

FUZZY INTEGRO-DIFFERENTIAL EQUATIONS: DISCRETE SOLUTION AND ERROR ESTIMATION

M. ZEINALI, S. SHAHMORAD AND K. MIRNIA

ABSTRACT. This paper investigates existence and uniqueness results for the first order fuzzy integro-differential equations. Then numerical results and error bound based on the left rectangular quadrature rule, trapezoidal rule and a hybrid of them are obtained. Finally an example is given to illustrate the performance of the methods.

1. Introduction

The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) in both theoretical and numerical points of view have been developed in recent years. Prior to discussing fuzzy integro-differential equations and their numerical treatments, it is necessary to present a brief introduction of the previous works about FDE and FIE. Goetschel, Voxman and Kaleva (see e.g [13, 15, 16]) studied initial value problems of fuzzy differential equations for the first time. The corresponding existence results for initial and boundary value problems have been obtained for fuzzy problems in [29, 18, 17, 21, 22, 24]. The main tool in studying existence and uniqueness of the solution for fuzzy integral equations is the Banach fixed point theorem which can be found in [23, 27, 5, 7, 6]. To solve fuzzy differential and integral equations numerically, several methods have been constructed in [3, 26, 19, 11, 1, 4, 11, 12, 2, 20]. Numerical procedures based on the trapezoidal quadrature rule along with convergence results have been investigated to solve fuzzy integral equations iteratively with arbitrary kernels in [11, 12]. In [9], the authors have introduced a general form for the quadrature rules and used it to solve linear fuzzy Fredholm integral equations by iterative method. Also they have investigated error estimation of the method. Bica (see e.g [10]) applied the general form to solve nonlinear fuzzy Fredholm integral equations numerically and also the error estimation of the method and a stopping criterion of the corresponding algorithm are given.

After a preliminary section, by using lemma 6.1 in [15], a fuzzy integro-differential equation is transformed to the corresponding Volterra integral equation of the second kind in section 3. Then, by using the metric appeared in [14], an existence theorem is proved irrespective to Lipschitz constant and also integration intervals in subsection 3.1. The fuzzy Volterra integral equation is solved by three methods

Received: August 2011; Revised: February 2012 and March 2012; Accepted: May 2012

Key words and phrases: Fuzzy integro-differential equation, Discrete solution, Fuzzy quadrature rule.

in subsection 3.2. The first method which is an explicit one uses the left rectangular quadrature rule and thus no equation is needed to be solved. The trapezoidal rule is used in the second method which is going to be an implicit method and thus a linear equation should be solved. The two rules mentioned above are combined to imply the third method which is an explicit one having an acceptable error bound. Finally error bounds of the methods are investigated and a numerical example is presented to illustrate the performance of the methods.

2. Preliminaries

Definition 2.1. (see e.g. [8]) Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $u : \mathbb{R} \rightarrow I = [0, 1]$, satisfying the following properties:

- (i) u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
- (ii) u is a convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0, 1], x, y \in \mathbb{R}$),
- (iii) u is upper semicontinuous on \mathbb{R} ,
- (iv) $\{x \in \mathbb{R}; u(x) > 0\}$ is compact, where \bar{A} denotes the closure of A .

Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_x; x \text{ is a usual real number}\}$. For $0 < r \leq 1$, r -cut of fuzzy number u is defined as $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Then it is easily established that u is a fuzzy number if and only if $[u]^r$ is a closed and bounded interval for each $r \in [0, 1]$, and $[u]^1 \neq \emptyset$ (see e.g. [13]). For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $[u + v]^r = [u]^r + [v]^r$ and $[\lambda.u]^r = \lambda[u]^r$, $\forall r \in [0, 1]$.

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+ \cup \{0\}$, $D(u, v) = \sup_{r \in [0, 1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-^r, u_+^r]$ and $[v]^r = [v_-^r, v_+^r]$. We define $\|\cdot\| = D(\cdot, \tilde{0})$, where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$, $\tilde{0} = \chi_{\{0\}}$. Then the following properties are satisfied (see e.g. [28])

- (i) $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space,
- (ii) $D(u + v, u + \omega) = D(v, \omega)$,
- (iii) $D(k.u, k.v) = |k|D(u, v)$,
- (ii) $D(u + v, \omega + e) \leq D(u, \omega) + D(v, e)$.

Definition 2.2. (see e.g. [9]) The function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is called a Lipschitz function if there exists a real constant $L \geq 0$ such that, for all $x, y \in [a, b]$

$$D(f(x), f(y)) \leq L|x - y|.$$

We refer to L as the Lipschitz constant of the function f .

Definition 2.3. (see e.g. [28]) Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$, $\delta : [a, b] \rightarrow \mathbb{R}_+$ and $a = x_0 < x_1 < \dots < x_n = b$ be a partition of the interval $[a, b]$ with the intermediate points $\psi_i \in [x_{i-1}, x_i]$. The partition $P = \{([x_{i-1}, x_i]; \psi_i); i = 1, \dots, n\}$ denoted by $P = (\Delta_n, \psi)$ is called δ -fine iff $[x_{i-1}, x_i] \subseteq (\psi_i - \delta(\psi_i), \psi_i + \delta(\psi_i))$.

Definition 2.4. (see e.g. [28]) The function f is called Henstock integrable if for every $\epsilon > 0$ there exists a function $\delta : [a, b] \rightarrow \mathbb{R}_+$ such that for any δ -fine partition P , we have $D(\sum_{i=1}^n (x_i - x_{i-1}) \cdot f(\psi_i), A) \leq \epsilon$ for some $A \in \mathbb{R}_{\mathcal{F}}$. Then A is called the Henstock integral of f and it is denoted by $(FH) \int_a^b f(t)dt$.

