Abstract. In this paper we present some properties of set-norm exhaustive set multifunctions and also of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity.

1. Introduction

The subject of the present paper concerns fuzzy set multifunctions. Non-additive set functions and fuzzy sets have been intensively studied by many authors (e.g., Asahina [1], Choquet [4], Daneshgar and Hashemi [7], Denneberg [9], Drewnowski [10], Dubois and Prade [11], Funikova [12], Li [16], Merghadi and Aliouche [17], Pap [18], Precupanu [19], Shafer [20], Sugeno [21], Suzuki [22], Zadeh [23], Vaezpour and Karini [24], Wen, Shi and Li [25], Wu and Bo [26]), due to its applications in statistics, economy, theory of games, human decision making.

In our previous papers [5,6,13-15] we extended and studied different concepts (such as pseudo-atom, Darboux property, continuity, exhaustivity, regularity) to the set-valued case.

In [5] we introduced the notion of set-norm on the family of non-empty subsets of a real linear space and studied different notions of continuous set multifunctions with respect to a set-norm.

This paper contains three sections. In the second section, properties of set-norm exhaustive set multifunctions are presented. These set multifunctions (such as probability multimeasures) are used in control, robotics, decision theory (in Bayesian estimation) or in statistical inference (Dempster [8]). In the third section, we present different properties of atoms and pseudo-atoms of set multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. Our results generalize to set-valued case important problems in measure theory, such as non-atomicity or pure atomicity (Aumann and Shapley [2]), that have applications in coincidence and rigidity phenomena (Chitescu [3]).

2. Non-Additive Set Multifunctions

Let $T$ be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of $T$ and $\mathcal{P}(T)$ the family of all subsets of $T$.

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In the sequel, \((X, +, 0)\) will be a commutative semigroup with unity 0 and \(\mathcal{P}_0(X)\) the family of non-empty subsets of \(X\). On \(\mathcal{P}_0(X)\) we consider an order relation denoted by "\(\leq\)". We write \(E < F\) if \(E \leq F\) and \(E \neq F\), for every \(E, F \in \mathcal{P}_0(X)\). The notation \(\bar{F} \geq E\) \((F \geq E\) respectively\) will often be used in the place of \(E \leq F\) \((E < F\) respectively\). We shall write \((\mathcal{P}_0(X), \leq)\). For every \(E, F \in \mathcal{P}_0(X)\), let \(E + F = \{x + y | x \in E, y \in F\} \).

**Example 2.1.** I. The usual set inclusion "\(\subseteq\)" is an order relation on \(\mathcal{P}_0(X)\).

If \(X\) is a normed space, then \(\mathcal{P}_c(X)\) is the family of non-empty closed subsets of \(X\) and \(\mathcal{P}_d(X)\) is the family of non-empty closed bounded subsets of \(X\).

The set of all real numbers is denoted by \(\mathbb{R}\). We denote \(\mathbb{N}^* = \mathbb{N}\setminus\{0\}\), where \(\mathbb{N}\) is the set of all positive integers.

**Definition 2.2.** [5] A function \(|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]\) is called a set-norm on \(\mathcal{P}_0(X)\) if it satisfies the conditions:

(i) \(|E| = 0 \iff E = \{0\}, \forall E \in \mathcal{P}_0(X)\).

(ii) \(|E + F| \leq |E| + |F|, \forall E, F \in \mathcal{P}_0(X)\).

**Definition 2.3.** [5] A set-norm \(|\cdot|\) on \((\mathcal{P}_0(X), \leq)\) is called monotone if for every sets \(E, F \in \mathcal{P}_0(X), E \leq F \Rightarrow |E| \leq |F|\). We denote \((\mathcal{P}_0(X), \leq, |\cdot|)\) when \((\mathcal{P}_0(X), \leq)\) is endowed with a monotone set-norm \(|\cdot|\).

**Example 2.4.** Let \((X, \| \cdot \|)\) be a real normed space and \(|E|_s = \sup_{x \in E} \| x \|\), for every \(E \in \mathcal{P}_0(X)\). Then the function \(|\cdot|_s\) is a monotone set-norm on \((\mathcal{P}_0(X), \subseteq)\) and we denote this by \((\mathcal{P}_0(X), \subseteq, |\cdot|_s)\).

**Definition 2.5.** A set multifunction \(\mu : \mathcal{C} \to \mathcal{P}_0(X)\) is called:

(i) a multimeasure if \(\mu(\emptyset) = \{0\}\) and \(\mu(A \cup B) = \mu(A) + \mu(B)\), for every \(A, B \in \mathcal{C}\), with \(A \cap B = \emptyset\).

(ii) null-additive if for every \(A, B \in \mathcal{C}\),

\[
\mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(A).
\]

(iii) null-null-additive if for every \(A, B \in \mathcal{C}\),

\[
\mu(A) = \mu(B) = \mu(\emptyset) \Rightarrow \mu(A \cup B) = \mu(\emptyset).
\]

**Definition 2.6.** Let \(\mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq)\) be a set multifunction. \(\mu\) is said to be:

(i) monotone if \(\mu(A) \leq \mu(B)\), for every \(A, B \in \mathcal{C}\), with \(A \subseteq B\).

(ii) fuzzy if \(\mu\) is monotone and \(\mu(\emptyset) = \{0\}\).

(iii) subadditive if \(\mu(A \cup B) \leq \mu(A) + \mu(B)\), for every \(A, B \in \mathcal{C}\).

