CHARACTERIZATIONS AND PROPERTIES OF BOUNDED L-FUZZY SETS

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Abstract. In 1997, Fang proposed the concept of boundedness of L-fuzzy sets in L-topological vector spaces. Since then, this concept has been widely accepted and adopted in the literature. In this paper, several characterizations of bounded L-fuzzy sets in L-topological vector spaces are obtained and some properties of bounded L-fuzzy sets are investigated.

1. Introduction

In 1997, Fang and Yan [3] proposed the notion of L-fuzzy topological vector space. According to the standardized terminology in [4], it should more accurately be called L-topological vector space (L-tvs). The concept of L-tvs unifies and generalizes that of classical topological vector space (i.e., $\{0,1\}$-tvs) and that of fuzzy topological vector space (i.e., $[0,1]$-tvs) due to Katsaras [5].

It is well known that boundedness of sets is a basic and important concept in the theory of classical topological vector spaces. So, it is natural to give a suitable definition of boundedness for L-fuzzy sets in the framework of L-tvs. In 1997, Fang [2] introduced the concept of boundedness of L-fuzzy sets in L-tvs. Since then, this concept has been widely accepted and adopted in the literature, such as in [9, 10, 11].

As we know, in the theory of classical topological vector spaces, bounded sets have some equivalent characterizations and nice properties. Naturally, the purpose of this paper is to investigate characterizations and properties of bounded L-fuzzy sets in L-tvs.

2. Preliminaries

Throughout this paper, L denotes a Hutton algebra [4], i.e., a complete and completely distributive lattice equipped with an order-reversing involution $' : L \to L$. Note that 0 and 1 are its lower and upper bounds, respectively. $M(L)$ denotes the set of all non-zero union-irreducible elements in L. The elements of $M(L)$ are also called molecules [6] in L. L^X denotes the family of all L-fuzzy sets on X. $M(L^X) = \{x_\lambda \mid x \in X,\ \lambda \in M(L)\}$, i.e., $M(L^X)$ is the set of all molecules in the lattice L^X. For $U \subseteq X$, $\chi_U : X \to L$ denotes the characteristic function of the set U. An L-fuzzy set that takes the constant value $\lambda \in L$ on X is denoted by $\underline{\lambda}$. An L-fuzzy set on X is called an L-fuzzy point if it takes the value 0 for all $y \in X$

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except a single $x \in X$. If its value at x is $\lambda \in L \setminus \{0\}$, we denote this L-fuzzy point by x_λ. An L-topology δ on X is called *stratified* if it contains all constant L-fuzzy sets on X. We assume that the L-topologies referred to in this paper are all stratified and that the lattice L is regular. That is, the intersection of each pair of non-zero elements in L is not zero, which is equivalent to $1 \in M(L)$. For other notations not mentioned here, please refer to [3, 6, 7].

Definition 2.1. [6, 7] Let (X, δ) be an L-topological space and $x_\lambda \in M(L^X)$. $P \in L^X$ is called a closed R-neighborhood of x_λ, if $P \in \delta'$ and $x_\lambda \nleqslant P$. The set of all closed R-neighborhoods of x_λ is denoted by $\eta^-(x_\lambda)$.

$A \in L^X$ is called an R-neighborhood of x_λ if there exists $P \in \eta^-(x_\lambda)$ such that $A \leqslant P$. The set of all R-neighborhoods of x_λ is denoted by $\eta(x_\lambda)$.

$\mathscr{U} \subseteq \eta(x_\lambda)$ is said to be an R-neighborhood base of x_λ if for each $P \in \eta(x_\lambda)$ there exists $Q \in \mathscr{U}$ such that $P \leqslant Q$.

Definition 2.2. [6, 7] An L-topological space (X, δ) is said to be Hausdorff, if for each $x_\lambda, y_\mu \in M(L^X)$ with $x \neq y$, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P \vee Q = \underline{1}$.