The integrals used in this paper are fuzzy Riemann integral which is a particular case of the Henstock integral.

Lemma 2.5. (see e.g. [9]) **(i)** Let f and g be Henstock integrable functions and $D(f(t), g(t))$ be Lebesgue integrable. Then

$$D\left((FH) \int_a^b f(t)dt, (FH) \int_a^b g(t)dt\right) \leq L \int_a^b D(f(t), g(t))dt.$$

(ii) Let the function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a Henstock integrable and bounded function. Then for every fixed point $u \in [a, b]$, the function $\phi_u : [a, b] \rightarrow \mathbb{R}_+$ defined by $\phi_u(t) = D(f(u), f(t))$ is Lebesgue integrable on $[a, b]$.

We recall the following preliminary results from [9, 25, 10].

Theorem 2.6. (see e.g. [9]) Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a Lipschitz function with the Lipschitz constant L . Then for any partition $a = x_0 < x_1 < \dots < x_n = b$ and $\xi_i \in [x_{i-1}, x_i]$, we have

$$\begin{aligned} D\left((FH) \int_a^b f(t)dt, \sum_{i=1}^n (x_i - x_{i-1}) \cdot f(\xi_i)\right) &\leq \frac{L}{2} \sum_{i=1}^n ((x_i - \xi_i)^2 + (\xi_i - x_{i-1})^2) \\ &\leq \frac{L}{2} \sum_{i=1}^n (x_i - x_{i-1})^2. \end{aligned}$$

Corollary 2.7. (see e.g. [9]) Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a Lipschitz function with the Lipschitz constant L . Then

$$D\left((FH) \int_a^b f(t)dt, ((x-a) \cdot f(t) + (b-x) \cdot f(s))\right) \leq L \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2\right], \quad (1)$$

for any $x \in [a, b]$, $t \in [a, x]$ and $s \in [x, b]$.

The following corollary gives a new error bound for the fuzzy variant of the classical trapezoidal rule.

Corollary 2.8. (i) (see e.g. [10]) If we put $t = a$, $s = b$ and $x = \frac{a+b}{2}$, then we obtain the error bound

$$D\left((FH) \int_a^b f(t)dt, \left(\frac{b-a}{2} \cdot f(a) + \frac{b-a}{2} \cdot f(b)\right)\right) \leq L \frac{(b-a)^2}{4},$$

for the fuzzy trapezoidal rule.

(ii) (see e.g. [10].) The generalization inequality for the Lipschitz fuzzy function is

$$D\left((FH) \int_a^b f(t)dt, \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{2} \cdot (f(x_i) + f(x_{i+1}))\right) \leq \frac{L(b-a)^2}{4n}. \quad (2)$$

Definition 2.9. (see e.g. [25]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the H -difference of x and y and it is denoted by $x - y$.

Definition 2.10. (see e.g. [25]) A function $f : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ is called H -differentiable at $x_0 \in (a, b)$ if for $h \geq 0$ sufficiently small, there exist the H -differences $f(x_0 + h) - f(x_0)$, $f(x_0) - f(x_0 - h)$ and an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\lim_{h \searrow 0} D \left(\frac{f(x_0 + h) - f(x_0)}{h}, f'(x_0) \right) = \lim_{h \searrow 0} D \left(\frac{f(x_0) - f(x_0 - h)}{h}, f'(x_0) \right) = 0.$$

(Here h at denominator means the multiplication with $\frac{1}{h}$.)

3. Main Results

Now we consider a linear fuzzy Volterra integro-differential equation of the form

$$y'(x) = f(x) + \int_a^x k(x, t).y(t)dt, \quad x \in I \quad (3)$$

with the initial condition

$$y(a) = y_0, \quad (4)$$

where $I = [a, b]$. Let $k : G \rightarrow \mathbb{R}$ be continuous with no sign changes being Lipschitz with respect to the second variable and $f : I \rightarrow \mathbb{R}_{\mathcal{F}}$ be continuous, where

$$G := \{(x, y) | x \in I, y \in [a, x]\} \subset I \times I.$$

A function $y : I \rightarrow \mathbb{R}_{\mathcal{F}}$ is a solution of the initial value problem (3)-(4) if and only if it is continuous and satisfies the integral equation

$$y(x) = y_0 + \int_a^x f(t)dt + \int_a^x \int_a^t k(t, s).y(s)dsdt,$$

for all $x \in I$ (see e.g. [15]).

By changing the order of the integration, we have

$$y(x) = y_0 + \int_a^x f(t)dt + \int_a^x \int_s^x k(t, s).y(s)dt ds.$$

Since the function k is with no sign changes by assumption, we have

$$\int_s^x k(t, s).y(s)dt = \left(\int_s^x k(t, s)dt \right).y(s)$$

and thus

$$y(x) = g(x) + \int_a^x k_1(x, s).y(s)ds,$$

where $k_1(x, s) = \int_s^x k(t, s)dt$ and $g(x) = y_0 + \int_a^x f(s)ds$.

Lemma 3.1. Let $k(t, s)$ be continuous in (t, s) and Lipschitz with respect to s . Then $k_1(x, s)$ is Lipschitz with respect to s .

