L-ENRICHED TOPOLOGICAL SYSTEMS—A COMMON FRAMEWORK OF L-TOPOLOGY AND L-FRAMES

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Abstract. Employing the notions of the strong L-topology introduced by Zhang and the L-frame introduced by Yao and the concept of L-enriched topological system defined in the present paper, we construct adjunctions among the categories **StL-Top** of strong L-topological spaces, **SL-Loc** of strict L-locales and **L-EnTopSys** of L-enriched topological systems. All of these concepts are essentially based on the theory of L-enriched categories, thus we obtain a unified enriched-categorical version of the classical adjunctions among the categories **Top** of topological spaces, **Loc** of locales and **TopSys** of topological systems, as well as a unified enriched-categorical approach to treating these concepts.

1. Introduction

Initiated by Lawvere's famous paper [16] in 1973, quantale-enriched categories (Ω-categories for short) become interesting objects for both mathematicians and theoretical computer scientists, which are the core objects in Quantitative Domain Theory [8, 11, 15, 14, 22, 24, 26]. All the categorical, order-theoretical, and topological aspects of Ω-categories have received lots of attention in the literatures, see, e.g. [2, 3, 5, 8, 11, 14, 15, 16, 21, 22, 24, 25, 26, 27, 28]. Many classical concepts have been developed in the frame-work of Ω-categories. Let us recall some of them.

In [27], Zhang presented an enriched category approach to many valued topology by interpreting the axioms (two logical formulas) for classical topological spaces in the fuzzy setting making use of the intrinsic fuzzy partial order on the fuzzy power set of X. In that paper, Zhang introduced the concept of strong Ω-topology. He also discussed some relations between the new concepts and related notions already available in the literatures.

Later, based on L-ordered sets (or equivalently, L-enriched categories) Yao [25] introduced a fuzzy version of frames, called L-frames. Then he constructed an adjunction between the category of stratified L-topological spaces and that of L-locales, which is a fuzzy counterpart of the Isbell-adjunction between topological spaces and locales.

Recently, motivated by questions arising from programming, J. T. Denniston, A. Melton and S. E. Rodabaugh [5] proposed an enriched category approach to study topological systems. They introduced the notion of enriched topological system.

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We know that the concepts of topology, frame and topological system have close relations with each other. Their categories (i.e., **Top** of topological spaces, **Loc** of locales and **TopSys** of topological systems) are connected by three well-know adjunctions [6, 23]. As reported above, all of these notions have been developed in the frame-work of enriched categories individually, so it is natural to ask that, can these concepts be connected with each other in a natural way just as in the crisp situation? In order to establish compatible relations among them, we extend the notion of topological system to the frame-work of L-enriched categories in a new way. Then employing the concepts of strong L-topology and L-frame and the concept of L-enriched topological system defined in the present paper, we construct adjunctions among their categories in a natural way. So we obtain a unified enriched-categorical approach to treat topologies, frames and topological systems.

The contents of the paper are organized as follows. Section 2 lists some preliminary notions and results about L-ordered sets. In Section 3, we recall the concepts of strong L-topology and L-frame. Then, we introduce the concept of L-enriched topological systems. In Section 4, we construct an adjunction between the category of strong L-topologies and the category of L-locales. In Section 5, we construct an adjunction between the category of strong L-topologies and the category of L-enriched topological systems. In Section 6, we construct an adjunction between the category of L-enriched topological systems and the category of L-locales.

2. L-ordered sets

We refer the reader to [1, 13] for general and enriched category theory. Throughout this paper, L denotes a *frame* [12, 20]. The greatest element of L is denoted 1 and the least element of L is denoted 0. For $A \subseteq L$, the least upper bound (resp., greatest lower bound) of A is written as $\bigvee A$ (resp., $\bigwedge A$). For $a, b \in L$, we define $a \to b = \bigvee \{c \in L | a \wedge c \leq b\}$. A map from a set X to L is called an *L-subset* of X. L^X denotes the set of all L-subsets of X.

Definition 2.1. [13, 14, 16, 24] An L-category, or an L-preordered set is a set with a binary function, called hom-functor, $A(\_,\_) : A \times A \longrightarrow L$ such that $A(a, a) = 1$ (reflexivity) and $A(a, b) \wedge A(b, c) \leq A(a, c)$ (transitivity) for all $a, b, c \in A$.

Two elements x and y in an L-category A are said to be *isomorphic* if $A(x, y) = 1 = A(y, x)$. An L-category is said to be *anti-symmetric* if different elements are always non-isomorphic.

Clearly, L-categories are a special case of Ω-categories, which are studied extensively in [2, 3, 8, 14, 15, 25, 26, 27, 28] from the fuzzy orders point of view . When we consider an L-category as a set with a many valued order, then it is usually said to be an *L-preordered set*, and an anti-symmetric L-category is usually said to be an *L-ordered set*. We often simply write A for an L-category $(A, A(\_,\_))$ if the hom-functor is clear from the context. For an L-category (resp., anti-symmetric L-category) A, the hom-functor $A(\_,\_)$ is said to be an *L-preorder* (resp., *L-order*) and will be often denoted by e.