Definition 2.3. [6, 7] Let $(X, \mathscr{T})$ be a topological space. A mapping $A : X \to L$ is called lower semi-continuous, if $\iota_{\lambda'}(A) = \{x \in X \mid A(x) \nleqslant \lambda'\} \in \mathscr{T}$ for each $\lambda \in M(L)$.

The set of all lower semi-continuous mappings from X to L, denoted by $\Omega_L(\mathscr{T})$, is a stratified L-topology, called induced L-topology by $\mathscr{T}$.

Lemma 2.4. [7] *Let $(X, \Omega_L(\mathscr{T}))$ be an L-topological space induced by the topological space $(X, \mathscr{T})$. Then $\beta = \{\chi_G \wedge \underline{r} \mid G \in \mathscr{T},\ r \in L\}$ is a base of $\Omega_L(\mathscr{T})$.*

By Lemma 2.4, we can easily obtain the following conclusion.

Proposition 2.5. *Let $(X, \Omega_L(\mathscr{T}))$ be an L-topological space induced by the topological space $(X, \mathscr{T})$.*

(1) If $\mathscr{U}$ is a neighborhood base of x in $(X, \mathscr{T})$, then for each $\lambda \in M(L)$,
$\mathscr{U}_\lambda = \{\chi_{U'} \vee \underline{\mu} \mid U \in \mathscr{U},\ \lambda \nleqslant \mu,\ \mu \in L\}$
is an R-neighborhood base of x_λ in $(X, \Omega_L(\mathscr{T}))$, where $U' = \{x \in X \mid x \notin U\}$;
(2) If $\mathscr{U}_\lambda$ is an R-neighborhood base of x_λ in $(X, \Omega_L(\mathscr{T}))$, then
$\mathscr{U} = \{\iota_{\lambda'}(U') \mid U \in \mathscr{U}_\lambda\}$
is a neighborhood base of x in $(X, \mathscr{T})$, where $\iota_{\lambda'}(U') = \{x \in X \mid U'(x) \nleqslant \lambda'\}$.

In the sequel, without special indication, X always denotes a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and θ denotes the zero element of X. $\mathbb{N}$ denotes the set of all positive integers.

Definition 2.6. [1, 3] Applying the extension principle of Zadeh to addition and scalar multiplication, we define the addition and scalar multiplication of L-fuzzy sets on X as follows. For $A,\ B \in L^X$ and $k \in \mathbb{K}$,

$$(A + B)(x) = \bigvee_{s+t=x} (A(s) \wedge B(t)),$$

$$(kA)(x) = A(x/k), \text{ whenever } k \neq 0,$$

$$(0A)(x) = \begin{cases} \bigvee\limits_{y\in X} A(y), & x = \theta, \\ 0, & x \neq \theta. \end{cases}$$

In particular, for L-fuzzy points x_λ, y_μ and $k \in \mathbb{K}$, note that

$$x_\lambda + y_\mu = (x+y)_{\lambda\wedge\mu} \text{ and } kx_\lambda = (kx)_\lambda.$$

Definition 2.7. [3] $A \in L^X$ is said to be balanced if $sA \leqslant A$ for each $s \in \mathbb{K}$ with $|s| \leqslant 1$. A is said to be co-balanced if A' is balanced.

The balanced hull $\mathrm{BH}(A)$ of an L-fuzzy set A on X is the intersection of all balanced L-fuzzy sets on X containing A.

It is clear that $\mathrm{BH}(A) = \bigvee\{kA \mid k \in \mathbb{K}, |k| \leqslant 1\}$ and A is balanced if and only if $\mathrm{BH}(A) = A$.

Definition 2.8. [3] Let δ be an L-topology on X. The pair (X, δ) is called an L-tvs if the two mappings (i.e., the addition and the scalar multiplication on X) are both continuous:

(1) $f : X \times X \to X,\ (x, y) \mapsto x + y$ and

(2) $g : \mathbb{K} \times X \to X,\ (k, x) \mapsto kx$

where $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product L-topologies $\delta \times \delta$ and $J_\mathbb{K} \times \delta$, respectively, and $J_\mathbb{K}$ denotes the usual topology on $\mathbb{K}$.

