and $\boxtimes(A)$ then

$$\begin{aligned} Z^{\omega,\theta}_{\alpha,\beta}(\boxplus(A)) &= \{< x, \beta\alpha\frac{\mu_A(x)}{2} + \beta\omega - \omega\beta\alpha, \alpha\beta\frac{\nu_A(x)+1}{2} + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &\sqsubseteq \{< x, \beta\alpha\frac{\mu_A(x)+1}{2} + \beta\omega - \omega\beta\alpha, \alpha\beta\frac{\nu_A(x)}{2} + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ &= Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes(A)) \end{aligned}$$

(2)If we use $\alpha\beta - \alpha\beta\tau \geq 0$ then

$$\begin{gathered} Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau}(A)) \\ = \{< x, (\beta\alpha\tau)\,\mu_A(x) + \beta\omega - \omega\beta\alpha, \\ (\beta\alpha\tau)\,\nu_A(x) + \alpha\beta - \alpha\beta\tau + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ \sqsubseteq \{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta - \alpha\beta\tau + \beta\omega - \omega\beta\alpha, \\ (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\ = Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A)) \end{gathered}$$

(3-4) are shown by the same way as above. □

Corollary 4.4. *Let $\alpha, \beta, \omega, \theta \in I, \beta \leq \alpha$ and $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(E_{\alpha,\beta}(A)) \sqsubseteq Z^{\omega,\theta}_{\beta,\alpha}(E_{\alpha,\beta}(A))$$

Theorem 4.5. *Let $\alpha, \beta, \omega, \theta, \tau_1, \tau_2 \in I, \max\{\tau_1, \tau_2\} + \tau_1 \in I$ and $\tau_1 \leq \tau_2, A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

Proof. If we use $\tau_1 + \tau_2 \leq 1$ then

$$\begin{aligned}
\alpha\beta - \alpha\beta\tau_1 &\geq \alpha\beta\tau_2 \\
&\Rightarrow (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta - \alpha\beta\tau_1 + \theta\alpha - \theta\alpha\beta \\
&\geq (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta\tau_2 + \theta\alpha - \theta\alpha\beta
\end{aligned}$$

so we get

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A))$$

on the other hand if we use $\tau_1 \leq \tau_2$ then

$$\begin{aligned}
\tau_2 - \tau_1 &\geq 0 \\
&\Rightarrow \tau_1\nu_A(x) + \tau_2 \geq \tau_2\nu_A(x) + \tau_1 \\
&\Rightarrow (\beta\alpha\tau_1)\,\nu_A(x) + \alpha\beta\tau_2 + \theta\alpha - \theta\alpha\beta \\
&\geq (\beta\alpha\tau_2)\,\nu_A(x) + \alpha\beta\tau_1 + \theta\alpha - \theta\alpha\beta
\end{aligned}$$

and we get

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

So,

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxplus_{\tau_1,\tau_2,\tau_1}(A))$$

□

Theorem 4.6. *Let $\alpha, \beta, \omega, \theta, \tau, \gamma \in I, \tau \leq \gamma$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A)) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))$$

Proof. If we use $\tau \leq \gamma$ then

$$\begin{aligned}
&Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau}(A)) \\
= &\{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\tau + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
\sqsubseteq &\{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\gamma + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
= &Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A))
\end{aligned}$$

on the other hand if we use $\tau + \gamma \leq 1$ then $\alpha\beta\gamma \leq \alpha\beta - \alpha\beta\tau$. with inequality, we get

$$\begin{aligned}
&Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau,\tau,\gamma}(A)) \\
= &\{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta\gamma + \beta\omega - \omega\beta\alpha, (\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
\sqsubseteq &\{< x, (\beta\alpha\tau)\,\mu_A(x) + \alpha\beta - \alpha\beta\tau + \beta\omega - \omega\beta\alpha, \\
&(\beta\alpha\tau)\,\nu_A(x) + \theta\alpha - \theta\alpha\beta >: x \in X\} \\
&= Z^{\omega,\theta}_{\alpha,\beta}(\boxtimes_{\tau}(A))
\end{aligned}$$