(iv) a multisubmeasure if \(\mu\) is fuzzy and subadditive.

**Remark 2.7.** I. If \(X\) is a normed space and \(\mu\) is \(\mathcal{P}_c(X)\)-valued, then in the definition of a multi(sub)measure, it usually appears "\(+\)" instead of "\(+\)", because the sum of two closed sets is not always closed.

II. The following implications hold:

(i) If \(\mu\) is a multisubmeasure, then \(\mu\) is null-additive.
(ii) If \( \mu \) is null-additive, then \( \mu \) is null-null-additive.

III. The concepts in Definitions 2.5 and 2.6 do not reduce to the usual single-valued case. The difficulty arises here since we have to consider an order relation on \( \mathcal{P}_0(X) \) and many classical measure theory proof methods fail. For instance, if \( \mu : (\mathcal{P}_0(\mathbb{R}), \subseteq) \) is single-valued and monotone, then \( \mu \) reduces in fact to a constant function \( \mu(A) = \{\mu(0)\}, \forall A \in \mathcal{C} \).

Moreover, \( \mathcal{P}_0(X) \) (and also \( \mathcal{P}_c(X) \)) is not a linear space since \( \mathcal{P}_0(X) \) is not a group with respect to the addition “+” defined by \( M + N = \{x + y \mid x \in M, y \in N\} \), for every \( M, N \in \mathcal{P}_0(X) \).

**Definition 2.8.** A set multifunction \( \mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \leq, |\cdot|) \) is said to be:

(i) set-norm exhaustive (shortly, sn-exhaustive) if \( \lim_{n \rightarrow \infty} |\mu(A_n)| = 0 \), for every pairwise disjoint sequence of sets \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \).

(ii) set-norm continuous (shortly, sn-continuous) if \( \lim_{n \rightarrow \infty} |\mu(A_n)| = 0 \), for every \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \), such that \( A_n \searrow \emptyset \) (i.e. \( A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^* \land \bigcap_{n=1}^{\infty} A_n = \emptyset \)).

(iii) strongly-set-norm continuous (shortly, strongly sn-continuous) if \( \lim_{n \rightarrow \infty} |\mu(A_n)| = 0 \) for every sequence \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_{n+1} \subseteq A_n, \forall n \in \mathbb{N}^* \) and \( \mu(\bigcap_{n=1}^{\infty} A_n) = \{0\} \).

(iv) null-continuous if \( \mu(\bigcup_{n=1}^{\infty} A_n) = \{0\} \) for every sequence \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\} \), for every \( n \in \mathbb{N}^* \).

We now establish some relationships among the set multifunctions introduced in Definition 2.8.

We recall that \( \mathcal{C} \) is a \( \sigma \)-ring if the following conditions hold:

(i) \( A \setminus B \in \mathcal{C} \), for every \( A, B \in \mathcal{C} \),

(ii) \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{C} \), for every \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \).

**Theorem 2.9.** If \( \mathcal{C} \) is a \( \sigma \)-ring and \( \mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \leq, |\cdot|) \) is fuzzy and sn-continuous, then \( \mu \) is sn-exhaustive.

**Proof.** Let \( (A_n)_{n \in \mathbb{N}^*} \) be a sequence of mutually disjoint sets of \( \mathcal{C} \) and let \( B_n = \bigcup_{k=n}^{\infty} A_k \), for all \( n \in \mathbb{N}^* \). Then \( B_n \in \mathcal{C} \), for every \( n \in \mathbb{N}^* \) and \( B_n \searrow \emptyset \). Since \( \mu \) is sn-continuous, it results \( |\mu(B_n)| \rightarrow 0 \), which implies \( |\mu(A_n)| \rightarrow 0 \). So \( \mu \) is sn-exhaustive. \( \square \)

**Theorem 2.10.** Suppose \( \mathcal{C} \) is a \( \sigma \)-ring and \( \mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \leq, |\cdot|) \) is fuzzy. If \( \mu \) is null-null-additive and strongly-sn-continuous, then \( \mu \) is null-continuous.

**Proof.** Let \( (A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\} \), for every \( n \in \mathbb{N}^* \).

Denote \( A = \bigcup_{n=1}^{\infty} A_n \). We recurrently define a subsequence \( (A_{n_k}) \) of \( (A_n) \) as follows.
Let \( n_1 = 1 \). For every \( k \in \mathbb{N}^* \), since \( \mu(A_{n_k}) = \{0\} \) and \( A_{n_k} \cup (A \setminus A_{n_k}) \setminus A_{n_k} \) when \( n \to \infty \), by the fact that \( \mu \) is strongly-sn-continuous, we can choose \( n_{k+1} \) so that \( n_{k+1} > n_k \)
\[
|\mu(A_{n_k} \cup (A \setminus A_{n_k+1}))| < \frac{1}{k}.
\]
Denote \( B = \bigcup_{k=1}^{\infty} \left( A_{n_{2k}} \setminus A_{n_{2k-1}} \right) \) and \( C = A \setminus B = A_{n_1} \cup \bigcup_{k=1}^{\infty} \left( A_{n_{2k+1}} \setminus A_{n_{2k}} \right) \). For all \( k \in \mathbb{N}^* \), since \( B \subseteq A_{n_{2k}} \cup (A \setminus A_{n_{2k+1}}) \), it results:
\[
|\mu(B)| \leq |\mu(A_{n_{2k}} \cup (A \setminus A_{n_{2k+1}}))| < \frac{1}{2k}.
\]
It follows \( \mu(B) = \{0\} \). Analogously, for every \( k \in \mathbb{N}^* \), since \( C \subseteq A_{n_{2k-1}} \cup (A \setminus A_{n_{2k}}) \), it follows that:
\[
|\mu(C)| \leq |\mu(A_{n_{2k-1}} \cup (A \setminus A_{n_{2k}}))| < \frac{1}{2k - 1},
\]
which implies that \( \mu(C) = \{0\} \). Since \( \mu \) is null-null-additive, we obtain \( \mu(A) = \mu(B \cup C) = \{0\} \). So, \( \mu \) is null-continuous. \( \square \)