By Lemma 2.4 and Proposition 2.5, it is not difficult to prove the following result.

Proposition 2.9. *Let $(X, \Omega_L(\mathscr{T}))$ be an L-topological space induced by the topological space $(X, \mathscr{T})$. Then $(X, \Omega_L(\mathscr{T}))$ is an L-tvs if and only if $(X, \mathscr{T})$ is a tvs.*

Definition 2.10. [2] Let (X, δ) be an L-tvs. An L-fuzzy set B on X is said to be λ-bounded ($\lambda \in M(L)$) if for each $Q \in \eta(\theta_\lambda)$, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant tQ'$. B is said to be bounded if it is λ-bounded for each $\lambda \in M(L)$.

Lemma 2.11. [3] *Each L-tvs (X, δ) has an R-neighborhood base of θ_λ consisting of closed and co-balanced L-fuzzy sets on X for each $\lambda \in M(L)$.*

Lemma 2.12. [9] *Let (X, δ) be an L-tvs. Then the following statements are mutually equivalent:*

(1) (X, δ) is Hausdorff;

(2) For each $\lambda \in M(L)$, θ_λ is a closed L-fuzzy set;

(3) For each $\lambda \in M(L)$ and $x \in X$ with $x \neq \theta$, there exists $P \in \eta(\theta_\lambda)$ such that $P(x) = 1$.

Lemma 2.13. [8] *(1) Let $\{(X_i, \delta_i)\}_{i\in\Gamma}$ be a family of L-tvses, $X = \Pi_{i\in\Gamma} X_i$ and $\delta = \Pi_{i\in\Gamma}\delta_i$ be the product L-topology of $\{\delta_i\}_{i\in\Gamma}$. Then (X, δ) is also an L-tvs.*

(2) Let (X, δ) be an L-tvs, E a linear subspace of X and $X/E = \{\overline{x} \mid x \in X\}$ be the quotient space, where $\overline{x} = \{y \in X \mid y - x \in E\}$. Put $\delta/E = \{q(G) \mid G \in \delta\}$, where $q : X \to X/E$ $(x \to \overline{x})$ is the quotient map. Then $(X/E, \delta/E)$ is also an L-tvs.

3. Characterizations of Bounded L-fuzzy Sets

Theorem 3.1. *Let (X,δ) be an L-tvs, $B\in L^X$ and $\lambda\in M(L)$. Then the following statements are equivalent:*

(1) B is λ-bounded;

(2) For each $Q\in\eta(\theta_\lambda)$, there exists $\mu\in L$ with $\mu\nleqslant\lambda'$ such that $\alpha_n(x_n)_{\lambda_n}\nleqslant Q$ is eventually true (i.e., there exists $N\in\mathbb{N}$ such that $\alpha_n(x_n)_{\lambda_n}\nleqslant Q$ for all $n\geqslant N$) for all $\{\alpha_n\}_{n=1}^\infty\subseteq\mathbb{K}$ satisfying $|\alpha_n|>0$ $(n=1,2,\ldots)$ and $\alpha_n\to 0$ $(n\to\infty)$, and for all $\{(x_n)_{\lambda_n}\}_{n=1}^\infty\subseteq M(L^X)$ satisfying $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$ $(n=1,2,\ldots)$;

(3) For each $Q\in\eta(\theta_\lambda)$, there exists $\mu\in L$ with $\mu\nleqslant\lambda'$ such that $\frac{1}{n}(x_n)_{\lambda_n}\nleqslant Q$ is eventually true for all $\{(x_n)_{\lambda_n}\}_{n=1}^\infty\subseteq M(L^X)$ satisfying $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$ $(n=1,2,\ldots)$.

