FUZZY RULES FOR FUZZY $\overline{X}$ AND R CONTROL CHARTS

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Abstract. Statistical process control (SPC), an internationally recognized technique for improving product quality and productivity, has been widely employed in various industries. SPC relies on the use of control charts to monitor a manufacturing process for identifying causes of process variation and signaling the necessity of corrective action for the process. Fuzzy data exist ubiquitously in the modern manufacturing process, and in this paper, two alternative approaches to fuzzy control charts are developed for monitoring sample averages and range. These approaches are based on "fuzzy mode" and "fuzzy rules" methods, when the measures are expressed by non-symmetric triangular fuzzy numbers. In contrast to the existing fuzzy control charts, the proposed approach does not require the use of the defuzzification and this prevents the loss of information included in samples. A numeric example illustrates the performance of the method and interprets the results.

1. Introduction

Statistical process control (SPC) is a large class of methods aiming at evaluating, monitoring and possibly reducing variability in industrial production processes and it plays an important role to assure the process is in statistical control. The two of the SPC functions are control charts and process capability analyses (PCA). Even though the first control chart was proposed by Shewhart [19], today they are still subject to new application areas that deserve further attention. Control charts are used to monitor whether or not the process is in statistical control. These charts are based on data representing one or several quality-related characteristics of the product or service. If these characteristics are measurable on a numerical scale, then variable control charts are used. If the quality-related characteristics cannot be easily represented in a numerical form, then attribute control charts are useful.

Shewhart control charts consist of a center line, the estimated process nominal level, and two control lines, the upper control limit and the lower control limit (usually set at away from the center line with a distance of $\pm$ three-sigma), indicating the boundaries of the normal variability that are used to verify whether the majority of the observations (approximately 99.73%) are lying within control limits. A process is in statistical control, if the control chart displays known patterns of variation and if the control chart points deviate from these known patterns, the process is considered to be out of control.

Precise data are not always available. So, the theory of fuzzy sets can adequately model processes where observed data are uncertain. Therefore, some researchers

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have constructed fuzzy control charts that accommodate uncertainty due to fuzziness.

To monitor a manufacturing process with fuzzy sample data, few researchers, Gulbay and Kahraman [11], Shu and Wu [21],Wang and Raz [24,25], Tannock [22], have used the fuzzy set theory for the construction of fuzzy control charts. Since fuzzy data ubiquitously exist in the modern manufacturing process, for monitoring its sample average and variance, Shu and Wu [21], offered the fuzzy $\overline{X}$ and R control charts, whose fuzzy control limits are obtained on the basis of the results of the resolution identity, a well-known theory in the fuzzy set field. By using the fuzzy dominance approach, which compares the fuzzy averages and variances to their respective fuzzy control limits, they are capable of determining whether the manufacturing process is needed to be adjusted or not. Kaya and Kahraman [15], introduce fuzzy rules method for symmetrical fuzzy numbers. In this paper, we construct the Shewhart $\overline{X}$ and R control charts for monitoring the process average and variability, respectively using new fuzzy rules.

This paper is organized as follows. In Section 2, the literature of fuzzy control charts is briefly reviewed. In Section 3, we briefly review the development for constructing the control limits for the $\overline{X}$ and R charts. Section 4, includes the main principles of fuzzy control charts and rules when the measures are expressed as symmetrical or asymmetrical triangular fuzzy number to check the process situation. The results of the several approaches are compared and interpreted by a numerical example in Section 5. Finally, Section 6 provides a conclusion.

2. **Literature Review**

After the inception of the notion of fuzzy sets by Zadeh [27], many authors have applied this approach to very different areas such as statistics, quality control, and optimization techniques. These studies also statistical process control and control chart. In recent years, some papers which have concentrated on different area of SPC using fuzzy set theory have been published. To monitor a manufacturing process with fuzzy sample data, several researchers, Kaya and Kahraman [15], and Ruan [10], Gulbay and Kahraman [11,12], Kanagawa et al. [14] and Wang and Raz [24, 25], have used fuzzy set theory for the construction of fuzzy control charts. Rowlands and Wang [18] introduced fuzzy SPC methods based on the application of fuzzy logic to the SPC-zone rules. For monitoring the process average, two fuzzy approaches, called the membership function approach and the fuzzy-probabilistic approach, are proposed by Wang and Raz [24, 25], where, using defuzzification methods, fuzzy sample data are first converted into real-valued sample data, then the center line and control limits are established as in the traditional, Shewhart control charts. For monitoring the process average as well as the process variability, Kanagawa et al. [14] adopted the fuzzy-probabilistic approach; however, they used the probability density function of a fuzzy random variable to form the center line and control limits of the fuzzy control chart, which is much more complicated than the method of Wang and Raz [24,25]. Taleb and Limam [23] discussed different procedures of constructing control charts for linguistic data, based on the fuzzy set and probability theories. A comparison between the fuzzy and probabilistic approaches, based on the average run length and the samples under control, is made by using real data. Tannock [22] proposed a fuzzy control chart for individual observations. Hsu and Chen [13] described a new diagnosis system based on fuzzy reasoning to monitor the performance of a discrete manufacturing process and to justify the possible causes. Fazel Zarandi et al. [6] use fuzzy adaptive sampling rules and fuzzy run rules to increase the sensitivity of the control charts. These approaches use fuzzy logic only in analyzing the control chart patterns to determine the process state. Diagnosing the assignable cause from the signals from the patterns has not been explored in these approaches.