3.1. Existence Result. Consider the space of functions

$$X = \{f : I \rightarrow \mathbb{R}_{\mathcal{F}} | f \text{ is continuous}\}$$

along with the metric defined by

$$D^*(f, g) = \sup_{t \in I} D(f(t), g(t)).$$

It is worth to remind that (X, D^*) is a complete metric space (see e.g. [15]) with the norm defined by

$$\|f\|_X = D^*(f, \tilde{0}), \quad \forall f \in X.$$

Theorem 3.4. *Let $k : G \rightarrow \mathbb{R}$ be a continuous function with no sign changes and $f : I \rightarrow \mathbb{R}_{\mathcal{F}}$ be continuous. Then problem (4) has a unique solution.*

Proof. Let the operator A be defined on X by

$$(Ay)(x) = g(x) + \int_a^x k_1(x, s).y(s)ds, \quad \forall x \in I.$$

We claim that $A : X \rightarrow X$. To do this, it is evident that $(Ay)(x) \in \mathbb{R}_{\mathcal{F}}$ for all $x \in I$ and thus $Ay : I \rightarrow \mathbb{R}_{\mathcal{F}}$. It is sufficient to prove it's continuity. For this, let $t_1, t_2 \in I$, $t_1 \leq t_2$ and $y \in X$. Then

$$\begin{aligned} D(Ay(t_1), Ay(t_2)) &= D\left(g(t_1) + \int_a^{t_1} k_1(t_1, s).y(s)ds, g(t_2) + \int_a^{t_2} k_1(t_2, s).y(s)ds\right) \\ &= D(g(t_1), g(t_2)) \\ &\quad + D\left(\int_a^{t_1} k_1(t_1, s).y(s)ds, \int_a^{t_1} k_1(t_2, s).y(s)ds + \int_{t_1}^{t_2} k_1(t_2, s).y(s)ds\right) \\ &\leq D(g(t_1), g(t_2)) + D\left(\int_a^{t_1} k_1(t_1, s).y(s)ds, \int_a^{t_1} k_1(t_2, s).y(s)ds\right) \\ &\quad + D\left(\tilde{0}, \int_{t_1}^{t_2} k_1(t_2, s).y(s)ds\right) \\ &\leq D(g(t_1), g(t_2)) + \int_a^{t_1} D(k_1(t_1, s).y(s), k_1(t_2, s).y(s))ds \\ &\quad + \int_{t_1}^{t_2} D(\tilde{0}, k_1(t_2, s).y(s))ds. \end{aligned} \tag{5}$$

Since $k_1(t_2, s).y(s)$ is continuous (see lemma 3.3), there exists $M > 0$ such that $\|k_1(t_2, s).y(s)\|_X \leq M$. Let $\epsilon > 0$ be arbitrary, then there are $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$ such that $\epsilon_1 + \epsilon_2(b-a) + \epsilon_3M < \epsilon$. For $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that $D(g(t_1), g(t_2)) \leq \epsilon_1$ for $|t_1 - t_2| \leq \delta_1$. From lemma 3.3, $\{k_1(x, s).y(s) | s \in I\}$ is an equicontinuous family of functions, thus for $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that

$$\sup_{s \in I} D(k_1(t_1, s).y(s), k_1(t_2, s).y(s)) < \epsilon_2,$$

for $|t_1 - t_2| \leq \delta_2$. Let $\delta = \min\{\delta_1, \delta_2, \epsilon_3\}$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \leq \delta$. Then from (5) we have

$$D(Ay(t_1), Ay(t_2)) \leq \epsilon_1 + \epsilon_2(b-a) + \epsilon_3M < \epsilon.$$

To prove that A is a contraction mapping, we set

$$L_0 = \max_{(x,s) \in G} |k_1(x,s)|$$

and define

$$D_\beta(f,g) := \sup_{x \in I} e^{-\beta L_0 x} D(f(x), g(x)), \quad \beta \geq 1$$

as a metric on X which is equivalent to the metric D^* (see e.g. [14]). Thus for $y_1, y_2 \in X$ and $t \in I$, we have

$$\begin{aligned} D(Ay_1(t), Ay_2(t)) &= D\left(g(t) + \int_a^t k_1(t,s).y_1(s)ds, g(t) + \int_a^t k_1(t,s).y_2(s)ds\right) \\ &= D\left(\int_a^t k_1(t,s).y_1(s)ds, \int_a^t k_1(t,s).y_2(s)ds\right) \\ &\leq \int_a^t D(k_1(t,s).y_1(s), k_1(t,s).y_2(s))ds \\ &\leq \int_a^t L_0 D(y_1(s), y_2(s))ds \\ &= \int_a^t L_0 e^{\beta L_0 s} e^{-\beta L_0 s} D(y_1(s), y_2(s))ds \\ &\leq \int_a^t L_0 e^{\beta L_0 s} D_\beta(y_1, y_2)ds \\ &= D_\beta(y_1, y_2) \frac{e^{\beta L_0 t} - e^{\beta L_0 a}}{\beta} \\ &= D_\beta(y_1, y_2) e^{\beta L_0 t} \frac{1 - e^{\beta L_0 (a-t)}}{\beta}, \end{aligned}$$

which implies

$$\begin{aligned} D(Ay_1(t), Ay_2(t)) e^{-\beta L_0 t} &\leq D_\beta(y_1, y_2) \frac{1 - e^{\beta L_0 (a-t)}}{\beta} \\ &\leq D_\beta(y_1, y_2) \frac{1 - e^{\beta L_0 (a-b)}}{\beta} \end{aligned}$$

and consequently

$$D_\beta(Ay_1, Ay_2) \leq D_\beta(y_1, y_2) \frac{1 - e^{\beta L_0 (a-b)}}{\beta}.$$