Proof. (1) $\Longrightarrow$ (2) For each $Q\in\eta(\theta_\lambda)$, by Lemma 2.11, there exists $P\in\eta(\theta_\lambda)$ such that $Q\leqslant P$ and P is co-balanced. Since B is λ-bounded, there exist $t>0$ and $\mu\in L$ with $\mu\nleqslant\lambda'$ such that $B\wedge\underline{\mu}\leqslant tP'$. From $\alpha_n\to 0$ we know that there exists $N\in\mathbb{N}$ such that $0<|\alpha_n|\leqslant\frac{1}{t}$ whenever $n\geqslant N$. Hence, $\alpha_n(B\wedge\underline{\mu})\leqslant\alpha_n tP'\leqslant P'\leqslant Q'$ whenever $n\geqslant N$, which implies $Q\leqslant\alpha_n(B'\vee\underline{\mu'})$. On the other hand, we have $\alpha_n(x_n)_{\lambda_n}\nleqslant\alpha_n(B'\vee\underline{\mu'})$ since $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$. Therefore, $\alpha_n(x_n)_{\lambda_n}\nleqslant Q$ whenever $n\geqslant N$.

(2) $\Longrightarrow$ (3) Trivial.

(3) $\Longrightarrow$ (1) Suppose that B is not λ-bounded. Then there exists $Q\in\eta(\theta_\lambda)$ such that $B\wedge\underline{\mu}\nleqslant nQ'$ for each $n\in\mathbb{N}$ and $\mu\in L$ with $\mu\nleqslant\lambda'$, which implies $nQ\nleqslant B'\vee\underline{\mu'}$ for each $n\in\mathbb{N}$ and $\mu\in L$ with $\mu\nleqslant\lambda'$. Hence, for each $\mu\in L$ with $\mu\nleqslant\lambda'$ there exists $\{(x_n)_{\lambda_n}\}_{n=1}^\infty\subseteq M(L^X)$ such that $(x_n)_{\lambda_n}\leqslant nQ$, but $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$ $(n=1,2,\ldots)$. Therefore, $\frac{1}{n}(x_n)_{\lambda_n}\leqslant Q$ for each $n\in\mathbb{N}$, which contradicts with (3). □

By Theorem 3.1, we can easily obtain the following corollary, which characterizes bounded L-fuzzy sets.

Corollary 3.2. *Let (X,δ) be an L-tvs and $B\in L^X$. Then the following statements are mutually equivalent:*

(1) B is bounded;

(2) For each $\lambda\in M(L)$ and $Q\in\eta(\theta_\lambda)$, there exists $\mu\in L$ with $\mu\nleqslant\lambda'$ such that $\alpha_n(x_n)_{\lambda_n}\nleqslant Q$ is eventually true for all $\{\alpha_n\}_{n=1}^\infty\subseteq\mathbb{K}$ satisfying $|\alpha_n|>0$ $(n=1,2,\ldots)$ and $\alpha_n\to 0$ $(n\to\infty)$, and for all $\{(x_n)_{\lambda_n}\}_{n=1}^\infty\subseteq M(L^X)$ satisfying $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$ $(n=1,2,\ldots)$;

(3) For each $\lambda\in M(L)$ and $Q\in\eta(\theta_\lambda)$, there exists $\mu\in L$ with $\mu\nleqslant\lambda'$ such that $\frac{1}{n}(x_n)_{\lambda_n}\nleqslant Q$ is eventually true for all $\{(x_n)_{\lambda_n}\}_{n=1}^\infty\subseteq M(L^X)$ satisfying $(x_n)_{\lambda_n}\nleqslant B'\vee\underline{\mu'}$ $(n=1,2,\ldots)$.

Theorem 3.3. *Let $(X,\Omega_L(\mathscr{T}))$ be an L-tvs induced by the tvs $(X,\mathscr{T})$ and $B\subseteq X$. Then χ_B is bounded in $(X,\Omega_L(\mathscr{T}))$ if and only if B is bounded in $(X,\mathscr{T})$.*

Proof. Necessity: Let χ_B be bounded in $(X,\Omega_L(\mathscr{T}))$ and $\lambda\in M(L)$. Assume that $\mathscr{U}_\lambda$ is an R-neighborhood base of θ_λ in $(X,\Omega_L(\mathscr{T}))$, then, by Proposition 2.5,

