SOME REMARKS ON GENERALIZED SEQUENCE SPACE OF BOUNDED VARIATION OF SEQUENCES OF FUZZY NUMBERS

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Abstract. The idea of difference sequences of real (or complex) numbers was introduced by Kızmaz [8]. In this paper, using the difference operator and a lacunary sequence, we introduce and examine the class of sequence $bv_\theta(\Delta, \mathcal{F})$. We study some of its properties like solidity, symmetricity, etc.

1. Introduction

The theory of sequences of fuzzy numbers was first introduced by Matloka [10]. Matloka [10] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. Since then, there has been increasing interest in the study of sequences of fuzzy numbers (see [3, 13, 14]). Lacunary sequences of fuzzy numbers and real (or complex) numbers has been studied by Nuray [11], Kwon and Sung [9], Altinok *et al.* [1], and many others [7, 12]. Recently, Başar *et al.* [4] and Tripathy and Dutta [14] have studied the spaces of bounded variation sequences.

2. Preliminaries

In this section we give the basic notions related to fuzzy numbers and a brief information about difference sequences and lacunary sequences

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

i) u is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

iii) u is upper semicontinuous;

iv) $\operatorname{supp} u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]^0$, is compact.

$\alpha-$level set $[u]^\alpha$ of a fuzzy number u is defined by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ \operatorname{supp} u, & \text{if } \alpha = 0. \end{cases}$$

It is clear that u is a fuzzy number if and only if $[u]^\alpha$ is a closed interval for each $\alpha \in [0,1]$ and $[u]^1 \neq \emptyset$.

A real number r can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \begin{cases} 1, & x = r \\ 0, & x \neq r \end{cases}.$$

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If $u \in L(\mathbb{R})$, then u is called a fuzzy number, and $L(\mathbb{R})$ is said to be a fuzzy number space.

Let $u, v \in L(\mathbb{R})$, $k \in \mathbb{R}$ and the $\alpha-$level sets of fuzzy numbers u and v be $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$, $\alpha \in (0,1]$. Then, a partial ordering " $\leq$ " in $L(\mathbb{R})$ is defined by $u \leq v \Leftrightarrow \underline{u}^\alpha \leq \underline{v}^\alpha$ and $\overline{u}^\alpha \leq \overline{v}^\alpha$ for all $\alpha \in (0,1]$. Some arithmetic operations for $\alpha-$level sets are defined as follows:

$$[u+v]^\alpha = [\underline{u}^\alpha + \underline{v}^\alpha, \overline{u}^\alpha + \overline{v}^\alpha]$$

$$[u-v]^\alpha = [\underline{u}^\alpha - \overline{v}^\alpha, \overline{u}^\alpha - \underline{v}^\alpha]$$

$$[ku]^\alpha = \begin{cases} [k\underline{u}^\alpha, k\overline{u}^\alpha], & \text{if } k \geq 0 \\ [k\overline{u}^\alpha, k\underline{u}^\alpha], & \text{otherwise} \end{cases}.$$

In order to calculate the distance between two fuzzy numbers u and v, we use the metric

$$d(u,v) = \sup_{0 \leq \alpha \leq 1} d_H\left([u]^\alpha, [v]^\alpha\right)$$

where d_H is the Hausdorff metric defined as

$$d_H\left([u]^\alpha, [v]^\alpha\right) = \max\left\{\left|\underline{u}^\alpha - \underline{v}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right|\right\}.$$

It is known that d is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), d)$ is a complete metric space.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. Thus, a sequence of fuzzy numbers (X_k) is a correspondence from the set of positive integers to a set of fuzzy numbers.

Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded and convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$. Let $\ell_\infty(F)$ and $c(F)$ denote the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively [10].

The difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x = (x_k)$ such that $\Delta x = \Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [8]. The idea of difference sequences was generalized by Et and Çolak [6], Et *et al.* [5], Altinok *et al.* [2] and Tripathy *et al.* [15–17].

Let $w(F)$ be the set of all sequences of fuzzy numbers. The operator $\Delta : w(F) \to w(F)$ is defined by $\Delta X_k = X_k - X_{k+1}$.

A sequence space $E(F)$ is said to be normal (or solid) if $(X_k) \in E(F)$ and (Y_k) is such that $d(Y_k, \bar{0}) \leq d(X_k, \bar{0})$ implies $(Y_k) \in E(F)$.