Since $\frac{1 - e^{\beta L_0 (a-b)}}{\beta} < 1$, the operator A is a contraction mapping. By the Banach fixed point theorem we conclude that the initial value problem (3)-(4) has a unique solution. \square

3.2. Numerical Solution. In order to get a particular method, we choose a uniform partition $\Delta : a = x_0 < x_1 < \dots < x_n = b$ of the interval $[a, b]$ with step size $h = \frac{b-a}{n}$ such that all resulting quadrature rules involve only intervals of the form $[a, x_i]$ using just the points x_0, x_1, \dots, x_i . Then we approximate the unknown function y at the points $x_i, \forall i = 0, \dots, n$ as follows:

(i) $y_0 = y(a)$,

(ii) $y_i = g(x_i) + Q_i := g(x_i) + Q_{[a,x_i]}(k_1(x_i, \cdot). \hat{y}(\cdot))$, $i = 1, 2, \dots$,
 where we define $\hat{y}(x_l) := y_l$ and

$$Q_i := Q_{[a,x_i]}(f) = \sum_{l=0}^i \omega_{l,i} f(x_l)$$

as a quadrature rule. We have explicit or implicit equation for y_i either $\omega_{i,i} = 0$ which corresponds to the left rectangular and hybrid quadrature rules or $\omega_{i,i} \neq 0$ which corresponds to the trapezoidal quadrature rule. In the rectangular rule, we do not need to solve any equation but the accuracy is low and its error is bounded by

$$\|R(Q_i)\| \leq \frac{L}{2} \sum_{j=1}^i (x_j - x_{j-1})^2 \leq \frac{L}{2} \sum_{j=1}^n (x_j - x_{j-1})^2 = \frac{L(b-a)}{2} h, \quad (6)$$

where $R(Q_i)$ is the quadrature error and L is a Lipschitz constant (see theorem 2.6). From lemma 3.2, $k_1(x, s).y(s)$ is Lipschitz with respect to s with a Lipschitz constant.

For more accurate results, we apply trapezoidal rule which leads to implicit method needing the equation

$$y_i = g(x_i) + \frac{h}{2}.k_1(x_i, x_0).y(x_0) + h. \sum_{l=1}^{i-1} k_1(x_i, x_l).y_l + \frac{h}{2}.k_1(x_i, x_i).y_i, \quad (7)$$

to be solved for $y_i, i = 1, 2, \dots, n$.

It is required to verify the existence and uniqueness of solution of this equation. To this end we prove the following lemma.

Lemma 3.5. *Let the step size h be sufficiently small:*

$$h < 2/L_0.$$

Then equation (7) has the unique solution y_i .

Proof. To simplify notations, let

$$c_i := g(x_i) + \frac{h}{2}.k_1(x_i, x_0).y(x_0) + h. \sum_{l=1}^{i-1} k_1(x_i, x_l).y_l$$

and define the operator $T : \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ by

$$T(\zeta) := c_i + \frac{h}{2}.k_1(x_i, x_i).\zeta.$$

Since $k_1(x_i, x_i).\zeta$ is a Lipschitz function with the Lipschitz constant L_0 with respect to ζ , we have

$$D(T(\zeta), T(\xi)) = \frac{h}{2} D(k_1(x_i, x_i).\zeta, k_1(x_i, x_i).\xi) \leq \frac{L_0 h}{2} D(\zeta, \xi)$$

and since $h < 2/L_0$, it follows that T is contraction in the complete metric space $(\mathbb{R}_{\mathcal{F}}, D)$. The Banach fixed point theorem can be applied to complete the proof. \square

Furthermore, the Banach fixed point theorem offers an iterative method to obtain approximate solution of (7), i.e.

$$y_i^{(0)} = y_{i-1}, \quad y_i^{(m+1)} = c_i + \frac{h}{2}.k_1(x_i, x_i).y_i^{(m)}, \quad m = 0, 1, 2, \dots \quad (8)$$

For the trapezoidal quadrature rule, from (2), we have

$$\| R(Q_i) \| \leq \frac{L}{4} h(b-a).$$

To avoid solving any equation, we divide the interval $[a, x_i]$ into the intervals $[a, x_{i-1}]$ and $[x_{i-1}, x_i]$. For the first interval we use trapezoidal rule and for the second interval we use left-side rectangular. This quadrature rule is called a hybrid method. The resulting explicit equation is given by

$$y_i = g(x_i) + \frac{h}{2} \cdot k_1(x_i, x_0) \cdot y(x_0) + h \cdot \sum_{l=1}^{i-2} k_1(x_i, x_l) \cdot y_l + \frac{3h}{2} \cdot k_1(x_i, x_{i-1}) \cdot y_{i-1},$$

for $i = 1, 2, \dots, n$ with the upper error bound

$$\begin{aligned} \| R(Q_i) \| &= D \left(\int_a^{x_i} f(t) dt, \frac{h}{2} \cdot f(x_0) + \sum_{l=1}^{i-2} h \cdot f(x_l) + \frac{3h}{2} \cdot f(x_{i-1}) \right) \\ &\leq D \left(\int_a^{x_{i-1}} f(t) dt, \frac{h}{2} \cdot f(x_0) + \sum_{l=1}^{i-2} h \cdot f(x_l) + \frac{h}{2} \cdot f(x_{i-1}) \right) \\ &\quad + D \left(\int_{x_{i-1}}^{x_i} f(t) dt, h \cdot f(x_{i-1}) \right), \end{aligned}$$

which is summarized by using (1) and (2) as

$$\| R(Q_i) \| \leq \frac{L(b-a)}{4} h + \frac{L}{2} h^2 = \frac{L(b-a)}{2} h \left(\frac{1}{2} + \frac{1}{n} \right).$$

Therefore for the given quadrature rules, we have

$$\| R(Q_i) \| \leq C_R h. \tag{9}$$

where C_R takes the values $\frac{L(b-a)}{2}$, $\frac{L(b-a)}{4}$ and $\frac{L(b-a)}{2} \left(\frac{1}{2} + \frac{1}{n} \right)$ for left-side rectangular, trapezoidal and hybrid quadrature rules respectively.