$\mathscr{U} = \{\iota_{\lambda'}(U') \mid U \in \mathscr{U}_\lambda\}$ is a neighborhood base of θ in $(X, \mathscr{T})$. Hence, for each neighborhood V of θ in $(X, \mathscr{T})$, there exists $U \in \mathscr{U}_\lambda$ such that $\iota_{\lambda'}(U') \subseteq V$. Since χ_B is λ-bounded in $(X, \Omega_L(\mathscr{T}))$, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $\chi_B \wedge \underline{\mu} \leqslant tU'$. Therefore, $B = \iota_{\lambda'}(\chi_B) \cap \iota_{\lambda'}(\underline{\mu}) = \iota_{\lambda'}(\chi_B \wedge \underline{\mu}) \subseteq \iota_{\lambda'}(tU') = t[\iota_{\lambda'}(U')] \subseteq tV$, which implies that B is bounded in $(X, \mathscr{T})$.

Sufficiency: Let B be bounded in $(X, \mathscr{T})$. Assume that $\mathscr{U}$ is a neighborhood base of θ in $(X, \mathscr{T})$, then, by Proposition 2.5, for each $\lambda \in M(L)$, $\mathscr{U}_\lambda = \{\chi_{U'} \vee \underline{\mu} \mid U \in \mathscr{U},\ \lambda \nleqslant \mu,\ \mu \in L\}$ is an R-neighborhood base of θ_λ in $(X, \Omega_L(\mathscr{T}))$. Hence, for each R-neighborhood Q of θ_λ, there exist $U \in \mathscr{U}$ and $\mu \in L$ with $\lambda \nleqslant \mu$ such that $Q \leqslant \chi_{U'} \vee \underline{\mu}$. Since B is bounded in $(X, \mathscr{T})$, there exists $t > 0$ such that $B \subseteq tU$. Thus,

$$\chi_B \wedge \underline{\mu'} \leqslant \chi_{tU} \wedge \underline{\mu'} = t\chi_U \wedge \underline{\mu'} = t(\chi_U \wedge \underline{\mu'}) \leqslant tQ'.$$

Note that $\mu' \nleqslant \lambda'$. Therefore, χ_B is λ-bounded in $(X, \Omega_L(\mathscr{T}))$. By the arbitrariness of $\lambda \in M(L)$, χ_B is bounded in $(X, \Omega_L(\mathscr{T}))$. □

4. **Properties of Bounded L-fuzzy Sets**

Theorem 4.1. *Let (X, δ) be an L-tvs, $B, B_i \in L^X$ $(i = 1, 2, \ldots, n, n \geqslant 2)$ and $\lambda \in M(L)$. Suppose that they are all λ-bounded. Then*

(1) Every L-fuzzy point x_α is λ-bounded; (2) If $A \leqslant B$, then A is λ-bounded; (3) $\bigvee_{i=1}^{n} B_i$ is λ-bounded; (4) $B_1 + B_2 + \cdots + B_n$ is λ-bounded; (5) For each $k \in \mathbb{K}$, kB is λ-bounded; (6) The balanced hull $BH(B)$ of B is λ-bounded.

Proof. Since (1) and (3) have been proved in [2] and (2) is trivial by the definition of λ-boundedness of L-fuzzy sets, we need only to prove (4)−(6).

(4) It suffices to show that $B_1 + B_2$ is λ-bounded.

In fact, for each $Q \in \eta(\theta_\lambda)$, by Theorem 3.1 in [3] and Lemma 2.11, there exists $P \in \eta(\theta_\lambda)$ such that P is co-balanced and $P' + P' \leqslant Q'$. Hence, there exist $t_i > 0$ $(i = 1, 2)$ and $\mu_i \in L$ with $\mu_i \nleqslant \lambda'$ $(i = 1, 2)$ such that $B_i \wedge \underline{\mu_i} \leqslant t_i P'$. Put $t = \max\{t_1, t_2\}$ and $\mu = \mu_1 \wedge \mu_2$. Then $\mu \nleqslant \lambda'$ and

$(B_1 + B_2) \wedge \underline{\mu} = (B_1 \wedge \underline{\mu}) + (B_2 \wedge \underline{\mu}) \leqslant (t_1 P') + (t_2 P') \leqslant (tP') + (tP') = t(P' + P') \leqslant tQ'$. Therefore, $B_1 + B_2$ is λ-bounded.