A sequence space $E(F)$ is said to be symmetric if $\left(X_{\pi(n)}\right) \in E(F)$, whenever $(X_k) \in E(F)$, where π is a permutation of $\mathbb{N}$.

A sequence space $E(F)$ is said to be convergence free if $(Y_k) \in E(F)$, whenever $(X_k) \in E(F)$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, ...,$ where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence (x_k) is said to be of bounded variation if $\sum_{k=1}^{\infty} |\Delta x_k| < \infty$. It is denoted by

$$bv = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} |\Delta x_k| < \infty \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The class of fuzzy real valued bounded variation sequences $bv\,(F)$ is given by

$$bv\,(F) = \left\{ (X_k) \in w\,(F) : \sum_{k=1}^{\infty} d\,(X_k, X_{k+1}) < \infty \right\}.$$

3. Main Results

In this section, we introduce the class of sequence $bv_\theta\,(\Delta, \mathcal{F})$ and give an inclusion relation. In addition, we show that this class of sequence is neither solid nor symmetric nor convergence free.

We define the class of sequence $bv_\theta\,(\Delta, F)$ as follows:

$$bv_\theta\,(\Delta, F) = \left\{ X = (X_k) \in w(F) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,(\Delta X_k, \Delta X_{k+1}) \right) < \infty \right\}.$$

In the special case $\theta = 2^r$, we shall write $bv\,(\Delta, F)$ instead of $bv_\theta\,(\Delta, F)$.

Theorem 3.1. *The class of sequence $bv_\theta\,(\Delta, \mathcal{F})$ is closed under the operations of addition and scalar multiplication.*

Proof. **(i)** Let (X_k) and (Y_k) be two sequences of fuzzy numbers in $bv_\theta\,(\Delta, \mathcal{F})$ and $\theta = (k_r)$ be a lacunary sequence. Then we can write the following inequality:

$$\begin{aligned} &\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,[(\Delta X_k + \Delta Y_k), (\Delta X_{k+1} + \Delta Y_{k+1})] \right) \\ &\quad \leq \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,(\Delta X_k, \Delta X_{k+1}) \right) + \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,(\Delta Y_k, \Delta Y_{k+1}) \right) \end{aligned}$$

Since (X_k), $(Y_k) \in bv_\theta\,(\Delta, \mathcal{F})$, the right side of inequality is less than infinity, so the left side is. Hence $bv_\theta\,(\Delta, \mathcal{F})$ is closed under the operation of addition.

(ii) Let (X_k) be a sequence of fuzzy numbers in $bv_\theta\,(\Delta, \mathcal{F})$ and $\theta = (k_r)$ be a lacunary sequence and $c \in R$. Then we can write

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,(c\Delta X_k, c\Delta X_{k+1}) \right) = |c| \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d\,(\Delta X_k, \Delta X_{k+1}) \right) < \infty.$$

Since $(X_k) \in bv_\theta\,(\Delta, \mathcal{F})$, the right side of inequality is less than infinity. Therefore we obtain $(c\Delta X_k) \in bv_\theta\,(\Delta, \mathcal{F})$ and so $bv_\theta\,(\Delta, \mathcal{F})$ is closed under the operation of scalar multiplication. □

Theorem 3.2. *Let θ be a lacunary sequence such that $h_r \neq r$. If $\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \right) < \infty$, then $bv\,(\mathcal{F}) \subset bv\,(\Delta, \mathcal{F}) \subset bv_\theta\,(\Delta, \mathcal{F})$.*

Proof. Let (X_k) be a sequence of fuzzy numbers and $(X_k) \in bv(\mathcal{F})$. Then we have $\sum\limits_{k=1}^{\infty} d(X_k, X_{k+1}) < \infty$. From the definition of difference operator and metric d, we can write

$$\begin{aligned}\sum_{k=1}^{\infty} d(\Delta X_k, \Delta X_{k+1}) &= \sum_{k=1}^{\infty} d(X_k - X_{k+1}, X_{k+1} - X_{k+2}) \\ &\leq \sum_{k=1}^{\infty} d(X_k, X_{k+1}) + \sum_{k=1}^{\infty} d(X_{k+1}, X_{k+2}) < \infty.\end{aligned}$$

Hence $bv(\mathcal{F}) \subset bv(\Delta, \mathcal{F})$.