3.3. Error Estimation. In this section, we discuss upper bound of the error function

$$d_i := D(y_i, y(x_i)),$$

where y_i is the approximate value of $y(x_i)$.

Lemma 3.6. (see e.g. [14]) Assume that there are the numbers $\alpha, \beta_l \geq 0$ and $0 \leq M_0 \leq 1$ such that

$$0 \leq d_i \leq \alpha + \sum_{l=0}^{i-1} \beta_l d_l + M_0 d_i \quad i \geq 0.$$

Then for all $i \geq 0$, we have

$$d_i \leq \frac{\alpha}{1 - M_0} \prod_{l=0}^{i-1} \left(1 + \frac{\beta_l}{1 - M_0} \right) \leq \frac{\alpha}{1 - M_0} \exp \left(\sum_{l=0}^{i-1} \frac{\beta_l}{1 - M_0} \right).$$

Theorem 3.7. (rectangular and hybrid quadrature rules) Assume that $k(x, t)$ is continuous and has no sign changes, $y(x)$ is H -differentiable and $f(x)$ is continuous. Then the error estimate

$$d_i \leq Ch$$

holds for the approximation y_i defined in subsection 3.2 by left-side rectangular or hybrid quadrature rule with $C = C_R \exp(L_0ih)$.

Proof. Due to (4) the initial error vanishes:

$$d_0 = 0$$

while for $i > 0$, we have

$$\begin{aligned} d_i &= D \left(g(x_i) + \sum_{l=0}^{i-1} \omega_{l,i} \cdot k_1(x_i, x_l) \cdot y_l, g(x_i) + \int_a^x k_1(x_i, s) \cdot y(s) ds \right) \\ &= D \left(\sum_{l=0}^{i-1} \omega_{l,i} \cdot k_1(x_i, x_l) \cdot y_l, \sum_{l=0}^{i-1} \omega_{l,i} \cdot k_1(x_i, x_l) \cdot y(x_l) + R(Q_i) \right) \\ &\leq D \left(\sum_{l=0}^{i-1} \omega_{l,i} \cdot k_1(x_i, x_l) \cdot y_l, \sum_{l=0}^{i-1} \omega_{l,i} \cdot k_1(x_i, x_l) \cdot y(x_l) \right) + D(0, R(Q_i)) \\ &\leq \sum_{l=0}^{i-1} |\omega_{l,i}| D(k_1(x_i, x_l) \cdot y_l, k_1(x_i, x_l) \cdot y(x_l)) + \|R(Q_i)\| \\ &\leq \sum_{l=0}^{i-1} |\omega_{l,i}| L_0 D(y_l, y(x_l)) + \|R(Q_i)\| \\ &\leq \sum_{l=0}^{i-1} |\omega_{l,i}| L_0 d_l + \|R(Q_i)\|, \end{aligned} \tag{10}$$

where $R(Q_i)$ is the quadrature error for the integrand

$$k_i(x_i, s) \cdot y(s).$$

From (9) and (10), we finally obtain

$$d_i \leq L_0 \sum_{l=0}^{i-1} |\omega_{l,i}| d_l + C_R h. \tag{11}$$

To simplify the inequality (11), we introduce the new constants $\beta_l = L_0 |\omega_{l,i}|$, $\alpha = C_R h$ and $M_0 = 0$. Then (11) can be written as

$$0 \leq d_i \leq \alpha + \sum_{l=0}^{i-1} \beta_l d_l + M_0 d_i \quad i > 0. \tag{12}$$

Consequently, we conclude from the lemma 3.6, equation (3.3) and $d_0 = 0$ that

$$d_i \leq \frac{\alpha}{1 - M_0} \exp \left(\sum_{l=0}^{i-1} \frac{\beta_l}{1 - M_0} \right).$$

Substituting $\alpha = C_R h$, $\beta_l = L_0 |\omega_{l,i}|$ and $M_0 = 0$, we obtain

$$d_i \leq C_R h \exp\left(\sum_{l=0}^{i-1} L_0 h\right) = C_R h \exp(L_0 i h),$$

which completes the proof. \square

For the error estimation of trapezoidal quadrature rule we state the following theorem, since in this case the equation for y_i is different from two other cases.

Theorem 3.8. (trapezoidal quadrature rule) Assume that $k(x, t)$ is continuous and does not change sign, $y(x)$ is H -differentiable and $f(x)$ is continuous. To get an approximation to the solution of equation(7) use the first iterate $\tilde{y}_i = y_i^{(1)}$ from (8). Then the following error estimate holds for \tilde{y}_i

$$D(y(x_i), \tilde{y}_i) \leq Ch,$$

where $C = (L_y + C_R) \exp(ihL_0)$.