**Example 2.11.** I. If \( \mathcal{C} \) is finite, then every set multifunction \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq, |\cdot|) \), with \( \mu(\emptyset) = \{0\} \), is sn-exhaustive and sn-continuous.

II. Let \( T = \mathbb{R} \), \( \mathcal{C} = \{A \subseteq T | A \text{ is finite} \} \) and \( \mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq, |\cdot|_\mathbb{R}) \) be defined by \( \mu(A) = [0, \nu(A)] \), where \( \nu(A) = \begin{cases} 0, & A = \emptyset \\ 1 + \text{card} \ A, & A \neq \emptyset \end{cases} \forall A \in \mathcal{C} \) and \( \text{card} \ A \) is the number of elements in \( A \). Then \( \mu \) is sn-continuous, but not sn-exhaustive.

III. Let \( T = \mathbb{N} \), \( \mathcal{C} = \mathcal{P}(\mathbb{N}) \) and \( \mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{N}), \subseteq, |\cdot|_\mathbb{N}) \) be defined by
\[
\mu(A) = \begin{cases} \{0\}, & A \neq \mathbb{N} \\ \{0,1\} \cup [3,7], & A = \mathbb{N} \end{cases}
\]
\( \mu \) is not null-null-additive, because there exist \( A = \{0\} \) and \( B = \mathbb{N}^* \) so that \( \mu(A) = \mu(B) = \{0\} \), but \( \mu(A \cup B) = \mu(\mathbb{N}) \neq \{0\} \).
\( \mu \) is not null-continuous since there is \( (A_n)_{n\in\mathbb{N}}, A_n = \{0,1,2,\ldots,n\} \), for every \( n \in \mathbb{N} \), such that \( A_n \subseteq A_{n+1} \) and \( \mu(A_n) = \{0\} \), for all \( n \in \mathbb{N} \), but \( \mu(\bigcup_{n=0}^{\infty} A_n) = \mu(\mathbb{N}) \neq \{0\} \).

3. **Atoms and Pseudo-Atoms**

We now give some properties regarding atoms and pseudo-atoms of set multifunctions taking values in the family of nonvoid subsets of a commutative semigroup with unity.

In the sequel, \( (X, +, 0) \) is a commutative semigroup with unity 0 and “\( \leq \)” is an order relation on \( \mathcal{P}_0(X) \).

**Definitions 3.1.** Let \( \mu : \mathcal{C} \to (\mathcal{P}_0(X), \leq) \) be a set multifunction.

(i) A set \( A \in \mathcal{C} \) is said to be an atom of \( \mu \) if \( \mu(A) > \mu(\emptyset) \) and for every \( B \in \mathcal{C} \), with \( B \subseteq A \), we have \( \mu(B) = \mu(\emptyset) \) or \( \mu(A \setminus B) = \mu(\emptyset) \).
(ii) A set \( A \in C \) is called a pseudo-atom of \( \mu \) if \( \mu(A) > \mu(\emptyset) \) and for every \( B \in C \), with \( B \subseteq A \), we have \( \mu(B) = \mu(\emptyset) \) or \( \mu(B) = \mu(A) \).

(iii) \( \mu \) is called non-atomic (non-pseudo-atomic respectively) if it has no atoms (pseudo-atoms respectively).

**Remark 3.2.** If \( \mu \) is fuzzy, then \( \mu \) is non-atomic (non-pseudo-atomic respectively) if and only if for every \( A \in C \) with \( \mu(A) > \{0\} \), there exists \( B \in C \), so that \( B \subseteq A \), \( \mu(B) > \{0\} \) and \( \mu(A \setminus B) > \{0\} \) (\( \mu(B) < \mu(A) \) respectively).

**Proposition 3.3.** Suppose \( \mu : C \to (P_0(X), \leq) \) is fuzzy and null-additive (or a multimeasure). Then every atom of \( \mu \) is a pseudo-atom of \( \mu \).

**Proof.** (i) Suppose \( \mu \) is fuzzy and null-additive and let \( A \in C \) be an atom of \( \mu \). Let \( B \in C \), \( B \subseteq A \) so that \( \mu(B) \neq \{0\} \). Since \( A \) is an atom of \( \mu \), it results \( \mu(A \setminus B) = \{0\} \). Since \( \mu \) is null-additive, it follows \( \mu(A) = \mu(B) \cup (A \setminus B)) = \mu(B) \) which proves that \( A \) is a pseudo-atom of \( \mu \).

(ii) Suppose \( \mu \) is a multimeasure and let \( A \in C \) be an atom of \( \mu \). Let \( B \in C \), \( B \subseteq A \) so that \( \mu(B) \neq \{0\} \). Since \( A \) is an atom of \( \mu \), it results in \( \mu(A \setminus B) = \{0\} \). Since \( \mu \) is a multimeasure, we have:

\[
\mu(A) = \mu(B \cup (A \setminus B)) = \mu(B) + \mu(A \setminus B) = \mu(B) + \{0\} = \mu(B).
\]

So \( A \) is pseudo-atom of \( \mu \).