Using the defuzzification method to simplify the fuzzy observations into real numbers causes the loss of original data information as has been pointed out by Cheng [2], Grzegorzewski [7, 8, 9] and Laviolette et al. [16]. Instead of using the defuzzification method, therefore, Gulbay and Kahraman [11] introduce a direct fuzzy approach (DFA), where fuzzy sample data and the imprecise number of nonconformities found in the manufacturing process are directly used to construct the fuzzy center line and fuzzy control limits for a fuzzy c control chart. Cheng [2] constructed fuzzy control charts for a process with fuzzy outcomes derived from the subjective quality ratings provided by a group of experts. The fuzzy quality ratings are then plotted on fuzzy control charts, whose construction and out-of-control conditions are developed using possibility theory.

Faraz and Moghadam [3] proposed a fuzzy control chart for monitoring the mean and variance of continuous (variable) quality characteristics with a warning line. The warning detects a quick shift in the process mean. To do so, a set of fuzzy if then rules are used, which classify the samples into linguistic terms or quality categories. As observations are normally distributed, they used the Gaussian membership function to describe linguistic terms. Faraz et al. [4] introduced a control chart for monitoring variables when uncertainty and randomness are combined, that is, the process mean value is a fuzzy number. They showed that when the Shewhart charts are used, the control limits must be adjusted to enhance the existing fuzziness. Also, Faraz and Shapiro [5] proposed an application of fuzzy random variables to control charts. Such that fuzzy statistical control chart is constructed based upon the notions of fuzzy random variables and graded inclusion measures. The proposed control chart is an extension of Shewhart $\overline{X} - S^2$ control charts in fuzzy space.

Also, Senturk and Erginel [20] proposed fuzzy $\overline{X} - R$ and $\overline{X} - S$ control charts with α-cuts. Their idea can be traced to traditional $\overline{X} - R$ and $\overline{X} - S$ control charts, which rely heavily on the properties of the normal distribution. Fuzzy multivariate exponentially weighted moving average control chart is developed for monitoring linguistic observations, proposed by Alipour and Noorossana [1]. The performance of the proposed control chart is evaluated using the average run length (ARL) criterion.

3. $\overline{X}$ and R and Charts for Real-Valued Data

The $\overline{X}$ and R charts are used for variable data (continuous data) with the assumption that the data follows a normal distribution with parameters μ and σ^2 The $\overline{X}$ monitors the process means, while the R chart monitors the within group variation at a given time point. The range of a sample is simply the difference between the largest and smallest observations. Let $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ $(R_1, R_2, \ldots, R_m)$ be the means (ranges) of m subgroups with size n, the grand average and average range are $\overline{\overline{X}} = \dfrac{\overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_m}{m}$ and $\overline{R} = \dfrac{R_1 + R_2 + \cdots + R_m}{m}$ respectively. Because parameters μ and σ^2 are usually unknown in practice, we use $\overline{\overline{X}}$ as an estimator μ and $\dfrac{\overline{R}}{d_2}$ as an estimator of σ. The three-sigma control limits for the $\overline{X}$ chart are as follows:

$$\begin{aligned} UCL(\overline{X}) &= \overline{\overline{X}} + \frac{3}{d_2\sqrt{n}}\overline{R} = \overline{\overline{X}} + A_2\overline{R}, \\ CL(\overline{X}) &= \overline{\overline{X}}, \\ LCL(\overline{X}) &= \overline{\overline{X}} - \frac{3}{d_2\sqrt{n}}\overline{R} = \overline{\overline{X}} - A_2\overline{R}. \end{aligned}$$

The constant A_2 is tabulated for various sample size [26].