Proof. The approximation \tilde{y}_i of y_i is obtained

$$\tilde{y}_0 := y_0 = y(x_0), \quad y_i^{(0)} := \tilde{y}_{i-1}, \quad \tilde{y}_i := y_i^{(1)}$$

and so

$$\begin{aligned} d_i &= D(\tilde{y}_i, y(x_i)) \\ &= D(g(x_i) + h \cdot \sum_{j=0}^{i-1} {}'k_1(x_i, x_j) \cdot \tilde{y}_j + \frac{h}{2} \cdot k_1(x_i, x_i) \cdot \tilde{y}_{i-1}, \\ &\quad g(x_i) + h \cdot \sum_{j=0}^i {}''k_1(x_i, x_j) \cdot y(x_j) + R(Q_i)) \\ &\leq h \sum_{j=0}^{i-1} {}'D(k_1(x_i, x_j) \cdot \tilde{y}_j, k_1(x_i, x_j) \cdot y(x_j)) \\ &\quad + \frac{h}{2} D(k_1(x_i, x_i) \cdot \tilde{y}_{i-1}, k_1(x_i, x_i) \cdot \tilde{y}(x_{i-1})) + D(\tilde{0}, R(Q_i)) \\ &\leq hL_0 \sum_{j=0}^{i-1} {}'D(\tilde{y}_j, y(x_j)) + \frac{hL_0}{2} D(\tilde{y}_{i-1}, y(x_i)) + \|R(Q_i)\| \\ &\leq hL_0 \sum_{j=0}^{i-1} {}'d_j + \frac{hL_0}{2} D(\tilde{y}_{i-1}, y(x_{i-1})) + D(y(x_{i-1}), y(x_i)) + C_R h \\ &\leq hL_0 \sum_{j=0}^{i-1} {}'d_j + \frac{hL_0}{2} d_{i-1} + L_y h + C_R h \\ &= hL_0 \sum_{j=0}^{i-2} {}'d_j + \frac{3hL_0}{2} d_{i-1} + (L_y + C_R)h. \end{aligned}$$

Here, \sum' denotes the sum with the weights 1 for $1 \leq j \leq i-1$ and $\frac{1}{2}$ for $j=0$ and \sum'' denotes the sum with the weights 1 for $1 \leq j \leq i-1$ and $\frac{1}{2}$ for $j=0, i$, and L_y is the Lipschitz constant of y (note that y is Lipschitz, since y is H -differentiable). Now lemma 3.6 is applicable with $M_0 = 0$, $\alpha = (L_y + C_R)h$ and $\sum_{j=0}^{i-1} \beta_j = ihL_0$ which yields the desired result. \square

Example 3.9. Consider the fuzzy number A along with the r -cuts $[A]^r = [r^2 + r, 4 - r^3 - r]$ for $r \in [0, 1]$. Let the functions $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ and $f : [0, 1] \rightarrow \mathbb{R}_{\mathcal{F}}$ be given by

$$\begin{aligned} k(x, t) &= 0.1 \sin\left(\frac{x}{2}\right) \sin(t), \\ f(x) &= \left(\frac{1}{2} \cos\left(\frac{x}{2}\right) - 0.1 \sin^2\left(\frac{x}{2}\right) + \frac{0.1}{3} \sin\left(\frac{x}{2}\right) \sin\left(\frac{3x}{2}\right)\right).A. \end{aligned}$$

Then the fuzzy integro-differential equation

$$\begin{aligned} y'(x) &= f(x) + \int_0^x k(x, t).y(t)dt, \quad x \in [0, \frac{\pi}{2}], \\ y(0) &= 0 \end{aligned}$$

has the exact solution $y(x) = \sin(\frac{x}{2}).A$. The results based on rectangular, trapezoidal and hybrid quadrature rules for $n = 10$ are shown in Figures 1-3. In the figures, the points $(x_i, [y_i]^{r_j})$ are displayed for $r_j = \frac{j}{4}$, $j = 0, \dots, 4$ where $[y_i]^{r_j}$ is r_j -cut of the approximate solution y_i at the points $x_i = i\frac{\pi}{20}$ for $i = 0, \dots, 10$. The lines show r -cuts of the exact solution for $r = 0, 1$. Numerical results for this example are reported in tables 1-3 for the rectangular, trapezoidal and hybrid methods respectively. In these tables, $E^r = \max\{|y_-^r(x_i) - y_i_-^r|, |y_+^r(x_i) - y_i_+^r|\}$. As you see, the results in Tables 1,2 are the same, since in converting the integro-differential equation to the corresponding integral equation the term $\cos(\frac{s}{2}) - \cos(\frac{x}{2})$ appears under integral sign and it is equal to zero at the points $x = x_i$, $i = 1, \dots, n$ for the trapezoidal rule. The *CPU time* reported at the end of each table is the running time of the program for getting all the results in that table.

4. Conclusions and Further Research

Approximation of the solution of linear fuzzy integro-differential equation in discrete form was studied and error estimates was obtained. We used the left rectangular and trapezoidal quadrature rules. First of them leads to an explicit method and the second one leads to an implicit and more accurate method. Finally we combined two mentioned quadrature rules to construct an explicit and accurate method. It is worth to note that, although the hybrid method is more time consuming but enjoys more accurate results. We have used H -differentiability which is (i) -differentiability defined in [8]. For the future works we will use (ii) or (iii) or (iv) differentiability and also discuss solvability of nonlinear fuzzy integro-differential equations.

Acknowledgements. The authors would like to thank the referees for their helpful comments which improved the paper.