An *L-functor*, or a *monotone map*, between L-categories A and B is a function $f : A \to B$ such that $A(a, b) \leq B(f(a), f(b))$ for all $a, b \in A$. A pair of L-functors

$f: A \to B$ and $g: B \to A$ is said to be an L-*adjunction* (or simply an *adjunction*) between L-categories A and B, if $B(f(a), b) = A(a, g(b))$ for all $a \in A$ and $b \in B$.

Remark 2.2. Let (X, e) be an L-ordered set. The L-order e induces a patrial order $\leq_e$ on X defined by $x \leq_e y$ iff $e(x, y) = 1$. We will denote $\leq_e$ simply by $\leq$, and denote the join operation in $(X, \leq_e)$ by $\vee$. Conversely, given a poset $(X, \leq)$, define $e_{\leq} : X \times X \longrightarrow L$ by $e_{\leq}(x, y) = 1$ if $x \leq y$ and $e_{\leq}(x, y) = 0$, otherwise, then $(X, e_{\leq})$ is a fuzzy poset. Let $\mathbf{2} = (\{0, 1\}, \wedge, 1)$, where $\{0, 1\}$ is a complete lattice with the ordering $0 < 1$. When $L = \mathbf{2}$, an L-ordered set is just a partially ordered set.

Example 2.3. (1) (The canonical L-order on L) Define $e_L : L \times L \longrightarrow L$ by $e_L(x, y) = x \to y$ for all $x, y \in L$. Then (L, e_L) is an L-ordered set.

(2) (Sub-L-ordered set) Let (X, e) be an L-ordered set and $Y \subseteq X$. Then $(Y, e|_Y)$ is an L-ordered set, where $e|_Y : Y \times Y \longrightarrow L$ is the restriction of e to $Y \times Y$, we will write $(Y, e|_Y)$ simply as (Y, e).

(3) Let X be a set. For $A, B \in L^X$, the subsethood degree [9] $\mathrm{sub}(A, B)$ of A in B is defined by $\mathrm{sub}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, sub) is an L-ordered set [3].

Definition 2.4. Let (X, e) be an L-ordered set and $A \in L^X$. An element $a \in X$ is called a join (resp., meet) of A, in symbols $a = \sqcup A$ or $a = \mathrm{sup}A$ (resp., $a = \sqcap A$ or $a = \mathrm{inf}A$), if $e(a, y) = \bigwedge_{x \in X}(A(x) \to e(x, y))$ for all $y \in X$ (resp., $e(y, a) = \bigwedge_{x \in X}(A(x) \to e(y, x))$ for all $y \in X$).

Definition 2.5. [21, 14] An L-ordered set is called complete if $\sqcup A$ exists for every $A \in L^X$, or equivalently, $\sqcap A$ exists for every A in L^X.

Complete L-ordered set can also be defined in a purely categorical manner. For more about complete L-ordered set, we refer to [2, 3, 14, 15, 28].

Let $L = \mathbf{2}$, (X, e) be an L-ordered set. Then for $A \subseteq X$, $\sqcup \chi_A$ (in (X, e)) exists iff $\bigvee A$ (in $(X, \leq_e)$) exists, in which case $\sqcup \chi_A = \bigvee A$. Thus, (X, e) is complete iff $(X, \leq_e)$ is complete.

Example 2.6. Let X be a set, then (L^X, sub) is complete. Particularly, (L, e_L) is complete. For their proofs, we refer to [13, 14, 15, 27].

For a map $f : X \longrightarrow Y$, the L-forward powerset operator $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and the L-backward powerset operator $f^{\leftarrow} : L^Y \longrightarrow L^X$ [19] are defined by $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) | f(x) = y\}$ $(\forall A \in L^X)$, $f^{\leftarrow}(B) = B \circ f$ $(\forall B \in L^Y)$. A monotone map $f : X \longrightarrow Y$ is said to *preserve joins* if, for every $A \in L^X$, $f(\sqcup A) = \sqcup f_L^{\rightarrow}(A)$, whenever $\sqcup A$ exists; Meet preserving maps are defined dually.