(5) If $k = 0$. Then $0B = \theta_{\vee_{x \in X} B(x)}$. By (1), $0B$ is λ-bounded;

Assume that $k \neq 0$. For each $Q \in \eta(\theta_\lambda)$, by Lemma 2.11, there exists $P \in \eta(\theta_\lambda)$ such that P is co-balanced and $Q \leqslant P$. Hence, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant tP'$, and so

$(kB) \wedge \underline{\mu} = k(B \wedge \underline{\mu}) \leqslant ktP' = |k|tP' \leqslant |k|tQ'$. Therefore, kB is λ-bounded.

(6) For each $Q \in \eta(\theta_\lambda)$, by Lemma 2.11, there exists $P \in \eta(\theta_\lambda)$ such that P is co-balanced and $Q \leqslant P$. Hence, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant tP'$, and so

$$\mathrm{BH}(B) \wedge \underline{\mu} = \mathrm{BH}(B \wedge \underline{\mu}) \leqslant \mathrm{BH}(tP') = t\mathrm{BH}(P') = tP' \leqslant tQ'.$$

Therefore, $\mathrm{BH}(B)$ is λ-bounded. □

Corollary 4.2. *Let (X, δ) be an L-tvs and $B, B_i \in L^X$ $(i = 1, 2, \ldots, n, n \geqslant 2)$. Suppose that they are all bounded. Then*

(1) Every L-fuzzy point x_α is bounded; (2) If $A \leqslant B$, then A is bounded; (3) $\bigvee_{i=1}^{n} B_i$ is bounded; (4) $B_1 + B_2 + \cdots + B_n$ is bounded; (5) For each $k \in \mathbb{K}$, kB is bounded; (6) The balanced hull $BH(B)$ of B is bounded.

Theorem 4.3. *Let (X, δ) be a Hausdorff L-tvs, $\lambda \in M(L)$ and B be λ-bounded in (X, δ). Then for each $x \in X$ with $x \neq \theta$, the set $S = \{k \in \mathbb{K} \mid kx_\lambda \nleqslant B'\}$ is bounded in $\mathbb{K}$.*

Proof. By Lemmas 2.11 and 2.12, there exists $Q \in \eta(\theta_\lambda)$ such that $Q(x) = 1$ and Q' is balanced. Since B is λ-bounded, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant tQ'$, which implies $\lambda \nleqslant \mu'$ and $tQ \leqslant B' \vee \underline{\mu'}$. Hence,

$S = \{k \in \mathbb{K} \mid kx_\lambda \nleqslant B' \vee \underline{\mu'}\} \subseteq \{k \in \mathbb{K} \mid kx_\lambda \nleqslant tQ\}$.

It suffices to prove that for each $k \in \{k \in \mathbb{K} \mid kx_\lambda \nleqslant tQ\}$, we have $|k| < t$.

In fact, if $k \in \{k \in \mathbb{K} \mid kx_\lambda \nleqslant tQ\}$ and $|k| \geqslant t$, since Q' is balanced, we have $tQ' \leqslant kQ'$, which implies $kQ \leqslant tQ$. Therefore, $kx_\lambda \nleqslant kQ$, i.e., $x_\lambda \nleqslant Q$, which contradicts with $Q(x) = 1$. □

Next, we shall investigate product and quotient properties of bounded L-fuzzy sets.