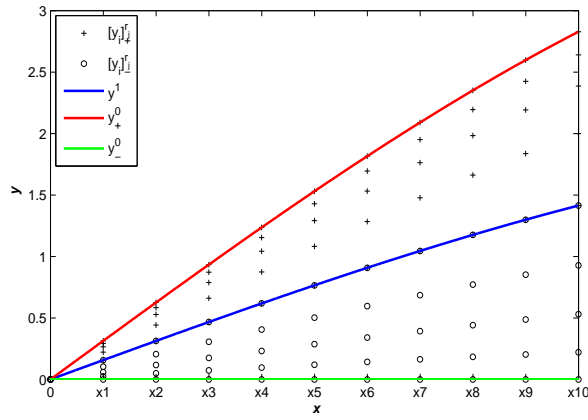


FIGURE 1. Discrete Approximation of the Solution by Rectangular Quadrature Rule

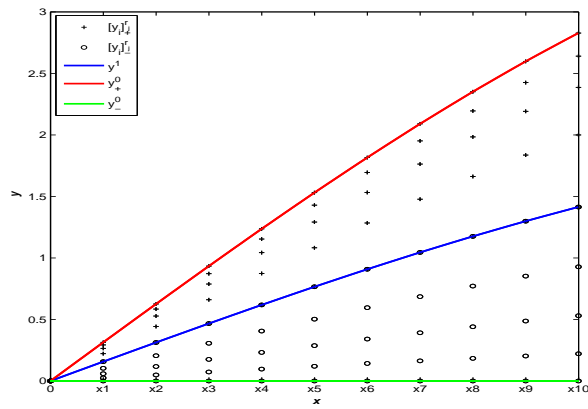


FIGURE 2. Discrete Approximation of the Solution by Trapezoidal Quadrature Rule

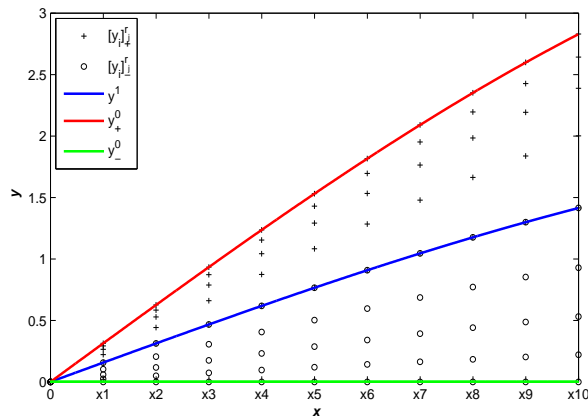


FIGURE 3. Discrete Approximation of the Solution by Hybrid Quadrature Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$.5929e-5	.5535e-5	.5002e-5	.4192e-5	.2964e-5
$3\pi/20$.1996e-4	.1864e-4	.1685e-4	.1412e-4	.9982e-5
$4\pi/20$.4575e-4	.4271e-4	.3860e-4	.3235e-4	.2287e-4
$5\pi/20$.8481e-4	.7918e-4	.7156e-4	.5997e-4	.4241e-4
$6\pi/20$.1370e-3	.1279e-3	.1156e-3	.9685e-4	.6849e-4
$7\pi/20$.2003e-3	.1870e-3	.1690e-3	.1416e-3	.10028e-3
$8\pi/20$.2712e-3	.2532e-3	.2289e-3	.1918e-3	.1356e-3
$9\pi/20$.3449e-3	.3219e-3	.2910e-3	.2438e-3	.1724e-3
$\pi/2$.4153e-3	.3878e-3	.3504e-3	.2937e-3	.2077e-3
CPU time:	2.408''				

TABLE 1. Numerical Results Based on Rectangular Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$.5929e-5	.5535e-5	.5002e-5	.4192e-5	.2964e-5
$3\pi/20$.1996e-4	.1864e-4	.1685e-4	.1412e-4	.9982e-5
$4\pi/20$.4575e-4	.4271e-4	.3860e-4	.3235e-4	.2287e-4
$5\pi/20$.8481e-4	.7918e-4	.7156e-4	.5997e-4	.4241e-4
$6\pi/20$.1370e-3	.1279e-3	.1156e-3	.9685e-4	.6849e-4
$7\pi/20$.2003e-3	.1870e-3	.1690e-3	.1416e-3	.1003e-3
$8\pi/20$.2712e-3	.2532e-3	.2289e-3	.1918e-3	.1356e-3
$9\pi/20$.3448e-3	.3219e-3	.2910e-3	.2438e-3	.1724e-3
$\pi/2$.4153e-3	.3878e-3	.3504e-3	.2937e-3	.2077e-3
CPU time:	2.437''				

TABLE 2. Numerical Results Based on Trapezoidal Rule

x	E^0	$E^{1/4}$	$E^{1/2}$	$E^{3/4}$	E^1
$\pi/20$.6357e-6	.5935e-6	.5363e-6	.4494e-6	.3178e-6
$2\pi/20$.1189e-5	.1110e-5	.1003e-5	.8404e-6	.5943e-6
$3\pi/20$.2656e-4	.2480e-4	.2241e-4	.1878e-4	.1328e-4
$4\pi/20$.9620e-4	.8981e-4	.8118e-4	.6801e-4	.4810e-4
$5\pi/20$.2255e-3	.2105e-3	.1902e-3	.1594e-3	.1127e-3
$6\pi/20$.4225e-3	.3944e-3	.3564e-3	.2987e-3	.2112e-3
$7\pi/20$.6867e-3	.6411e-3	.5794e-3	.4855e-3	.3433e-3
$8\pi/20$.1009e-2	.9418e-3	.8511e-3	.7132e-3	.5044e-3
$9\pi/20$.1371e-2	.1280e-2	.1159e-2	.9694e-3	.6856e-3
$\pi/2$.1749e-2	.1633e-2	.1476e-2	.1237e-2	.8747e-3
CPU time:	8.749''				