To compute the control limits of the R chart, we may estimate σ_R by $\hat{\sigma}_R = d_3\dfrac{\overline{R}}{d_2}$. The three-sigma control limits for the R chart are as follows:

$$\begin{aligned} UCL(R) &= \overline{R} + 3d_3\frac{\overline{R}}{d_2} = \overline{R}D_4, \\ CL(R) &= \overline{R}, \\ LCL(R) &= \overline{R} - 3d_3\frac{\overline{R}}{d_2} = \overline{R}D_3. \end{aligned}$$

where D_3 and D_4 for various sample sizes $2 \leq n \leq 25$ are also available [26].

4. Fuzzy $\overline{X}$ and R Charts for Triangular Fuzzy Numbers

Assume that a quality characteristic is defined as "approximately X". Considering the fuzzy sets concept, this value can be converted to the triangular fuzzy number (TFN) $\tilde{X} = (X_1, X_2, X_3)$. After measuring a sample of size n from triangular fuzzy numbers (X_{1j}, X_{2j}, X_{3j}) $j = 1, \ldots, n$, the average of this sample can be calculated by extension principle as follows:

$$\tilde{\overline{X}} = (\overline{X}_1, \overline{X}_2, \overline{X}_3) = \left(\frac{\sum_{j=1}^{n} X_{1j}}{n}, \frac{\sum_{j=1}^{n} X_{2j}}{n}, \frac{\sum_{j=1}^{n} X_{3j}}{n} \right).$$

Also considering extension principle, the range of the sample can be calculated by

$$\begin{aligned} \tilde{R} &= (R_1, R_2, R_3) \\ &= (\max X_{1j}, \max X_{2j}, \max X_{3j}) \ominus (\min X_{1j}, \min X_{2j}, \min X_{3j}) \\ &= (\max X_{1j} - \min X_{3j}, \max X_{2j} - \min X_{2j}, \max X_{3j} - \min X_{1j}), \end{aligned}$$

where $(\max X_{1j}, \max X_{2j}, \max X_{3j})$ and $(\min X_{1j}, \min X_{2j}, \min X_{3j})$ represent the maximum and minimum values of fuzzy measurements, respectively. One method to determine the maximum and minimum values of fuzzy measurements is assign from ranking method. In this paper, one of the most popular defuzzification methodologies called the "weighted average method" is used [17].

For m subgroups with size n, the fuzzy grand average and the average range of samples are

$$\tilde{\overline{\overline{X}}} = (\overline{\overline{X}}_1, \overline{\overline{X}}_2, \overline{\overline{X}}_3) = \left(\frac{\sum_{i=1}^{m} \overline{X}_{1i}}{m}, \frac{\sum_{i=1}^{m} \overline{X}_{2i}}{m}, \frac{\sum_{i=1}^{m} \overline{X}_{3i}}{m} \right)$$

$$\tilde{\overline{R}} = (\overline{R}_1, \overline{R}_2, \overline{R}_3) = \left(\frac{\sum_{i=1}^{m} R_{1i}}{m}, \frac{\sum_{i=1}^{m} R_{2i}}{m}, \frac{\sum_{i=1}^{m} R_{3i}}{m} \right), \qquad (1)$$

respectively. Therefore, the control limits for $\tilde{\overline{X}} - \tilde{R}$ control charts are calculated as follows:

$$\begin{aligned} \widetilde{UCL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} + A_2\tilde{\overline{R}} = (\overline{\overline{X}}_1 + A_2\overline{R}_1, \overline{\overline{X}}_2 + A_2\overline{R}_2, \overline{\overline{X}}_3 + A_2\overline{R}_3) \\ &= (UCL(\overline{X})_1, UCL(\overline{X})_2, UCL(\overline{X})_3), \\ \widetilde{CL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} = (\overline{\overline{X}}_1, \overline{\overline{X}}_2, \overline{\overline{X}}_3) = (CL(\overline{X})_1, CL(\overline{X})_2, CL(\overline{X})_3), \\ \widetilde{LCL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} - A_2\tilde{\overline{R}} = (\overline{\overline{X}}_1 - A_2\overline{R}_3, \overline{\overline{X}}_2 - A_2\overline{R}_2, \overline{\overline{X}}_3 - A_2\overline{R}_1) \\ &= (LCL(\overline{X})_1, LCL(\overline{X})_2, LCL(\overline{X})_3). \end{aligned} \qquad (2)$$