**Remark 3.4.** The converse of Proposition 3.3 is not valid (see Example 3.7). As we shall see in the sequel, if \( X \) is a normed space, then this converse is true. To prove this we need the following lemma.

**Lemma 3.5.** Let \((X, \|\cdot\|)\) be a normed space and \( A, B \in P_0(X) \) so that \( A + B = B \) and \( B \) is bounded. Then \( A = \{0\} \).

**Proof.** Since \( B \) is bounded, there is \( M > 0 \) so that \( \|y\| \leq M \), for every \( y \in B \). Let \( a \in A, b \in B \). It results in \( a + b \in A + B = B \). Then \( 2a + b = a + (a + b) \in A + B = B \).

By induction we obtain that \( na + b \in B \), for every \( n \in \mathbb{N} \). It follows \( \|na + b\| \leq M \). Consequently, we have \( \|a\| \leq \frac{M}{n} \), for every \( n \in \mathbb{N}^* \), which proves that \( a = 0 \). So \( A = \{0\} \).

**Proposition 3.6.** If \( X \) is a normed space and \( \mu : C \to (P_0(X), \leq) \) is a multimeasure, then \( A \in C \) is a pseudo-atom of \( \mu \) if and only if \( A \) is an atom of \( \mu \).

**Proof.** I. Suppose \( A \) is an atom of \( \mu \) and let \( B \in C \), \( B \subseteq A \) such that \( \mu(B) \neq \{0\} \). Since \( A \) is an atom of \( \mu \), it results in \( \mu(A \setminus B) = \{0\} \). Since \( \mu \) is a multimeasure, we have:

\[
\mu(A) = \mu(A \setminus B) \cup B = \mu(A \setminus B) + \mu(B) = \{0\} + \mu(B) = \mu(B),
\]

which proves that \( A \) is a pseudo-atom of \( \mu \).

II. Suppose \( A \) is a pseudo-atom of \( \mu \) and let \( B \in C \), \( B \subseteq A \) such that \( \mu(B) \neq \{0\} \). Since \( A \) is a pseudo-atom of \( \mu \), we have \( \mu(B) = \mu(A) \). Since \( \mu \) is a multimeasure, it results in:

\[
\mu(A) = \mu((A \setminus B) \cup B) = \mu(A \setminus B) + \mu(B) = \mu(A \setminus B) + \mu(B) = \mu(A \setminus B) + \mu(A) = \mu(A).
\]
According to Lemma 3.5, it follows \( \mu(A \setminus B) = \{0\} \). Consequently, \( A \) is an atom of \( \mu \). \( \square \)

**Example 3.7.** I. Let \( T = \{a, b, c\} \), \( C = \mathcal{P}(T) \), \( \mu : C \rightarrow (\mathcal{P}_0(\mathbb{R}), \subseteq) \) be defined for every \( A \in C \) by
\[
\mu(A) = \begin{cases} 
[0, 1], & \text{if } A \neq \emptyset, \\
\{0\}, & \text{if } A = \emptyset.
\end{cases}
\]
Then \( \mu \) is null-additive, \( A = \{a, b\} \) is a pseudo-atom of \( \mu \), but not an atom of \( \mu \).

II. Let \( T = \{a, b\} \), \( C = \mathcal{P}(T) \), \( \mu : C \rightarrow (\mathcal{P}_0(\mathbb{R}), \subseteq) \) be defined for every \( A \in C \) by
\[
\mu(A) = \begin{cases} 
[0, 2], & \text{if } A = T \\
[0, 1], & \text{if } A = \{b\} \\
\{0\}, & \text{if } A = \emptyset \text{ or } A = \{a\}.
\end{cases}
\]
Then \( \mu \) is not null-additive, \( T \) is an atom of \( \mu \), but not a pseudo-atom of \( \mu \).

III. Let \( T = 2\mathbb{N} = \{0, 2, 4, \ldots\} \), \( C = \mathcal{P}(T) \) and \( \mu : C \rightarrow (\mathcal{P}_0(\mathbb{X}), \subseteq) \) be defined for every \( A \in C \) by
\[
\mu(A) = \begin{cases} 
\{0\}, & \text{if } A = \emptyset \\
\frac{1}{2}A \cup \{0\}, & \text{if } A \neq \emptyset
\end{cases}
\]
where \( \frac{1}{2}A = \{\frac{x}{2} \mid x \in A\} \). Then \( \mu \) is a multisubmeasure.

If \( A \in C \) has \( \text{card}A = 1 \) and \( A \neq \{0\} \) or \( A \in C, A = \{0, 2n\}, n \in \mathbb{N}^* \), then \( A \) is an atom of \( \mu \) (and a pseudo-atom of \( \mu \) too).

If \( A \in C \) has \( \text{card}A \geq 2 \) and there exist \( a, b \in A \) such that \( a \neq b \) and \( ab \neq 0 \), then \( A \) is not a pseudo-atom of \( \mu \) (and not an atom of \( \mu \)).