On the other hand, we can find a positive integer N such that $\sum\limits_{k \in I_r} d(\Delta X_k, \Delta X_{k+1}) < 1$, for all $r \geq N$. Therefore, we can write

$$\sum_{r \geq N} \left(\frac{1}{h_r} \sum_{k \in I_r} d(\Delta X_k, \Delta X_{k+1}) \right) \leq \sum_{r \geq N} \frac{1}{h_r} < \infty$$

and so

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d(\Delta X_k, \Delta X_{k+1}) \right) < \infty.$$

Thus we have $(X_k) \in bv_\theta(\Delta, \mathcal{F})$. □

Theorem 3.3. *The class of sequence $bv_\theta(\Delta, \mathcal{F})$ is not solid.*

Proof. To prove the theorem it is sufficient to give the following example. □

Example 3.4. Let (X_k) be a sequence of fuzzy numbers and $\theta = (5^r)$ be a lacunary sequence. Define the sequence (X_k) of fuzzy numbers by

$$X_k(t) = \begin{cases} \frac{k}{2}t + 1, & -\frac{2}{k} \leq t \leq 0 \\ -\frac{k}{2}t + 1, & 0 \leq t \leq \frac{2}{k} \\ 0, & \text{otherwise} \end{cases}.$$

Using the membership function of (X_k), we calculate the $\alpha-$level sets of (X_k) and (ΔX_k), respectively, as

$$[X_k]^\alpha = \left[\frac{2\alpha - 2}{k}, \frac{2 - 2\alpha}{k} \right]$$

and

$$[\Delta X_k]^\alpha = \left[\frac{(2\alpha - 2)(2k+1)}{k(k+1)}, \frac{(2 - 2\alpha)(2k+1)}{k(k+1)} \right]$$

for $\alpha \in [0,1]$ and obtain the membership function of (ΔX_k) as

$$\Delta X_k(t) = \begin{cases} \frac{k(k+1)}{4k+2}t + 1, & -\frac{4k+2}{k(k+1)} \leq t \leq 0 \\ 1 - \frac{k(k+1)}{4k+2}t, & 0 \leq t \leq \frac{4k+2}{k(k+1)} \\ 0, & \text{otherwise} \end{cases}$$

Then we get

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r} d\left(\Delta X_k,\Delta X_{k+1}\right)\right)=\sum_{r=1}^{\infty}\left(\frac{1}{4.5^{r-1}}\sum_{k\in I_r}\frac{4}{k\left(k+2\right)}\right)<\infty.$$

Hence $(X_k)\in bv_\theta\left(\Delta,\mathcal{F}\right)$.

On the other hand, define the sequence (Y_k) by

$$Y_k=\begin{cases}\underline{X_k^\alpha}, & \text{for } k \text{ odd}\\ \bar{0}, & \text{for } k \text{ even}\end{cases}.$$

where $[X_k]^\alpha=\left[\underline{X_k^\alpha},\overline{X_k^\alpha}\right]$. That is, $(Y_k)=\left(\underline{X_1^\alpha},0,\underline{X_3^\alpha},0,\underline{X_5^\alpha},0,\underline{X_7^\alpha},0,...\right)$ and so we have difference sequence of (Y_k) as

$$(\Delta Y_k)=\left(\underline{X_1^\alpha},-\underline{X_3^\alpha},\underline{X_3^\alpha},-\underline{X_5^\alpha},\underline{X_5^\alpha},-\underline{X_7^\alpha},\underline{X_7^\alpha},...\right).$$

Then we get

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r} d\left(\Delta Y_k,\Delta Y_{k+1}\right)\right)>\sum_{r=1}^{\infty}\left(\frac{1}{4.5^{r-1}}\sum_{k\in I_r}\frac{2}{k}\right)=\infty.$$

Therefore it implies that $(Y_k)\notin bv_\theta\left(\Delta,\mathcal{F}\right)$ from the definition.