And similarly, for $\tilde{R}$ control chart,

$$\begin{aligned} \widetilde{UCL}_R &= \tilde{\overline{R}}D_4 = (\overline{R}_1 D_4, \overline{R}_2 D_4, \overline{R}_3 D_4) \\ &= (UCL(R)_1, UCL(R)_2, UCL(R)_3), \\ \widetilde{CL}_R &= \tilde{\overline{R}} = (\overline{R}_1, \overline{R}_2, \overline{R}_3) = (CL(R)_1, CL(R)_2, CL(R)_3), \\ \widetilde{LCL}_R &= \tilde{\overline{R}}D_3 = (\overline{R}_1 D_3, \overline{R}_2 D_3, \overline{R}_3 D_3) \\ &= (LCL(R)_1, LCL(R)_2, LCL(R)_3). \end{aligned} \qquad (3)$$

In the next section, two alternative approaches to fuzzy control charts are developed. The first approach is constructed based on the "fuzzy mode" and the second approach is constructed based on the "fuzzy rules".

4.1. **Mode Method for TFN Case.** There are several defuzzification methods, such as median, mod, midrange and mean. In this section we are going to review and modify the introduced mode by Kaya and Kahraman [15] and Alipour and Noorossana [1], That introduce in this section. The mode of a fuzzy set (f_{mod}) is the value of the base variable X, for which the membership value equal to 1, i.e. $f_{mod} = \{x \in X | \mu_f(x) = 1\}$. Therefore the fuzzy modes of $\tilde{\overline{X}} - \tilde{R}$ control charts are defined as follows which is equivalent to the classical $\overline{X} - R$ control limits on the basis of the core values of triangular fuzzy numbers $\tilde{X}_{ij}$'s:

$$\begin{cases} UCL_{mod} = UCL(\overline{X})_2 \\ CL_{mod} = CL(\overline{X})_2 \\ LCL_{mod} = LCL(\overline{X})_2 \end{cases}$$

$$\begin{cases} UCL_{mod} = UCL(R)_2 \\ CL_{mod} = CL(R)_2 \\ LCL_{mod} = LCL(R)_2 \end{cases}$$

Also the mode values of the sample mean and range are defined as follows:

$$\overline{X}mod_i = \overline{X}_{2i} \qquad i = 1, 2, \ldots, m$$
$$rmod_i = R_{2i} \qquad i = 1, 2, \ldots, m$$

The process situation ($C_{\overline{X}}$) for the $\tilde{\overline{X}}$ control chart and the process situation (C_R) for the $\tilde{R}$ control chart are defined as follows:

$$C_{\overline{X}} = \begin{cases} 1 & if(LCL(\overline{X})_2 \leq \overline{X}_2 \leq UCL(\overline{X})_2), \\ 0 & if(\overline{X}_2 \leq LCL(\overline{X})_2)or(UCL(\overline{X})_2 \leq \overline{X}_2), \end{cases}$$

$$C_R = \begin{cases} 1 & \text{if}(LCL(R)_2 \leq R_2 \leq UCL(R)_2), \\ 0 & \text{if}(R_2 \leq LCL(R)_2)or(UCL(R)_2 \leq R_2). \end{cases}$$

So finally, the decision of process controller can be defined by:

$$\text{Process is} \begin{cases} \text{in control} & \text{if}(C_R = 1)\text{and}(C_{\overline{X}} = 1), \\ \text{out of control} & \text{if}(C_R = 0)\text{or}(C_{\overline{X}} = 0). \end{cases}$$

This process is applied to all samples and continues until all sample means and ranges are within control limits. If $\tilde{\overline{X}} - \tilde{R}$ control charts indicate any out of control pattern, assignable causes are investigated and the control limits are recalculated.

4.2. **Fuzzy Rules Method for TFN Case.** The method is based on certain rules which define all possible patterns of a process. Kaya and Kahraman, explain several rules for control charts when measures are symmetric [15]. In this subsection, we investigated on process control by introducing the following four rules for non-symmetric $TFNs(X_{1j}, X_{2j}, X_{3j})$:

Rule-1 Takes into account the case where the sample means (or ranges) are between control limits, i.e. $X_1 \geq LCL(X)_3$ and $X_3 \leq UCL(X)_1$, as is shown in Figure.1 for an is in control process. Here ($C_R = 1$) and ($C_{\overline{X}} = 1$).

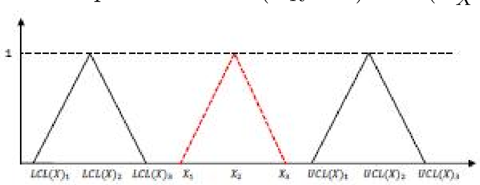

Figure 1. Fuzzy Rule-1

Rule-2 Takes into account the cases where a sample mean (or a range) is out of either $\widetilde{UCL}$ or $\widetilde{LCL}$, i.e. $X_3 \leq LCL(X)_1$ or $X_1 \geq UCL(X)_3$, as is shown in Figure.2 for an out of control process. Here ($C_R = 0$) or ($C_{\overline{X}} = 0$).