Thus,

$$Z^{\omega,\theta}_{\alpha,\beta}\left(\boxtimes_{\tau,\tau}(A)\right) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}\left(\boxtimes_{\tau,\tau,\gamma}(A)\right) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}\left(\boxtimes_{\tau}(A)\right)$$

□

Corollary 4.7. *Let $\alpha, \beta, \omega, \theta, \tau_1, \tau_2 \in I, \tau_1 + \tau_2 \in I$ and let $A \in IFS(X)$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}\left(\boxplus_{\tau_1}(A)\right) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}\left(\boxplus_{\tau_1,\tau_2}(A)\right) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}\left(\boxtimes_{\tau_1,\tau_2}(A)\right) \sqsubseteq Z^{\omega,\theta}_{\alpha,\beta}\left(\boxtimes_{\tau_1}(A)\right)$$

Theorem 4.8. *Let $\alpha, \beta, \omega, \theta, \tau_1, \tau_2 \in I$ and let $A \in IFS(X)$ then*

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use $\tau_1 + \tau_2 \leq 1$ then

$$\alpha\tau_1(\beta\nu_A(x) + \theta - \beta\theta) + \tau_2 \leq \alpha\tau_1(\beta\nu_A(x) + \theta - \beta\theta) + 1 - \tau_1$$

and if we use inequality $\beta\tau_1(\alpha\mu_A(x) + \omega - \alpha\omega) = \beta\tau_1(\alpha\mu_A(x) + \omega - \alpha\omega)$ then we get

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.9. *Let $\alpha, \beta, \omega, \tau_1, \tau_2 \in I,\ \tau_1 + \tau_2 \leq 1,\ A \in IFS(X)$ then*

$$\boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

Proof. If we use definition of τ_1, τ_2 we get $\tau_2 \leq 1 - \tau_1$ then

$$\begin{aligned}
& \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \\
= \ & \{< x, \beta\tau_1(\alpha\mu_A(x) + \omega - \alpha\omega) + \tau_2, \alpha\tau_1(\beta\nu_A(x) + \theta - \beta\theta) >: x \in X\} \\
\sqsubseteq \ & \{< x, \beta\tau_1(\alpha\mu_A(x) + \omega - \alpha\omega) + 1 - \tau_1, \alpha\tau_1(\beta\nu_A(x) + \theta - \beta\theta >: x \in X\} \\
= \ & \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))
\end{aligned}$$

Hence,

$$\boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

□

Theorem 4.10. *Let $\alpha, \beta, \omega, \theta, \tau \in I,\ A \in IFS(X)$ then*

(1) $\boxplus(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(2) $\boxplus_{\tau}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(3) $\tau_1 + \tau_2 \in I \Rightarrow \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

(4) $\max\{\tau_1, \tau_2\} + \tau_3 \in I \Rightarrow \boxplus_{\tau_1,\tau_2,\tau_3}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2,\tau_3}(Z^{\omega,\theta}_{\alpha,\beta}(A))$

Proof. We know that

$$\text{if } A \in IFS(X) \text{ then} \boxplus(A) \sqsubseteq \boxtimes(A).$$

So,

$$\boxplus(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes(Z^{\omega,\theta}_{\alpha,\beta}(A)).$$

Consequently, the other properties can be shown similarly. □

Corollary 4.11. *Let $\alpha, \beta, \omega,\ \tau_1, \tau_2 \in I,\ \tau_1 + \tau_2 \in I,\ A \in IFS(X)$ then*

$$\boxplus_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxplus_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1,\tau_2}(Z^{\omega,\theta}_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\tau_1}(Z^{\omega,\theta}_{\alpha,\beta}(A))$$

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