Recall that an L-category A is said to be *tensored* (resp., *cotensored*) [13, 15, 22] if for every $\alpha \in L$ and every $x \in A$, there is an element $\alpha \otimes x \in A$ (resp., $\alpha \rightarrowtail x \in A$) called the *tensor* (resp., *cotensor*) of α and x, such that $\alpha \to A(x, y) = A(\alpha \otimes x, y)$ (resp., $\alpha \to A(y, x) = A(y, \alpha \rightarrowtail x)$) for any $y \in L$. If X is a complete L-ordered set, then $(X, \leq_e)$ is a complete poset, and $\mathrm{sup}\mu = \bigvee_{x \in X}(\mu(x) \otimes x)$, $\mathrm{inf}\mu = \bigwedge_{x \in X}(\mu(x) \rightarrowtail x)$, for all $\mu \in L^X$. Particularly, for every subset $A \subseteq X$, we have $\sup\chi_A = \bigvee A, \inf\chi_A = \bigwedge A$. For more about tensored and cotensored L-categories, we refer to [15, 22].

Every complete L-ordered set is both tensored and cotensored. For instance, (L^X, sub) is both tensored and cotensored. For $\alpha \in L, \mu \in L^X, \alpha \otimes \mu = \alpha \wedge \mu, \alpha \rightarrowtail \mu = \alpha \to \mu$, where $(\alpha \wedge \mu)(x) = \alpha \wedge \mu(x), (\alpha \to \mu)(x) = \alpha \to \mu(x)$.

3. Basic Ideals of Strong L-topologies, L-frames and L-enriched Topological Systems Based on L-ordered Sets

In this section, firstly let us recall the definition of strong L-topologies introduced by Zhang [27] and the definition of L-frames introduced by Yao [25]. Then we will introduce the concept of L-enriched topological systems. All of the three concepts are based on L-ordered sets.

3.1. **Strong L-topologies.** We say an L-subset $A \in L^X$ is *finite* if the support set $\{x \in X : A(x) \neq 0\}$ of A is a finite set.

Definition 3.1. [27] A strong L-topology on a set X is a subset $\mathcal{T} \subseteq L^X$ such that:

(LT1) For every function $G : \mathcal{T} \longrightarrow L$,

$$\sup i_L^{\rightarrow}(G) = \bigvee_{\mu\in\mathcal{T}} G(\mu) \wedge \mu \in \mathcal{T}.$$

(LT2) For every function $G : \mathcal{T} \longrightarrow L$ with finite support,

$$\inf i_L^{\rightarrow}(G) = \bigwedge_{\mu\in\mathcal{T}} (G(\mu) \to \mu) \in \mathcal{T}.$$

Where, $i : \mathcal{T} \longrightarrow L^X$ is the inclusion function.

For a strong L-topology $\mathcal{T}$ on X, the pair $(X, \mathcal{T})$ is called a *strong L-topological space.*

Theorem 3.2. [27] *A subset $\mathcal{T} \subseteq L^X$ is a strong L-topology on X if and only if it satisfies the following axioms:*

(O1) $\bar{0} \in \mathcal{T}$.
(O2) $\bar{1} \in \mathcal{T}$.
(O3) $\bigvee_{j\in J} \lambda_j \in \mathcal{T}$ *for every family* $\{\lambda_j | j \in J\} \subseteq \mathcal{T}$.
(O4) $\lambda \wedge \mu \in \mathcal{T}$ *for every* $\lambda, \mu \in \mathcal{T}$.
(H1) $a \wedge \lambda \in \mathcal{T}$ *for every* $a \in L, \lambda \in \mathcal{T}$.
(H2) $a \to \lambda \in \mathcal{T}$ *for every* $a \in L, \lambda \in \mathcal{T}$.

According to the proof of the above theorem given in [27], we know (LT1) $\Longleftrightarrow$ (O1)+(O3)+(H1), (LT2) $\Longleftrightarrow$ (O2)+(O4)+(H2). In other words, (LT1) means that $(\mathcal{T}, \mathrm{sub})$ is closed in (L^X, sub) under the formation of (crisp) arbitrary joins and tensors, (LT2) means that $(\mathcal{T}, \mathrm{sub})$ is closed in (L^X, sub) under the formation of (crisp) finite meets and cotensors.

A subset $\delta \subseteq L^X$ satisfying (O1)—(O4) is a Chang-Goguen L-topology, or an L-topology in short, on X [4, 10]. For a strong L-topology $\mathcal{T}$, the axiom (H1) implies that every constant function $X \longrightarrow L$ belongs to $\mathcal{T}$. Such L-topologies were first introduced by Lowen [18] in the case $L = ([0, 1], \wedge, 1)$, and have been called stratified L-topologies.

A map $f : X \longrightarrow Y$ is called *continuous* respect to two given L-topological spaces (X, δ_X) and (Y, δ_Y) iff $f^{\leftarrow}(B) \in \delta_X$ for all $B \in \delta_Y$. The category of L-topological spaces with continuous maps is denoted by **L-Top** and by **SL-Top** its full subcategory of all stratified L-topological spaces. The category of strong L-topological spaces with continuous maps is denoted by **StL-Top**

3.2. **L-frames.**

Definition 3.3. [25] Let (P, e) be a complete L-ordered set and $\wedge$ is the meet operation in $(P, \leq_e)$. We call (P, e) an L-frame if for any $a \in P$, the map $\wedge_a : P \longrightarrow P$ $(b \mapsto a \wedge b)$ has a right adjoint, or equivalently, the following identity holds:

(FIDL) $\wedge_a(\sqcup S) = \sqcup(\wedge_a)^{\rightarrow}_{L}(S)$ $(\forall a \in P, \forall S \in L^P)$.