Theorem 3.5. *The class of sequence $bv_\theta\left(\Delta,\mathcal{F}\right)$ is not symmetric.*

Proof. We give the following example to show that $bv_\theta\left(\Delta,\mathcal{F}\right)$ is not symmetric. □

Example 3.6. Let (X_k) be a sequence of fuzzy numbers and $\theta=(5^r)$ be a lacunary sequence. Define the sequence (X_k) of fuzzy numbers by

$$X_k\left(t\right)=\begin{cases}\left.\begin{array}{cc} t+2, & -2\leq t\leq 0\\ 2-t, & 0\leq t\leq 2\\ 0, & \text{otherwise}\end{array}\right\}:=L, & \begin{array}{c}\text{if } k=5^m\\ m\in\mathbb{N}\end{array}\\ \bar{0}, & \text{otherwise}\end{cases}.$$

Then we calculate the $\alpha-$level sets of (X_k) and (ΔX_k), respectively, as

$$[X_k]^\alpha=\begin{cases}[\alpha-2,2-\alpha], & \text{if } k=5^m\\ \bar{0}, & \text{otherwise}\end{cases}.$$

and

$$[\Delta X_k]^\alpha=\begin{cases}[\alpha-2,2-\alpha], & \text{if } k=5^m \text{ and } k+1=5^m\\ \bar{0}, & \text{otherwise}\end{cases}$$

for $\alpha\in[0,1]$. That is, we write the difference sequence $\left([\Delta X_k]^\alpha\right)=(L^\alpha,\bar{0},\bar{0},L^\alpha,L^\alpha,\bar{0},...,\bar{0},L^\alpha,L^\alpha,\bar{0},...)$, where $L^\alpha=[\alpha-2,2-\alpha]$. Then $X_k\in bv_\theta\left(\Delta,\mathcal{F}\right)$ is clear from the following inequalities:

$$\begin{aligned}\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r} d\left(\Delta X_k,\Delta X_{k+1}\right)\right)&\leq\sum_{r=1}^{\infty}\left(\frac{1}{4.5^{r-1}}\sum_{k\in I_r}2\right)\\ &\leq\sum_{r=1}^{\infty}\left(\frac{4}{4.5^{r-1}}\right)<\infty.\end{aligned}$$

Now, we define a sequence (Y_k) which is the arrangement of the sequence (X_k) by

$$(Y_k) = (L, 0, 0, L, 0, 0, L, 0, 0, L, 0, 0, L, ...)$$

and calculate the difference sequence as

$$(\Delta Y_k) = (L, 0, L, L, 0, L, L, 0, L, L, 0, L, L, ...) .$$

Then

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d(\Delta Y_k, \Delta Y_{k+1}) \right) = \sum_{r=1}^{\infty} \left(\frac{1}{4.5^{r-1}} \left(4. \left[\left| \frac{4.5^{r-1}}{3} \right| \right] + 2 \right) \right) > \sum_{r=1}^{\infty} 1 = \infty.$$

Hence we get $Y_k \notin bv_\theta(\Delta, \mathcal{F})$. This shows that $bv_\theta(\Delta, \mathcal{F})$ is not symmetric.

Theorem 3.7. *The sequence space $bv_\theta(\Delta, \mathcal{F})$ is not convergence free.*

Proof. The proof follows from the following example. □

Example 3.8. Let (X_k) be a sequence of fuzzy numbers and $\theta = (5^r)$ be a lacunary sequence. Consider the sequence (X_k) of fuzzy numbers defined as follows:

$$X_k(t) = \begin{cases} \left. \begin{array}{cc} \frac{k^2}{2}t + 1, & -\frac{2}{k^2} \leq t \leq 0 \\ -\frac{k^2}{2}t + 1, & 0 \leq t \leq \frac{2}{k^2} \\ 0, & \text{otherwise} \end{array} \right\}, & \text{if } k \text{ is odd} \\ \bar{0}, & \text{if } k \text{ is even} \end{cases} .$$