IV. Let \( T = \mathbb{N}, C = \mathcal{P}(\mathbb{N}) \) and \( \mu : C \rightarrow (\mathcal{P}_0(\mathbb{R}), \subseteq) \) be defined for every \( A \in C \) by
\[
\mu(A) = \begin{cases} 
\{0\}, & \text{if } A \text{ is finite} \\
\{0\} \cup \{n_A, +\infty\}, & \text{if } A \text{ is infinite and } n_A = \min A
\end{cases}
\]
Then \( \mu \) is monotone and non-pseudo-atomic.

From definitions we obtain the following properties of pseudo-atoms.

**Proposition 3.8.** Let \( \mu : C \rightarrow (\mathcal{P}_0(\mathbb{X}), \subseteq) \) be a fuzzy set multifunction.

I. If \( A \in C \) is a pseudo-atom of \( \mu \) and \( B \in C, B \subseteq A \) is so that \( \mu(B) > \{0\} \), then \( B \) is a pseudo-atom of \( \mu \) and \( \mu(B) = \mu(A) \).

II. If \( A, B \in C \) are pseudo-atoms of \( \mu \) and \( \mu(A) \neq \mu(B) \), then \( \mu(A \cap B) = \{0\} \).

III. Moreover, suppose \( \mu \) is null-null-additive and let \( A, B \in C \) be pseudo-atoms of \( \mu \). Then the following statements hold:

(i) If \( \mu(A \setminus B) = \{0\} \), then \( A \setminus B \) and \( B \setminus A \) are pseudo-atoms of \( \mu \) and \( \mu(A \setminus B) = \mu(A), \mu(B \setminus A) = \mu(B) \).

(ii) If \( \mu(A \setminus B) > \{0\} \) and \( \mu(A \setminus B) = \mu(B \setminus A) = \{0\} \), then \( A \setminus B \) is a pseudo-atom of \( \mu \) and \( \mu(A \setminus B) = \{0\} \) (where \( A \setminus B = (A \setminus B) \cup (B \setminus A) \)).
Proof. I. Since $\mu(B) > \{0\}$ and $A$ is a pseudo-atom of $\mu$, it results in $\mu(B) = \mu(A)$. Let $C \in \mathcal{C}$, $C \subseteq B$ and suppose $\mu(C) > \{0\}$. Since $C \subseteq A$ and $A$ is a pseudo-atom, it follows $\mu(C) = \mu(A)$. This shows that $\mu(C) = \mu(B)$. So $B$ is a pseudo-atom of $\mu$.

II. Let $a, B \in \mathcal{C}$ be pseudo-atoms of $\mu$ such that $\mu(A) \neq \mu(B)$. Suppose, by contrary, that $\mu(A \cap B) > \{0\}$. Since $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $A, B$ are pseudo-atoms of $\mu$, according to Proposition 3.8-I, we have $\mu(A \cap B) = \mu(A)$ and $\mu(A \cap B) = \mu(B)$. It follows $\mu(A) = \mu(B)$, false!

III. (i) Suppose $\mu(A \setminus B) = \{0\}$. Since $\mu(A \cap B) = \{0\}$ and $\mu$ is null-null-additive, it results $\mu((A \cup B) \cap (A \setminus B)) = \mu(A) = \{0\}$, that is false because $A$ is a pseudo-atom. So $\mu(A \setminus B) > \{0\}$. By Proposition 3.8-I, it follows that $A \setminus B$ is a pseudo-atom and $\mu(A \setminus B) = \mu(A)$. In the same way it follows that $B \setminus A$ is a pseudo-atom and $\mu(B \setminus A) = \mu(B)$.

(ii) Since $A \cap B \subseteq A$ and $\mu(A \cap B) > \{0\}$, by Proposition 3.8-I, it results that $A \cap B$ is a pseudo-atom of $\mu$. Since $\mu$ is null-null-additive, we have $\mu(A \Delta B) = \{0\}$. □

Proposition 3.9. Suppose $\mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \subseteq)$ is a null-additive fuzzy set multifunction and $A \in \mathcal{C}$ is an atom (pseudo-atom respectively). If $E \in \mathcal{C}$ is so that $\mu(E) = \{0\}$, then $B = A \cup E$ is also an atom (pseudo-atom respectively).

Proof. Suppose $A$ is a pseudo-atom of $\mu$ and let $C \in \mathcal{C}$, $C \subseteq B = A \cup E$. We can set $C = (C \cap A) \cup (C \cap E)$. By the monotonicity of $\mu$, we have $\mu(C \cap E) = \{0\}$. Since $\mu$ is null-additive, the following relation holds:

$$\mu(C \cap A) = \mu(C).$$

But $A$ is a pseudo-atom of $\mu$ and $C \cap A \subseteq A$. Then we have $\mu(C \cap A) = \{0\}$ or $\mu(C \cap A) = \mu(A)$.

(i) If $\mu(C \cap A) = \{0\}$, by (1) it results $\mu(C) = \{0\}$.

(ii) If $\mu(C \cap A) = \mu(A)$, by (1) it results $\mu(A) = \mu(C)$. Since $\mu(E) = \{0\}$ and $\mu$ is null-additive, we have $\mu(B) = \mu(A)$. So $\mu(C) = \mu(B)$.

This shows that $B$ is a pseudo-atom of $\mu$.

The case when $A$ is an atom of $\mu$ analogously follows. □

Proposition 3.10. Suppose $\mu : \mathcal{C} \rightarrow (\mathcal{P}_0(X), \subseteq, | \cdot |)$ is a null-additive fuzzy set multifunction and let $A \in \mathcal{C}$ be an atom of $\mu$. If $\{B_i\}_{i=1}^n \subset \mathcal{C}$ is a partition of $A$, then there exists a unique $i_0 \in \{1, \ldots, n\}$ such that $\mu(B_{i_0}) = \mu(A)$ and $\mu(B_i) = \{0\}$, for $i \in \{1, \ldots, n\}$, $i \neq i_0$.