Theorem 4.4. *Let $\{(X_i, \delta_i)\}_{i\in\Gamma}$ be a family of L-tvses, $X = \Pi_{i\in\Gamma} X_i$, $\delta = \Pi_{i\in\Gamma}\delta_i$ be the product L-topology of $\{\delta_i\}_{i\in\Gamma}$ and $\lambda \in M(L)$. Then $B \in L^X$ is λ-bounded in (X, δ) if and only if each of its projections $P_i(B)$ is λ-bounded in (X_i, δ_i).*

Proof. Necessity: Suppose that $B \in L^X$ is λ-bounded in (X, δ). Let Q be an arbitrary R-neighborhood of θ_λ in (X_i, δ_i). Since P_i is continuous, ${P_i}^{-1}(Q)$ is an R-neighborhood of θ_λ in (X, δ). Hence, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $B \wedge \underline{\mu} \leqslant t[{P_i}^{-1}(Q)]' = t{P_i}^{-1}(Q')$, and so $P_i(B) \wedge \underline{\mu} = P_i(B \wedge \underline{\mu}) \leqslant P_i(t{P_i}^{-1}(Q')) \leqslant tQ'$. Therefore, $P_i(B)$ is λ-bounded in (X_i, δ_i).

Sufficiency: Suppose that each of the projections $P_i(B)$ of B is λ-bounded in (X_i, δ_i). Since $B \leqslant \Pi_{i\in\Gamma} P_i(B)$, it suffices to show that $\Pi_{i\in\Gamma} P_i(B)$ is λ-bounded.

In fact, for each closed R-neighborhood of θ_λ in (X, δ), $(\Pi_{i\in\Gamma} V_i)'$ say, we have $V_i \in \delta_i$ for each $i \in \Gamma$ and $V_i = X_i$ for each $i \in \Gamma$ except for $i = i_1, i_2, \ldots, i_n$. Since each $P_{i_k}(B)$ is λ-bounded, there exist $t > 0$ and $\mu \in L$ with $\mu \nleqslant \lambda'$ such that $P_{i_k}(B) \wedge \underline{\mu} \leqslant tV_{i_k}$ $(k = 1, 2, \ldots, n)$. Hence, $[\Pi_{i\in\Gamma} P_i(B)] \wedge \underline{\mu} \leqslant t\Pi_{i\in\Gamma} V_i$. This completes the proof. □

Theorem 4.5. *Let (X, δ) be an L-tvs, E a linear subspace of X, $\lambda \in M(L)$, and $X/E = \{\overline{x} \mid x \in X\}$ the quotient space, where $\overline{x} = \{y \in X \mid y - x \in E\}$. Put $\delta/E = \{q(G) \mid G \in \delta\}$, where $q : X \to X/E$ $(x \to \overline{x})$ is the quotient map. If $B \in L^X$ is λ-bounded in (X, δ), then $q(B)$ is λ-bounded in $(X/E, \delta/E)$.*

Proof. Similar to the proof of the necessity part of Theorem 4.4. □

Remark 4.6. In the theory of classical topological vector spaces, the converse of Theorem 4.5 doesn't necessarily hold, so it certainly doesn't hold in the case of L-topological vector spaces in general.

By Theorems 4.4 and 4.5, we obtain the following two corresponding corollaries, which uncover product and quotient properties of bounded L-fuzzy sets.

Corollary 4.7. *Let $\{(X_i, \delta_i)\}_{i\in\Gamma}$ be a family of L-tvses, $X = \Pi_{i\in\Gamma} X_i$, and $\delta = \Pi_{i\in\Gamma}\delta_i$ the product L-topology of $\{\delta_i\}_{i\in\Gamma}$. Then $B \in L^X$ is bounded in (X, δ) if and only if each of its projections $P_i(B)$ is bounded in (X_i, δ_i).*

Corollary 4.8. *Let (X, δ) be an L-tvs, E a linear subspace of X, and $X/E = \{\overline{x} \mid x \in X\}$ the quotient space, where $\overline{x} = \{y \in X \mid y - x \in E\}$. Put $\delta/E = \{q(G) \mid G \in \delta\}$, where $q : X \to X/E$ $(x \to \overline{x})$ is the quotient map. If $B \in L^X$ is bounded in (X, δ), then $q(B)$ is bounded in $(X/E, \delta/E)$.*

5. **Conclusion**

In this paper, we have obtained several equivalent characterizations and some properties of bounded L-fuzzy sets, which extend the corresponding results in classical topological vector space to L-topological vector space.

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