Remark 3.4. For every complete L-ordered set P and every $a \in P$, it can be checked that the map $\wedge_a : P \longrightarrow P$ $(b \mapsto a \wedge b)$ must be monotone. Indeed, for $x, y \in P$, we have $e(a \wedge x, a \wedge y) = e(a \wedge x, \sqcap \chi_{\{a,y\}}) = \bigwedge_{z \in P}(\chi_{\{a,y\}}(z) \rightarrow e(a \wedge x, z)) = e(a \wedge x, a) \wedge e(a \wedge x, y) = 1 \wedge e(a \wedge x, y) \geq e(a \wedge x, x) \wedge e(x, y) = e(x, y)$. Thus, the map $\wedge_a$ is monotone. So, the requirement that the map $\wedge_a$ is monotone in the original definition of the concept of L-frame given by Yao is omitted here.

Let $f : P \longrightarrow Q$ be a map between two L-frames (P, e_P), and (Q, e_Q). (1) f is said to be an *L-frame homomorphism* [25] if $f : (P, \leq_{e_P}) \longrightarrow (Q, \leq_{e_Q})$ is a crisp frame homomorphism and $f : (P, e_P) \longrightarrow (Q, e_Q)$ preserves joins of arbitrary L-subsets of P. (2) f is said to be a *strict L-frame homomorphism* if f preserves joins of arbitrary L-subsets and preserves meets of finite L-subsets of P. The category of L-frames and L-frame homomorphisms (resp., strict L-frame homomorphisms) is denoted by **L-Frm** (resp., **SL-Frm**). Their opposite categories are denoted by **L-Loc** and **SL-Loc** respectively.

Remark 3.5. Let $f : P \longrightarrow Q$ be a map between two L-frames (P, e_P) and (Q, e_Q).

(1) f is an L-frame homomorphism iff $f : (P, \leq_{e_P}) \longrightarrow (Q, \leq_{e_Q})$ is a crisp frame homomorphism and f preserves tensors in P.

(2) f is a strict L-frame homomorphism iff $f : (P, \leq_{e_P}) \longrightarrow (Q, \leq_{e_Q})$ is a crisp frame homomorphism and f preserves tensors and cotensors in P.

3.3. **L-enriched Topological Systems.**

Definition 3.6. An L-enriched topological system is an ordered triple $(X, P, \models)$, where X is a set, P is an L-frame and $\models : X \times P \longrightarrow L$ is an L-valued relation such that:

(1) For every $A \in L^P$, $(x \models \sqcup A) = \sqcup(x \models_{-})^{\rightarrow}_{L}(A)$.

(2) For every L-subset $B \in L^P$ with finite support, $(x \models \sqcap B) = \sqcap(x \models_{-})^{\rightarrow}_{L}(B)$.

Proposition 3.7. *Let X be a set, P an L-frame and $\models : X \times P \longrightarrow L$ an L-relation. Then the following conditions are equivalent:*

(1) $(X, P, \models)$ is an L-enriched topological system.

(2) $x \models_{-} : P \longrightarrow L$ is a strict L-frame homomorphism for every $x \in X$.

(3) $(X, P, \models)$ is an L-topological system in the sense of [6] *and $x \models_{-} : P \longrightarrow L$ preserves tensors and cotensores for every $x \in X$.*

Recently, motivated by questions arising from programming, J. T. Denniston, A. Melton and S. E. Rodabaugh [5] proposed an enriched category approach to study topological system. They introduced the notion of enriched topological system. Both the concept L-enriched topological system and the concept of enriched topological system in the sense of [5] are based on enriched category theory and both of them generalize the concept of L-topological system in the sense of [6]. But those two approaches are obviously different in many aspects.

Definition 3.8. Let $(X, P, \models)$ and $(Y, Q, \models')$ be two L-enriched topological systems. An ordered pair (f, φ) is said to be a continuous map from $(X, P, \models)$ to $(Y, Q, \models')$ (in symbols, $(f, \varphi) : (X, P, \models) \longrightarrow (Y, Q, \models')$) provided that:

(1) $f : X \longrightarrow Y$ is a function.

(2) $\varphi : P \longrightarrow Q$ is a strict L-locale homomorphism (which means that $\varphi^{op} : Q \longrightarrow P$ is a strict L-frame homomorphism).

(3) $(x \models \varphi^{op}(b)) = (f(x) \models' b)$ for every $x \in X, b \in Q$.

The class of all L-enriched topological systems as objects with continuous maps between them as morphisms form a category, which will be denoted by L**-EnTopSys**.