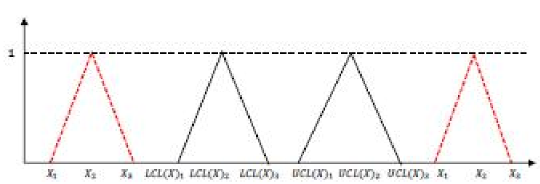

Figure 2. Fuzzy Rule-2

Rule-3 Analyzes the cases where X_2 is between $LCL(X)_3$ and $UCL(X)_1$, $(LCL(X)_3 \leq X_2 \leq UCL(X)_1)$, i.e. a sample mean (or a range) is partially included by either of control limits or by both of control limits and or is partially by one or both of control limits and partially is out of control limit as different cases are shown in Figure.3. Let S denote the area under the graph of the membership of $\tilde{\overline{X}}$ (or $\tilde{R}$) i.e. the area of the dotted triangle. In this case, the linguistic decisions such as "rather in control" and "rather out of control" are made. If the percentage area of a sample mean (or a range) which remains between the fuzzy control limits (i.e. $C_{\overline{X}}(C_R) = \frac{S_1}{S}$), is equal or greater than a specified acceptable percentage (β_1), then the process is in control, otherwise if percentage total area of a sample mean (or a range) which remains between the fuzzy control limits and inside the control limits (i.e. $C_{\overline{X}}(C_R) = \frac{S_1+S_2'+S_2''}{S}$), is equal or greater than a specified acceptable percentage $\beta_0(\beta_0 < \beta_1)$, then the process is "rather in control" and otherwise it can be stated as "rather out of control".

$$\text{Therefore Process is} \begin{cases} \text{in control} & \text{if } (C_R \geq \beta_1)\text{and}(C_{\overline{X}} \geq \beta_1), \\ \text{rather in control} & \text{if } (C_R \geq \beta_0)\text{and}(C_{\overline{X}} \geq \beta_0), \\ \text{rather out of control} & \text{if } (C_R < \beta_0)\text{or}(C_{\overline{X}} < \beta_0). \end{cases}$$

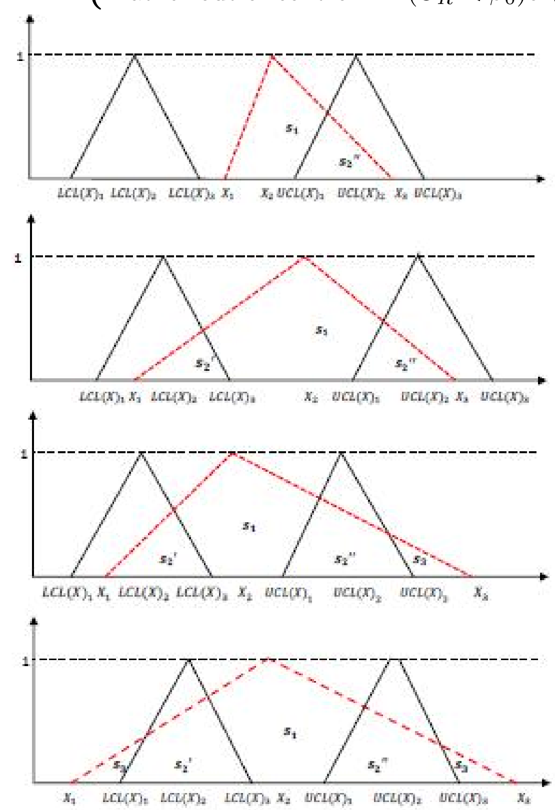

FIGURE 3. Different Cases for Fuzzy Rule-3

Rule-4 Analyzes the cases where a sample mean (or a range) is partially out of a control limits, i.e. $X_2 \leq LCL(X)_2$ or $X_2 \geq UCL(X)_2$ as different cases are shown in Figure.4. In this case, if the percentage area of a sample mean (or a range) which remains out of fuzzy control limits (i.e. $C_{\overline{X}}(C_R) = \frac{S_3}{S}$), is equal or greater than β_1, then the process is out of control and if percentage total area of a sample mean (or a range) which remains outside the fuzzy control limits and inside the control limits (i.e. $C_{\overline{X}}(C_R) = \frac{S_3+S_2'+S_2''}{S}$), is equal or greater than a β_0, then the process is "rather out of control" and otherwise it can be stated as "rather in control".