Then we obtain $\alpha-$level sets of (X_k) and (ΔX_k) as follows

$$[X_k]^\alpha = \begin{cases} \left[\frac{2(\alpha-1)}{k^2}, \frac{2(1-\alpha)}{k^2} \right], & \text{if } k \text{ is odd} \\ \bar{0}, & \text{if } k \text{ is even} \end{cases} .$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[\frac{2(\alpha-1)}{k^2}, \frac{2(1-\alpha)}{k^2} \right], & \text{if } k \text{ is odd} \\ \left[\frac{2(\alpha-1)}{(k+1)^2}, \frac{2(1-\alpha)}{(k+1)^2} \right], & \text{if } k \text{ is even} \end{cases}$$

for $\alpha \in [0, 1]$ and obtain the membership function of (ΔX_k) as

$$\Delta X_k(t) = \begin{cases} \left. \begin{array}{cc} \frac{k^2}{2}t + 1, & -\frac{2}{k^2} \leq t \leq 0 \\ 1 - \frac{k^2}{2}t, & 0 \leq t \leq \frac{2}{k^2} \\ 0, & \text{otherwise} \end{array} \right\} & \text{if } k \text{ is odd} \\ \left. \begin{array}{cc} \frac{(k+1)^2}{2}t + 1, & -\frac{2}{(k+1)^2} \leq t \leq 0 \\ 1 - \frac{(k+1)^2}{2}t, & 0 \leq t \leq \frac{2}{(k+1)^2} \\ 0, & \text{otherwise} \end{array} \right\} & \text{if } k \text{ is even} \end{cases}$$

Then we have $X_k \in bv_\theta(\Delta, \mathcal{F})$ from the following inequality:

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} d(\Delta X_k, \Delta X_{k+1}) \right) = \sum_{r=1}^{\infty} \left(\frac{2}{4.5^{r-1}} \left[\left| \frac{4.5^{r-1}}{2} \right| \right] \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \right) < \infty.$$

Now, we consider a sequence (Y_k) of fuzzy numbers defined by

$$Y_k(t)=\begin{cases}\left.\begin{array}{cc}\frac{\sqrt{k}}{2}t+1, & -\frac{2}{\sqrt{k}}\leq t\leq 0\\ 1-\frac{\sqrt{k}}{2}t, & 0\leq t\leq\frac{2}{\sqrt{k}}\\ 0, & \text{otherwise}\end{array}\right\}, & \text{if } k \text{ is odd}\\ \qquad\bar{0}, & \text{if } k \text{ is even}\end{cases}.$$

To show $Y_k\notin bv_\theta(\Delta,\mathcal{F})$, firstly, we calculate $\alpha-$level sets of (Y_k) and (ΔY_k) as follows:

$$[Y_k]^\alpha=\begin{cases}\left[\frac{2(\alpha-1)}{\sqrt{k}},\frac{2(1-\alpha)}{\sqrt{k}}\right], & \text{if } k \text{ is odd}\\ \bar{0}, & \text{if } k \text{ is even}\end{cases}$$

and

$$[\Delta Y_k]^\alpha=\begin{cases}\left[\frac{2(\alpha-1)}{\sqrt{k}},\frac{2(1-\alpha)}{\sqrt{k}}\right], & \text{if } k \text{ is odd}\\ \left[\frac{2(\alpha-1)}{\sqrt{k+1}},\frac{2(1-\alpha)}{\sqrt{k+1}}\right], & \text{if } k \text{ is even}\end{cases}.$$

for $\alpha\in[0,1]$. From this, we obtain the membership function of (ΔY_k) as follows:

$$\Delta Y_k(t)=\begin{cases}\left.\begin{array}{cc}\frac{\sqrt{k}}{2}t+1, & -\frac{2}{\sqrt{k}}\leq t\leq 0\\ 1-\frac{\sqrt{k}}{2}t, & 0\leq t\leq\frac{2}{\sqrt{k}}\\ 0, & \text{otherwise}\end{array}\right\} & \text{if } k \text{ is odd}\\ \left.\begin{array}{cc}\frac{\sqrt{k+1}}{2}t+1, & -\frac{2}{\sqrt{k+1}}\leq t\leq 0\\ 1-\frac{\sqrt{k+1}}{2}t, & 0\leq t\leq\frac{2}{\sqrt{k+1}}\\ 0, & \text{otherwise}\end{array}\right\} & \text{if } k \text{ is even}\end{cases}$$

We can write

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}d\left(\Delta Y_k,\Delta Y_{k+1}\right)\right)=\sum_{r=1}^{\infty}\left(\frac{2}{4.5^{r-1}}\left[\left|\frac{4.5^{r-1}}{2}\right|\right]\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)\right)=\infty.$$

Thus it implies that $(Y_k)\notin bv_\theta(\Delta,\mathcal{F})$ from the definition.

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