TABLE 3. Numerical Results Based on Hybrid Rule

REFERENCES

- [1] S. Abbasbandy and T. Allahviranloo, *Numerical solution of fuzzy differential equation by Runge-Kutta method*, Nonlinear Stud., **11** (2004), 117-129.
- [2] S. Abbasbandy and T. Allahviranloo, *The adomian decomposition method applied to the fuzzy system of Fredholm integral equations of the second kind*, Int. J. Uncertain. Fuzziness Knowl.-Based Syst., **14**(1) (2006), 101-110.
- [3] T. Allahviranloo and M. Afshar Kermani, *Numerical methods for fuzzy linear partial differential equations under new definition for derivative*, Iranian journal of fuzzy systems, **7** (2010), 33-50.
- [4] E. Babolian, H. Sadeghi and S. Javadi, *Numerically solution of fuzzy differential equations by Adomian method*, Appl. Math. Comput., **149** (2004), 547-557.
- [5] K. Balachandran and K. Kanagarajan, *Existence of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations*, J. Appl. Math. Stochast. Anal., **3** (2005), 333-343.
- [6] K. Balachandran and P. Prakash, *Existence of solutions of nonlinear fuzzy integral equations in Banach spaces*, Libertas Math., **21** (2001), 91-97.
- [7] K. Balachandran and P. Prakash, *Existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations*, Indian J. Pure Appl. Math., **33**(3) (2002), 329-343.
- [8] B. Bede and S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, **151** (2005), 581-599.
- [9] B. Bede and S. G. Gal, *Quadrature rules for integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems, **145** (2004), 359-380.
- [10] A. M. Bica, *Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations*, Information Sciences, **178** (2008), 1279-1292.
- [11] M. Friedman, M. Ma and A. Kandel, *Numerical solutions of fuzzy differential and integral equations*, Fuzzy Sets and Systems, **106** (1999), 35-48.
- [12] M. Friedmann, M. Ming and A. Kandel, *Solution to fuzzy integral equations with arbitrary kernels*, Int. J. Approx. Reason., **20** (1999), 249-262.
- [13] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, **18** (1986), 31-43.
- [14] W. Hackbusch, *Integral equations: theory and numerical treatment*, Birkhuser Verlag, Basel, 1995.
- [15] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24** (1987), 301-317.
- [16] O. Kaleva, *The Cauchy problem for fuzzy differential equations*, Fuzzy Sets and Systems, **35** (1990), 389-396.
- [17] V. Lakshmikantham and R. N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, Taylor and Francis, London, 2003.
- [18] V. Lakshmikantham, K. N. Murty and J. Turner, *Two point boundary value problems associated with nonlinear fuzzy differential equations*, Math. Inequal. Appl., **4** (2003), 527-533.
- [19] M. Ma, M. Friedman and Abraham Kandel, *Numerical solutions of fuzzy differential equations*, Fuzzy Sets and Systems, **105** (1999), 133-138.
- [20] A. Molabahrami, A. Shidfar and A. Ghyasi, *An analytical method for solving linear Fredholm fuzzy integral equations of the second kind*, Computers and Mathematics with applications, **61** (2011), 2754-2761.
- [21] J. J. Nieto, *The Cauchy problem for continuous fuzzy differential equations*, Fuzzy Sets and Systems, **102** (1999), 259-262.
- [22] D. O'Regan, V. Lakshmikantham and J. J. Nieto, *Initial and boundary value problems for fuzzy differential equations*, Nonlinear Anal., **54** (2003), 405-415.
- [23] J. Y. Parka and J. U. Jeong, *A note on fuzzy integral equations*, Fuzzy Sets and Systems, **108** (1999), 193-200.
- [24] P. Prakash, J. J. Nieto, J. H. Kim and R. Rodriguez-Lopez, *Existence of solutions of fuzzy neutral differential equations in Banach spaces*, Dyn. Syst. Appl., **14**(3-4) (2005), 407-417.

- [25] M. Puri and D. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552-558.
- [26] O. Solaymani Fard, A. Esfahani and A. Vahidian Kamyad, *On solution of a class of fuzzy BVPs*, Iranian journal of fuzzy systems, **9** (2012), 49-60.
- [27] P. V. Subrahmanyam and S. K. Sudarsanam, *A note on fuzzy Volterra integral equations*, Fuzzy Sets and Systems, **81** (1996), 237-240.
- [28] C. Wu and Z. Gong, *On Henstock integral of fuzzy-number-valued functions I*, Fuzzy Sets and Systems, **120** (2001), 523-532.
- [29] C. Wu and M. Ma, *Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions*, Information Sciences, **108** (1998), 123-134.

MASOUMEH ZEINALI, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN

E-mail address: zeynali@tabrizu.ac.ir

SEDAGHAT SHAHMORAD*, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN

E-mail address: Shahmorad@tabrizu.ac.ir

KAMAL MIRNIA, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN

E-mail address: Mirnia-kam@tabrizu.ac.ir

*CORRESPONDING AUTHOR