$$\text{Process is} \begin{cases} \text{out of control} & \text{if } (C_R \geq \beta_1)\text{or}(C_{\overline{X}} \geq \beta_1), \\ \text{rather out of control} & \text{if } (C_R \geq \beta_0)\text{and}(C_{\overline{X}} \geq \beta_0), \\ \text{rather in control} & \text{if } (C_R < \beta_0)\text{or}(C_{\overline{X}} < \beta_0). \end{cases}$$

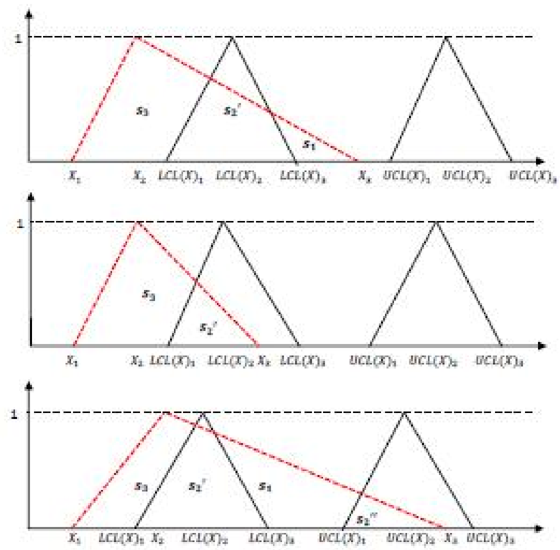

FIGURE 4. Different Cases for Fuzzy Rule-4

5. Numerical Example

Fifteen samples of size $n = 4$ were taken from a production process. The data are TFN as are shown in Table 1, and consider the minimum predefined acceptable percentage as $\beta_1 = 0.7$ and $\beta_0 = 0.5$.

Using equation (1), $\tilde{\overline{\overline{X}}}$ and $\tilde{\overline{R}}$ are determined as follows:

$$\tilde{\overline{\overline{X}}} = \left(\frac{\sum_{i=1}^{m} \overline{X}_{1i}}{m}, \frac{\sum_{i=1}^{m} \overline{X}_{2i}}{m}, \frac{\sum_{i=1}^{m} \overline{X}_{3i}}{m} \right) = (\overline{\overline{X}}_1, \overline{\overline{X}}_2, \overline{\overline{X}}_3) = (74, 74.001, 74.003),$$

$$\tilde{\overline{R}} = \left(\frac{\sum_{i=1}^{m} R_{1i}}{m}, \frac{\sum_{i=1}^{m} R_{2i}}{m}, \frac{\sum_{i=1}^{m} R_{3i}}{m} \right) = (0.019359, 0.022359, 0.025359).$$

$\tilde{X}_1 = (X_{11}, X_{21}, X_{31})$	$\tilde{X}_2 = (X_{12}, X_{22}, X_{32})$	$\tilde{X}_3 = (X_{13}, X_{23}, X_{33})$	$\tilde{X}_4 = (X_{14}, X_{24}, X_{34})$
(73.992, 73.993, 73.995)	(73.992, 73.993, 73.995)	(74.003, 74.004, 74.006)	(73.998, 73.999, 74.001)
(73.987, 73.988, 73.990)	(74.003, 74.004, 74.006)	(73.985, 73.986, 73.988)	(73.989, 73.990, 73.992)
(74.007, 74.008, 74.010)	(73.978, 73.979, 73.981)	(73.996, 73.997, 73.999)	(74.006, 74.007, 74.009)
(73.982, 73.983, 73.985)	(74.003, 74.004, 74.006)	(74.013, 74.014, 74.016)	(74.009, 74.010, 74.012)
(74.008, 74.009, 74.011)	(73.984, 73.985, 73.987)	(73.993, 73.994, 73.996)	(74.002, 74.003, 74.005)
(74.012, 74.013, 74.015)	(74.032, 74.033, 74.035)	(74.013, 74.014, 74.016)	(73.990, 73.991, 73.993)
(73.986, 73.987, 73.989)	(73.991, 73.992, 73.994)	(73.985, 73.986, 73.988)	(73.981, 73.982, 73.984)
(74.015, 74.016, 74.018)	(74.024, 74.025, 74.027)	(74.012, 74.013, 74.015)	(73.994, 73.995, 73.997)
(73.988, 73.989, 73.991)	(73.997, 73.998, 73.910)	(73.996, 73.997, 73.999)	(74.021, 74.022, 74.024)
(74.017, 74.018, 74.020)	(74.008, 74.009, 74.011)	(74.006, 74.007, 74.009)	(74.015, 74.016, 74.018)
(74.001, 74.002, 74.004)	(74.017, 74.018, 74.020)	(73.996, 73.997, 73.999)	(73.988, 73.989, 73.991)
(74.014, 74.015, 74.017)	(74.004, 74.005, 74.007)	(74.021, 74.022, 74.024)	(74.001, 74.002, 74.004)
(74.009, 74.010, 74.012)	(73.994, 73.995, 73.997)	(73.998, 73.999, 74.002)	(73.995, 73.996, 73.998)
(73.987, 73.988, 73.989)	(73.997, 73.998, 74.000)	(74.003, 74.004, 74.006)	(73.997, 73.998, 74.000)
(74.004, 74.005, 74.007)	(74.005, 74.006, 74.008)	(73.989, 73.990, 73.992)	(73.995, 73.996, 73.998)