4. An Adjunction Between StL-Top and SL-Loc

In this section we will construct an enriched-categorical version of the Isbell-adjunction between topological spaces and locales. Firstly, let us recall the adjunction $\Omega_L \dashv Pt_L$: **S**L**-Top**$\rightharpoonup$ L**-Loc** between the category of stratified L-topological spaces and that of L-locales constructed by Yao [25].

Proposition 4.1. [25] *(1) If (X, δ) is a stratified L-topological space, then (δ, sub_X) is an L-frame.*

(2) If $f : (X, \delta) \longrightarrow (Y, \eta)$ is a morphism in **S**L**-Top**, *then $f_L^{\leftarrow} : (\eta, sub_Y) \longrightarrow (\delta, sub_X)$ is a morphism in* L**-Frm**.

For an L-frame (A, e), an L-frame homomorphism (resp., a strict L-frame homomorphism) from A to L is called an *L-fuzzy point* (resp., a *strict L-fuzzy point*) [17] of A. Let $\mathrm{pt}_L(A)$ (resp., $\mathrm{spt}_L(A)$) denote the set of all L-fuzzy points (resp., strict L-fuzzy points) of A. For $a \in A$, define $\Phi_L(a) : \mathrm{pt}_L(A) \longrightarrow L$ by $(p \mapsto p(a))$. Denote the restriction of $\Phi_L(a)$ to $\mathrm{spt}_L(A)$ by $S\Phi_L(a)$. In [25], it is proved that $\Phi_L(A) =: \{\Phi_L(a) | a \in A\}$ is a stratified L-topology on $\mathrm{pt}_L(A)$.

Recall that the functor Ω_L : **S**L**-Top**$\longrightarrow$ L**-Loc** is defined by $(f : (X, \delta) \longrightarrow (Y, \eta)) \mapsto ((f_L^{\leftarrow})^{op} : (\delta, \mathrm{sub}_X) \longrightarrow (\eta, \mathrm{sub}_Y))$. The functor Pt_L : L**-Loc** $\longrightarrow$**S**L**-Top** is defined by sending an L-locale (A, e) to the stratified L-topological space $(\mathrm{pt}_L(A), \Phi_L(A))$ and sending an L**-Loc** morphism $f : A \longrightarrow B$ to $\mathrm{Pt}_L(f) : \mathrm{pt}_L(A) \longrightarrow \mathrm{pt}_L(B)(p \mapsto p \circ f^{op})$.

Proposition 4.2. *If $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is a continuous map between strong topological spaces, then $f_L^{\leftarrow} : (\mathcal{T}_Y, sub_Y) \longrightarrow (\mathcal{T}_X, sub_X)$ is a strict L-frame homomorphism.*

Proof. By Proposition 4.1 and Remark 3.5, we only need to show that $f_L^{\leftarrow}$ preserves cotensors. Suppose $\alpha \in L, \mu \in \mathcal{T}_Y$. Then $f^{\leftarrow}(\alpha \overset{\mathcal{T}_Y}{\rightarrowtail} \mu) = f^{\leftarrow}(\alpha \overset{L^Y}{\rightarrowtail} \mu) = (\alpha \to \mu) \circ f = \alpha \to (\mu \circ f) = \alpha \overset{\mathcal{T}_X}{\rightarrowtail} f^{\leftarrow}(\mu)$. □

By this proposition we know the functor Ω_L : **S**L**-Top**$\longrightarrow$ L**-Loc** restricts to a functor $S\Omega_L$: **St**L**-Top**$\longrightarrow$ **S**L**-Loc**.

Proposition 4.3. *Let (P, e) be an L-frame. Then*

(1) $S\Phi_L(P) = \{S\Phi_L(a) | a \in P\}$ is a strong L-topology on $spt_L(P)$.

(2) $(\Phi_L(P), sub_{spt_L(P)})$ is an L-frame and the map $S\Phi_L : P \longrightarrow S\Phi_L(P)$ $(a \mapsto S\Phi_L(a))$ is a strict L-frame homomorphism.

Proof. (1) It is easy to check that $S\Phi_L(P)$ satisfies (O1)—(O4).

(H1) Suppose $\alpha \in L$, $S\Phi_L(a) \in S\Phi_L(P)$. Then, for every $f \in \mathrm{spt}_L(P)$, $(\alpha \wedge S\Phi_L(a))(f) = \alpha \wedge (S\Phi_L(a)(f)) = \alpha \wedge f(a) = f(\alpha \otimes a) = S\Phi_L(\alpha \otimes a)(f)$. Thus, $\alpha \wedge S\Phi_L(a) = S\Phi_L(\alpha \otimes a) \in S\Phi_L(P)$.

Similarly, we can check that $S\Phi_L(P)$ satisfies (H2). Thus, by Theorem 3.2 we know $S\Phi_L(P)$ is a strong L-topology on $\mathrm{spt}_L(P)$.