Proof. We have two cases:

I. $\mu(B_i) = \{0\}$, for every $i \in \{1, \ldots, n\}$. From the null-additivity of $\mu$, it results $\mu(A) = \{0\}$, which is false.

II. There exists $i_0 \in \{1, \ldots, n\}$ so that $\mu(B_{i_0}) > \{0\}$. Suppose without loss in generality that $\mu(B_1) > \{0\}$ and $\mu(B_2) > \{0\}$. Since $A$ is an atom of $\mu$, it follows $\mu(A \setminus B_1) = \{0\}$. Since $B_2 \subseteq A \setminus B_1$ and $\mu$ is fuzzy, $\mu(B_2) = \{0\}$, which is false.
According to the Remark 3.12-I, we only have to prove:

**Proposition 3.14.**

If \( A \) is an atom of \( \mu \), then \( \mu(A) = \{0\} \). From the null-additivity of \( \mu \) we obtain \( \mu(A) = \mu(B_i) \). Since \( B_i \subseteq A \setminus B_i \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) and \( \mu \) is fuzzy, it follows that \( \mu(B_i) = \{0\} \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) and the proof is finished. \( \square \)

**Definition 3.11.** For a set multifunction \( \mu : C \to (P_0(X), \leq, |\cdot|) \), the following set function (called the variation of \( \mu \)) is introduced:

\[
\overline{\mu} : P(T) \to [0, +\infty], \quad \overline{\mu}(E) = \sup \left( \sum_{i=1}^{n} |\mu(A_i)| ; A_i \subseteq E, A_i \in C, \cap A_i = \emptyset, i \neq j, i, j \in \{1, \ldots, n\}, n \in \mathbb{N}^\ast \right), \quad \forall E \in P(T).
\]

**Remark 3.12.** Let \( \mu : C \to P_0(X) \) be a set multifunction. Then the following statements hold:

I. \( |\mu(A)| \leq \overline{\mu}(A) \), for every \( A \in C \).

The inequality may be strict (see Example 3.13-II). It becomes an equality in supplementary hypothesis (see Proposition 3.14).

II. \( \overline{\mu}(A) = 0 \Rightarrow |\mu(A)| = 0 \), for every \( A \in C \).

III. Moreover, if \( \mu : C \to (P_0(X), \leq, |\cdot|) \) is fuzzy, then we also have:

\[
|\mu(A)| = 0 \Rightarrow \overline{\mu}(A) = 0, \quad \forall A \in C.
\]

Indeed, let \( \{B_i\}_{i=1}^{n} \subset C \) be a partition of \( A \). Since \( \mu \) is fuzzy, \( \sum_{i=1}^{n} |\mu(B_i)| = 0 \) and so, \( \overline{\mu}(A) = 0 \).

If \( \mu \) is not fuzzy, then (2) may be false as we can see in Example 3.13-I.

**Example 3.13.** I. Let \( T = \{a, b\} \), \( C = P(T) \) and \( \mu : C \to (P_0(\mathbb{R}), \leq, |\cdot|) \) be defined by:

\[
\mu(A) = \begin{cases} 
\{0\}, & A = \emptyset \text{ or } A = T \\
[0, 1] \cup \{2, 3\}, & A = \{a\} \text{ or } A = \{b\}.
\end{cases}
\]

We have \( |\mu(T)| = 0 \) and \( \overline{\mu}(T) = 6 \).

II. Let \( T = \{1, 2, 3\} \), \( C = P(T) \), \( X = \{f|f : [0, +\infty) \to [0, +\infty]\} \) and \( |E| = \sup_{f \in E} \|f\|_u \), for every \( E \in P_0(X) \), where \( \|f\|_u = \sup_{x \in E} |f(x)| \). Let \( \mu : C \to (P_0(X), |\cdot|) \) be defined by

\[
\mu(A) = \{X_i|_{[0, \infty]} | n \in A\}, \quad \forall A \in C,
\]

where \( \chi_{[0, \infty]} \) is the characteristic function of \([0, \infty]\).

In this setting we have \( |\mu(T)| = 3 < 6 = \overline{\mu}(T) \).

**Proposition 3.14.** If \( \mu : C \to (P_0(X), \leq, |\cdot|) \) is a fuzzy set multifunction and \( A \in C \) is an atom of \( \mu \), then \( \overline{\mu}(A) = |\mu(A)| \).

**Proof.** According to the Remark 3.12-I, we only have to prove:

\[
\overline{\mu}(A) \leq |\mu(A)|.
\]
Let \( \{B_i\}_{i=1}^n \subset C \) be an arbitrary partition of \( A \), where \( n \in \mathbb{N}^* \). We have two cases:

I. \( \mu(B_i) = \{0\} \), for every \( i \in \{1, \ldots, n\} \). Then \( \sum_{i=1}^n |\mu(B_i)| = 0 \leq |\mu(A)| \).

II. There exists \( i_0 \in \{1, \ldots, n\} \) so that \( \mu(B_{i_0}) > \{0\} \).

Suppose without loss of generality that \( \mu(B_1) > \{0\} \) and \( \mu(B_2) > \{0\} \). Since \( A \) is an atom of \( \mu \), it follows that \( \mu(A \setminus B_1) = \{0\} \). Since \( B_2 \subseteq A \setminus B_1 \) and \( \mu \) is fuzzy, it results in \( \mu(B_2) > \{0\} \). So that \( \mu(B_{i_0}) > \{0\} \). Since \( B_i \subseteq A \setminus B_{i_0} \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) so that \( \mu(B_i) > \{0\} \). Since \( B_i \subseteq A \setminus B_{i_0} \), for every \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) so that \( \sum_{i=1}^n |\mu(B_i)| \leq |\mu(A)| \).