Table 1. Triangular Fuzzy Numbers

$D_3 = 0$ and $D_4 = 2.115$, and using equation (3), $\widetilde{CL}_R$, $\widetilde{UCL}_R$, $\widetilde{LCL}_R$ are determined as follows:

$$\begin{aligned}\widetilde{UCL}_R &= \tilde{\overline{R}} D_4 = (\overline{R}_1 D_4, \overline{R}_2 D_4, \overline{R}_3 D_4) = (0.04417, 0.05102, 0.05787),\\ \widetilde{CL}_R &= \tilde{\overline{R}} = (\overline{R}_1, \overline{R}_2, \overline{R}_3) = (0.019359, 0.022359, 0.025359),\\ \widetilde{LCL}_R &= \tilde{\overline{R}} D_3 = (\overline{R}_1 D_3, \overline{R}_2 D_3, \overline{R}_3 D_3) = (0, 0, 0).\end{aligned}$$

$A_2 = 0.729$, and using equation (2), $\widetilde{CL}_{\overline{X}}, \widetilde{UCL}_{\overline{X}}, \widetilde{LCL}_{\overline{X}}$ are determined as follows:

$$\begin{aligned}\widetilde{UCL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} + A_2 \tilde{\overline{R}} = (74.0141, 74.0163, 74.0185),\\ \widetilde{CL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} = (\overline{\overline{X}}_1, \overline{\overline{X}}_2, \overline{\overline{X}}_3) = (74, 74.001, 74.003),\\ \widetilde{LCL}_{\overline{X}} &= \tilde{\overline{\overline{X}}} - A_2 \tilde{\overline{R}} = (73.9815, 73.9837, 73.9859).\end{aligned}$$

The decisions about the process control results from each sample based on the fuzzy mode and fuzzy rules method are given in Table 2. The process was in control for each sample. So these control limits can be used to control the process.

sample	Mode method for $\tilde{\overline{X}}$ chart	Mode method for $\tilde{R}$ chart	Process control	Fuzzy rules for $\tilde{\overline{X}}$ chart	Fuzzy rules for $\tilde{R}$ chart	Process control	$C_{\overline{X}}$	C_R
1	in control	in control	in control	in control	in control	in control	1	1
2	in control	in control	in control	in control	in control	in control	1	1
3	in control	in control	in control	in control	in control	in control	1	1
4	in control	in control	in control	in control	in control	in control	1	1
5	in control	in control	in control	in control	in control	in control	1	1
6	in control	in control	in control	in control	in control	in control	0.98	0.99
7	in control	in control	in control	in control	in control	in control	0.94	1
8	in control	in control	in control	in control	in control	in control	1	1
9	in control	in control	in control	in control	in control	in control	1	1
10	in control	in control	in control	in control	in control	in control	1	1
11	in control	in control	in control	in control	in control	in control	1	1
12	in control	in control	in control	in control	in control	in control	1	1
13	in control	in control	in control	in control	in control	in control	1	1
14	in control	in control	in control	in control	in control	in control	1	1
15	in control	in control	in control	in control	in control	in control	1	1

Table 2. The Results for Process Control

After determining that the process is in control, seven samples with a sample size of 4 were taken from the process for monitoring the process. The values of samples are given in Table 3. $C_{\overline{X}}$ and C_R were calculated for seven new samples using fuzzy rules method. The results are given in Table 4. The conditions of process control can be seen in Table 4. According to the results of these methods of fuzzy rules sounds an earlier alarm.