(2) By Proposition 4.1 we know $(\Phi_L(P), \mathrm{sub}_{\mathrm{spt}_L(P)})$ is an L-frame. By a similar arguments as in the proof of part (1) we can check that the map $S\Phi_L$ is a strict L-frame homomorphism. □

Proposition 4.4. *If $f : A \longrightarrow B$ is a* **S**L**-Loc** *morphism, then $SPt_L(f) : spt_L(A) \longrightarrow spt_L(B) (p \mapsto p \circ f^{op})$ is a continuous map between strong topological spaces $(spt_L(A), S\Phi_L(A))$ and $(spt_L(B), S\Phi_L(B))$.*

Proof. This is similar to the proof of the fact that $\mathrm{Pt}_L(f)$ is a continuous map given in [25]. □

By the above two propositions, we know the assignment SPt_L which associates an L-locale A with the strict L-topological space $(\mathrm{spt}_L(A), S\Phi_L(A))$ and a morphism $f : A \longrightarrow B$ in **S**L**-Loc** with the map $\mathrm{SPt}_L(f)$ is a functor **S**L**-Loc** $\longrightarrow$ **St**L**-Top**. Thus, we can obtain the following theorem [17].

Theorem 4.5. $S\Omega_L \dashv SPt_L$: **St**L**-Top**$\rightharpoonup$ **S**L**-Loc**.

Proof. This is similar to the proof of the fact that $\Omega_L \dashv Pt_L$. □

5. An Adjunction Between L-EnTopSys and StL-Top

In this section, we will construct an enriched version of the adjunction between the category of topological systems and the category of topological spaces. It is useful to recall the adjunction $E_L \dashv Ext_L$: L**-Top** $\rightharpoonup$ L**-TopSys** between the category of L-topological spaces and the category of L-topological systems constructed in [6].

The functor E_L : L**-Top** $\longrightarrow$ L**-TopSys** is defined as follows:

For an L-topological space (X, δ), $E_L(X, \delta) = (X, \delta, \models)$, where $(x \models \mu) = \mu(x)$, and for a morphism $f : (X, \delta) \longrightarrow (Y, \eta)$, $E_L(f) : (X, \delta, \models) \longrightarrow (Y, \eta, \models)$ is given by

$E_L(f) = (f, ((f_L^{\leftarrow})_{|\eta})^{op})$, where $(f_L^{\leftarrow})_{|\eta} : \eta \longrightarrow \delta$ is the restriction and co-restriction of $f_L^{\leftarrow}$.

The functor $Ext_L : L$-**TopSys**$\longrightarrow L$-**Top** is defined as follows:

For an L-topological system $(X, A, \models)$, $Ext_L(X, A, \models) = (X, ext_L(A))$, where $ext_L : A \longrightarrow L^X$ is given by $ext_L(a)(x) = (x \models a)$; and for a morphism $(f, \varphi) : (X, A, \models_1) \longrightarrow (Y, B, \models_2)$, $Ext_L(f, \varphi) = f : (X, ext_L(A)) \longrightarrow (Y, ext_L(B))$.

Proposition 5.1. *If $(X, \mathcal{T})$ is a strong L-topological space, then $(X, \mathcal{T}, \models)$ is an L-enriched topological system, where $\models: X \times \mathcal{T} \longrightarrow L$ is defined by $(x \models \mu) = \mu(x)$ for every $x \in X$ and $\mu \in \mathcal{T}$.*

Proof. By Proposition 3.7, we only need to show that $x \models\_ : P \longrightarrow L$ preserves tensors and cotensores for every $x \in X$. These are direct consequences of axioms (H1) and (H2), and we leave the details to the reader. □

Proposition 5.2. *If $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is a continuous map between two strong L-topological spaces, then $E_{L\text{-}en}(f) =: (f, ((f_L^{\leftarrow})_{|\eta})^{op}) : (X, \mathcal{T}_X, \models) \longrightarrow (Y, \mathcal{T}_Y, \models)$ is a continuous map between L-enriched topological systems.*

Proof. If we consider $(X, \mathcal{T}_X, \models)$ and $(Y, \mathcal{T}_Y, \models)$ as L-topological systems, then $(f, ((f_L^{\leftarrow})_{|\eta})^{op})$ is a continuous map between them. Thus, by the definition of continuous map between L-enriched topological system and Remark 3.5, we need to show that $(f_L^{\leftarrow})_{|\eta} : \mathcal{T}_Y \longrightarrow \mathcal{T}_X$ preserves tensors and cotensors.