Since \( \{B_i\}_{i=1}^n \) is an arbitrary partition of \( A \), it results (3). \( \square \)

**Proposition 3.15.** Suppose \( \mu : C \to (\mathcal{P}_0(X), \subseteq, |·|) \) is a sn-exhaustive fuzzy set multifunction. Then for every \( E \in \mathcal{P}(T) \) and every \( \varepsilon > 0 \), there is a \( A \subset C \) so that \( A \subseteq E \) and \( |\mu(B \setminus A)| < \varepsilon \), for all \( B \subset C, A \subset B \subset E \).

**Proof.** Suppose on the contrary that there exist \( E_0 \in \mathcal{P}(T) \) and \( \varepsilon > 0 \) so that for all \( A \subset C \), with \( A \subset E_0 \), there is \( B_0 \in C \) so that \( A \subset B_0 \subset E_0 \) and \( |\mu(B_0 \setminus A)| \geq \varepsilon \).

We construct by recurrence a sequence of mutual disjoint sets \( (L_n) \subset C \) such that \( L_n \subset E_0 \) for every \( n \in \mathbb{N} \) and \( |\mu(L_n)| \geq \varepsilon \).

Suppose we obtained \( L_1, L_2, \ldots, L_n \) and let \( K = \bigcup_{i=1}^n L_i \). Obviously, \( K \subset C \) and \( K \subset E_0 \). Then there is \( B \subset C \) so that \( K \subset B \subset E_0 \) and \( |\mu(B \setminus K)| \geq \varepsilon \). If we set \( L_{n+1} = B \setminus K \), then we have \( L_{n+1} \subset C, L_{n+1} \subset E_0, |\mu(L_{n+1})| \geq \varepsilon \) and \( L_{n+1} \cap L_i = \emptyset \), for every \( i \in \{1, \ldots, n\} \). Since \( \mu \) is sn-exhaustive, we have \( \lim_{n \to \infty} |\mu(L_n)| = 0 \), that is a contradiction. \( \square \)

**Definition 3.16.** Suppose \( |·| \) is a monotone set-norm on \( (\mathcal{P}_0(X), \subseteq) \) and let \( \mathcal{R} \) be a non-empty subset of \( \mathcal{P}_0(X) \).

(i) A net \( (Y_i) \subset \mathcal{P}_0(X) \) is called sn-Cauchy if the net \( (|Y_i|) \) is Cauchy (i.e., for every \( \varepsilon > 0 \), there is \( i_\varepsilon \) such that \( |Y_i| - |Y_j| < \varepsilon \), for every \( i, j \geq i_\varepsilon \)).

(ii) A net \( (Y_i) \subset \mathcal{R} \) is called sn-convergent in \( \mathcal{R} \) if there is a unique \( Y_0 \in \mathcal{R} \) so that \( \lim_{i \to \infty} |Y_i| = |Y_0| \). We denote this by \( \lim_{i \to \infty} Y_i = Y_0 \).

(iii) \( \mathcal{R} \) is called sn-complete if every sn-Cauchy net of \( \mathcal{R} \) is convergent in \( \mathcal{R} \).

**Example 3.17.** Let \( \mathcal{R} = \{[0, x] | x \in [0, +\infty)\} \). Then \( \mathcal{R} \) is a sn-complete subspace of \( (\mathcal{P}_0([0, +\infty)), \subseteq, |·|) \).

**Theorem 3.18.** Suppose \( |·| \) is a monotone set-norm on \( (\mathcal{P}_0(X), \subseteq) \), let \( \mathcal{R} \) be a sn-complete subset of \( (\mathcal{P}_0(X), \subseteq, |·|) \) and let \( \mu : C \to \mathcal{R} \) be an sn-exhaustive subadditive fuzzy set multifunction. Then the following statements hold:

(i) For every \( E \in \mathcal{P}(T) \), there exists \( \lim_{A \subset C} \mu(A) = \mu^*(E) \in \mathcal{R} \), where the net \( (\mu(A))_{A \subset \mathcal{R}} \) is directed by the usual inclusion “\( \subseteq \)” of sets and the limit is in the sense of Definition 3.16-(ii). We obtain a set multifunction \( \mu^* : \mathcal{P}(T) \to \mathcal{R} \).
(ii) \( |\mu^*(A)| = |\mu(A)|, \forall A \in \mathcal{C} \).

(iii) \( \forall E_1, E_2 \in \mathcal{P}(T), E_1 \subseteq E_2 \Rightarrow |\mu^*(E_1)| \leq |\mu^*(E_2)|. \)

(iv) \( \mu^* \) is sn-exhaustive.

(v) If \( \mu \) is non-atomic, then \( \mu^* \) is non-atomic.