$\tilde{X}_1 = (X_{11}, X_{21}, X_{31})$	$\tilde{X}_2 = (X_{12}, X_{22}, X_{32})$	$\tilde{X}_3 = (X_{13}, X_{23}, X_{33})$	$\tilde{X}_4 = (X_{14}, X_{24}, X_{34})$
(73.980, 73.989, 73.999)	(73.999, 74.001, 74.002)	(74.001, 74.002, 74.003)	(73.970, 73.980, 73.990)
(73.983, 73.985, 73.986)	(73.989, 73.990, 73.991)	(74.001, 74.003, 74.005)	(74.002, 74.004, 74.007)
(73.990, 74.001, 74.010)	(74.041, 74.042, 74.050)	(74.009, 74.010, 74.012)	(73.999, 74.007, 74.009)
(73.970, 73.980, 74.000)	(74.020, 74.029, 74.040)	(74.019, 74.029, 74.039)	(74.020, 74.030, 74.040)
(73.999, 74.003, 74.005)	(74.050, 74.059, 74.069)	(74.002, 74.004, 74.005)	(74.006, 74.007, 74.008)
(74.003, 74.004, 74.005)	(73.999, 74.000, 74.002)	(74.047, 74.053, 74.061)	(74.025, 74.035, 74.045)
(73.999, 74.001, 74.002)	(74.005, 74.006, 74.007)	(74.029, 74.031, 74.033)	(74.064, 74.065, 74.066)

TABLE 3. Seven Samples with Sample Size of 4

sample	Mode method for $\tilde{\overline{X}}$ chart	Mode method for $\tilde{R}$ chart	Process control	Fuzzy rules for $\tilde{\overline{X}}$ chart	Fuzzy rules for $\tilde{R}$ chart	Process control
1	in control	in control	in control	in control	in control	in control
2	in control	in control	in control	in control	in control	in control
3	in control	in control	in control	rather in control	rather in control	rather in control
4	in control	in control	in control	rather out control	rather in control	rather out control
5	Out of control	Out of control	Out of control	rather out control	rather out control	rather out control
6	Out of control	Out of control	Out of control	Out of control	rather out control	Out of control
7	Out of control	Out of control	Out of control	Out of control	Out of control	Out of control

sample	$C_{\overline{X}} = \frac{S_1}{S}(\frac{S_1+S_2'+S_2''}{S})$	$\frac{S_3}{S}(\frac{S_3+S_2'+S_2''}{S})$	rule	$C_R = \frac{S_1}{S}(\frac{S_1+S_2'+S_2''}{S})$	$\frac{S_3}{S}(\frac{S_3+S_2'+S_2''}{S})$	rule
1	1	0	1	1	0	1
2	1	0	1	1	0	1
3	0.5349 (0.925)	0.0749	3	0.6609 (0.9868)	0.0131	3
4	0.2768	0.5306 (0.72316)	4	0.5252	0.2046 (0.4747)	4
5	0	0.5897 (1)	4	0	0.6315 (1)	4
6	0	0.981 (1)	4	0	0.2941 (1)	4
7	0	1	2	0	1	2

TABLE 4. Results of Process Control for Seven Samples

For example, explain decision making for third and fourth sample. In third sample is $X_2 < UCL(X)_2$, then used from third rules. At this sample, there are for $\tilde{\overline{X}}$ control chart:

$$\left(C_{\overline{X}} = \frac{S_1}{S} = 0.5349\right) < (\beta_1 = 0.7),$$

Then,

$$\left(C_{\overline{X}} = \frac{S_1 + S_2' + S_2''}{S} = 0.925\right) > (\beta_0 = 0.5).$$

So, this sample is rather in control.

In four sample is $X_2 > UCL(X)_2$, then used from fourth rules. At this sample, there are for $\tilde{\overline{X}}$ control chart:

$$\left(C_{\overline{X}} = \frac{S_3}{S} = 0.53061\right) < (\beta_1 = 0.7),$$

Then,

$$\left(C_{\overline{X}} = \frac{S_3 + S_2' + S_2''}{S} = 0.72316\right) > (\beta_0 = 0.5).$$

So, this sample is rather out of control.

6. Conclusions and Future Works

Fuzzy control charts in the literature are commonly based on transformation of fuzzy to crisp cases and unnatural patterns analyses for fuzzy control charts have not been studied. In this paper, we have developed a direct fuzzy approach to fuzzy control charts without any defuzzification, and then defined fuzzy unnatural pattern rules. The proposed fuzzy control chart is illustrated by a numerical example. We have defined fuzzy unnatural pattern rules based on the fuzzification of the crisp rules. There are only two types of decisions when we used defuzzification methods either the process is in control or out of control. However in our method, we may also decide that the process is almost in control or almost out of control modes. Therefore, the decision about the process is more accurate. Study on control charts on the basis of a fuzzy measure for being in control or out of control is a potential subject for further research. As another related subject, one can investigate and extend other control charts from the same point of view.

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