Suppose $\alpha \in L, \mu \in \mathcal{T}_Y$. Then, for every $x \in X$, we have $(f_L^{\leftarrow})_{|\eta}(\alpha \wedge \mu)(x) = ((\alpha \wedge \mu) \circ f)(x) = (\alpha \wedge \mu)(f(x)) = \alpha \wedge (\mu(f(x))) = \alpha \wedge (f_L^{\leftarrow})_{|\eta}(\mu)(x) = (\alpha \wedge (f_L^{\leftarrow})_{|\eta}(\mu))(x)$. Thus $(f_L^{\leftarrow})_{|\eta}$ preserves tensors. That it preserves cotensors is similar. □

Proposition 5.3. *If $(X, P, \models)$ is an L-enriched topological system, then*

$$Ext_{L\text{-}en}(X, P, \models) =: (X, ext_{L\text{-}en}(P))$$

is a strong L-topological space, where $ext_{L\text{-}en} : P \longrightarrow L^X$ is defined by $ext_{L\text{-}en}(a)(x) = (x \models a)$ for every $x \in X$ and $a \in P$.

Proof. Since every L-enriched topological system can be considered as an L-topological system, we know from the functor Ext_L that $(X, ext_{L\text{-}en}(P))$ is an L-topological space. Suppose $\alpha \in L, a \in P$. Then, for every $x \in X$, $(\alpha \wedge ext_{L\text{-}en}(a))(x) = \alpha \wedge ext_{L\text{-}en}(a)(x) = \alpha \wedge (x \models a) = (x \models \alpha \otimes a) = ext_{L\text{-}en}(\alpha \otimes a)(x)$. Thus $\alpha \wedge ext_{L\text{-}en}(a) = ext_{L\text{-}en}(\alpha \otimes a) \in ext_{L\text{-}en}(P)$. A similar argument implies that $\alpha \to ext_{L\text{-}en}(a) = ext_{L\text{-}en}(\alpha \rightarrowtail a) \in ext_{L\text{-}en}(P)$. Thus $Ext_{L\text{-}en}(X, P, \models)$ is a strong L-topological space. □

By Propositions 5.1 and 5.2, the functor $E_L : L$-**Top** $\longrightarrow L$-**TopSys** restricts to a functor $E_{L\text{-}en}$: **St**L-**Top**$\longrightarrow L$-**EnTopSys**. By Proposition 5.3, the functor $Ext_L : L$-**TopSys**$\longrightarrow L$-**Top** restricts to a functor $Ext_{L\text{-}en} : L$-**EnTopSys** $\longrightarrow$**St**L-**Top**. Thus, the restrictions of the adjunction $E_L \dashv Ext_L$ establish an enriched-categorical version of the adjunction between the category of topological spaces and the category of topological systems.

Theorem 5.4. $E_{L\text{-}en} \dashv Ext_{L\text{-}en}$: **St**L-**Top**$\rightharpoonup L$-**EnTopSys**.

6. An Adjunction Between L-EnTopSys and SL-Loc

In this section, we will give an enriched-categorical version of the adjunction $\Omega_V \dashv E_{\mathbf{Loc}}$ between the category of topological systems and the category locales (see [6, 23]).

Proposition 6.1. *(1) If P is an L-frame, then $(spt_L(P), P, \models)$ is an L-enriched topological system, where $\models$: $spt_L(P) \times P \longrightarrow L$ is defined by $(f \models x) = f(x)$.*

(2) Suppose $\varphi : P \longrightarrow Q$ is an **SL-Loc** *morphism. Define $spt(\varphi) : spt_L(P) \longrightarrow spt_L(Q)$ by $spt(\varphi)(f) = f \circ \varphi^{op}$ for every $f \in spt_L(P)$. Then*

$$(spt(\varphi), \varphi) : (spt_L(P), P, \models) \longrightarrow (spt_L(Q), Q, \ \models)$$

is a continuous map.

Proof. (1) For every $f \in \mathrm{spt}_L(P)$, the map $(f \models\_) : P \longrightarrow L$ is just the map f, thus $(f \models\_) : P \longrightarrow L$ is a strict L-frame homomorphism. From Proposition 3.7, we conclude that $(\mathrm{spt}_L(P), P, \models)$ is an L-enriched topological system.

(2) Since the composition of two strict L-frame homomorphisms is also a strict L-frame homomorphism, the map $\mathrm{spt}(\varphi)$ is well-defined. For any $f \in \mathrm{spt}_L(P)$ and any $y \in Q$, $(\mathrm{spt}(\varphi)(f) \models y) = (f \circ \varphi^{op} \models y) = f(\varphi^{op}(y)) = (f \models \varphi^{op}(y))$. This completes the proof. □

By Proposition 6.1, we immediately have

Proposition 6.2. *The assignment $E_{\mathbf{S}L\text{-}\mathbf{Loc}}$ which associates with an L-locale P the L-enriched topological system $(spt_L(P), P, \models)$ and with an* **SL-Loc** *morphism $\varphi : P \longrightarrow Q$ the continuous map $E_{\mathbf{S}L\text{-}\mathbf{Loc}}(\varphi) = (spt(\varphi), \varphi) : (spt_L(P), P, \models) \longrightarrow (spt_L(Q), Q, \models)$ in* **L-EnTopSys** *is a functor* **SL-Loc$\longrightarrow$$L$-EnTopSys**.