**Proof.** (i) According to Proposition 3.15, for every \( E \in \mathcal{P}(T) \) and \( \varepsilon > 0 \), there exists \( A_0 \in \mathcal{C} \) so that \( A_0 \subseteq E \) and for every \( A \in \mathcal{C} \) with \( A_0 \subseteq A \subseteq E \), we have

\[
0 \leq |\mu(A)| - |\mu(A_0)| \leq |\mu(A \setminus A_0)| < \varepsilon.
\]

So \( (\mu(A))_{A \subseteq E} \) is sn-Cauchy in \( \mathcal{R} \) and since \( \mathcal{R} \) is sn-complete, the net \( (\mu(A))_{A \subseteq E} \) is sn-convergent in \( \mathcal{R} \).

(ii) Let \( A \in \mathcal{C} \). For every \( \varepsilon > 0 \) there is \( B_0 \in \mathcal{C} \) so that \( B_0 \subseteq A \) and for every \( B \in \mathcal{C} \) with \( B_0 \subseteq B \subseteq A \) we have \( |\mu(B)| - |\mu^*(A)| < \varepsilon \). Particularly, \( |\mu(A)| = |\mu^*(A)| \) for every \( \varepsilon > 0 \). Hence \( |\mu^*(A)| = |\mu(A)| \).

Let \( A = \emptyset \) it results in \( |\mu^*(\emptyset)| = |\mu(\emptyset)| = 0 \) which implies that \( \mu^*(\emptyset) = \{0\} \).

(iii) Let \( E_1, E_2 \in \mathcal{P}(T) \) be so that \( E_1 \subseteq E_2 \). Consider an arbitrary \( \varepsilon > 0 \). Then there exists \( A_1 \in \mathcal{C} \) so that \( A_1 \subseteq E_1 \) and for every \( A \in \mathcal{C} \), with \( A_1 \subseteq A \subseteq E_1 \), we have \( |\mu^*(A_1)| - |\mu^*(E_1)| < \varepsilon. \) Particularly we have

\[
|\mu^*(E_1)| - |\mu(A_1)| < \frac{\varepsilon}{2}.
\]

Analogously there exists \( A_2 \subseteq \mathcal{C} \) so that \( A_2 \subseteq E_2 \) and for every \( A \in \mathcal{C} \), with \( A_2 \subseteq A \subseteq E_2 \) we have

\[
|\mu^*(E_2)| - |\mu(A)| < \frac{\varepsilon}{2}.
\]

Let \( A_0 = A_1 \cup A_2 \in \mathcal{C} \). Then \( A_2 \subseteq A_0 \subseteq E_2 \) and we have

\[
|\mu^*(E_2)| - |\mu(A_0)| < \frac{\varepsilon}{2},
\]

Now, from (4) and (5) we obtain:

\[
|\mu^*(E_1)| < |\mu(A_1)| + \frac{\varepsilon}{2} \leq |\mu(A_0)| + \frac{\varepsilon}{2} < |\mu^*(E_2)| + \varepsilon
\]

for every \( \varepsilon > 0 \). Hence \( |\mu^*(E)| \leq |\mu^*(E_2)| \).

(iv) Let \( (E_n) \subseteq \mathcal{P}(T) \) be so that \( E_n \cap E_m = \emptyset \), for every \( m \neq n \). For every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), there exists \( A_n \in \mathcal{C} \) such that \( A_n \subseteq E_n \) and \( |\mu(A_n)| - |\mu^*(E_n)| < \frac{\varepsilon}{2} \).

Since \( A_n \cap A_m = \emptyset \), for every \( n \neq m \) and \( \mu \) is sn-exhaustive, it results that there is \( n_0 \in \mathbb{N} \) so that \( |\mu(A_n)| < \frac{\varepsilon}{2} \), for every \( n \geq n_0 \). So we have:

\[
|\mu^*(E_n)| \leq |\mu^*(E_n)| - |\mu(A_n)| + |\mu(A_n)| < \varepsilon, \ \forall n \geq n_0,
\]

which proves that \( \mu^* \) is sn-exhaustive.

(v) On the contrary suppose there is \( E \in \mathcal{P}(T) \) an atom of \( \mu^* \). Then \( \mu^*(E) > \{0\} \) and so \( |\mu^*(E)| > 0 \). It results that there is \( A \in \mathcal{C} \) so that \( A \subseteq E \) and \( |\mu(A)| > 0 \). Since \( \mu(A) > \{0\} \) and \( \mu \) is non-atomic, there exists \( B \in \mathcal{C} \) such that \( B \subseteq A \), \( \mu(B) > \{0\} \) and \( \mu(A \setminus B) > \{0\} \). Since \( B \subseteq E \) and \( E \) is an atom of \( \mu^* \), it follows that \( \mu^*(B) = \{0\} \) or \( \mu^*(E \setminus B) = \{0\} \).
I. If $\mu^*(B) = \{0\}$, then by (ii) we have $|\mu(B)| = |\mu^*(B)| = 0$ and so $\mu(B) = \{0\}$ which is false.

II. If $\mu^*(E \setminus B) = \{0\}$, then by (ii) and (iii) we have:

$$0 < |\mu(A \setminus B)| = |\mu^*(A \setminus B)| \leq |\mu^*(E \setminus B)| = 0$$

that is a contradiction. Consequently, $\mu^*$ is non-atomic. □

4. Conclusion

In this paper we presented properties of set-norm exhaustive set multifunctions and also some properties of atoms and pseudo-atoms of set-multifunctions taking values in the family of non-empty subsets of a commutative semigroup with unity. It would be interesting to see what non-atomicity becomes in the absence of fuzzyness of $\mu$.

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