Proposition 6.3. *The assignment L-en-Ω_V which associates with an L-enriched topological system $(X, P, \models)$ the L-locale P and with a continuous map $(f, \varphi) : (X, P, \models_1) \longrightarrow (Y, Q, \models_2)$ in* **L-EnTopSys** *the* **SL-Loc** *morphism $\varphi : P \longrightarrow Q$ is a functor* **L-EnTopSys $\longrightarrow$ SL-Loc**.

Proof. This follows immediately from the definition of L-enriched topological system. □

Theorem 6.4. *L-en-$\Omega_V \dashv E_{\mathbf{S}L\text{-}\mathbf{Loc}}$:* **L-EnTopSys $\rightharpoonup$ SL-Loc**.

Proof. Let $(X, P, \models)$ be an L-enriched topological system. Define $f : X \longrightarrow \mathrm{spt}_L(P)$ by $f(x) = (x \models\_)$ for every $x \in X$. Then $(f, \mathrm{id}_P) : (X, P, \models) \longrightarrow (\mathrm{spt}_L(P), P, \models)$ is a continuous map. Let Q be an L-frame and $(g, \varphi) : (X, P, \models) \longrightarrow (\mathrm{spt}_L(Q), Q, \models)$ a continuous map. Then, for every $b \in Q$ and every $x \in X$, $((\mathrm{spt}(\varphi) \circ f)(x))(b) = (\mathrm{spt}(\varphi)(f(x)))(b) = (f(x) \circ \varphi^{op})(b) = f(x)(\varphi^{op}(b)) = (x \models \varphi^{op}(b)) = (g(x) \models b) = g(x)(b)$, i.e., $g = \mathrm{spt}(\varphi) \circ f$. Thus $(g, \varphi) = (\mathrm{spt}(\varphi), \varphi) \circ (f, \mathrm{id}_P)$. This shows that $(f, \mathrm{id}_P) : (X, P, \models) \longrightarrow (\mathrm{spt}_L(P), P, \models)$ is an $E_{\mathbf{S}L\text{-}\mathbf{Loc}}$-universal arrow. Thus, $E_{\mathbf{S}L\text{-}\mathbf{Loc}}$ is an adjoint for L-en-Ω_V. □

It is easy to check that $S\Omega_L = L\text{-}en\text{-}\Omega_V \circ E_{L\text{-}en}$, $SPt_L = Ext_{L\text{-}en} \circ E_{\mathbf{S}L\text{-}\mathbf{Loc}}$, so we have

Theorem 6.5. *As adjunctions* $[S\Omega_L \dashv SPt_L] = [L\text{-}en\text{-}\Omega_V \circ E_{L\text{-}en} \dashv Ext_{L\text{-}en} \circ E_{\mathbf{S}L\text{-}\mathbf{Loc}}]$.

When $L = \mathbf{2}$, the categories **St**L**-Top**, **S**L**-Loc** and L**-EnTopSys** are the categories **Top**, **Loc** and **TopSys** respectively. Thus, the three classical adjunctions among these categories [6, 23] can be recovered from the results in this paper.

7. Conclusions

In this paper, we establish a unified enriched-categorical version of the classical adjunctions among the categories **Top**, **Loc** and **TopSys**. Several concepts based on enriched category theory that are studied individually in the literature previously have been connected with each other by adjunctions. Thus we obtain a unified enriched-categorical approach to treat these concepts.

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References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.
[2] R. Bělohlávek, *Concept lattices and order in fuzzy logic*, Annals of Pure and Applied Logic, **128(1-3)** (2004), 277-298.
[3] R. Bělohlávek, *Fuzzy relational systems: foundations and principles*, Kluwer Academic/Plenum Publishers, New York, 2002.
[4] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24(1)** (1968), 182-190.
[5] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Enriched topological systems and variable-basis enriched functors*, In: U. Höhle, L. N. Stout, E. P. Klement, Abstracts of the 33th Linz Seminar on Fuzzy Set Theory, Universitätsdirektion Johannes Kepler Universität (Linz, Austria), 14-18 February (2012), 16-20.
[6] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Interweaving algebra and topology: Lattice-valued topological systems*, Fuzzy Sets and Systems, **192** (2012), 58-103.
[7] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued topological systems*, In: U. Bodenhofer, B. De Baets, E. P. Klement, Abstracts of the 30th Linz Seminar, Universitätsdirektion Johannes Kepler Universität (Linz, Austria), 3-7 February (2009), 24-31.
[8] L. Fan, *A new approach to quantitative domain theory*, Electronic Notes in Theoretical Computer Science, **45** (2001), 77-87.
[9] J. A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18(1)** (1967), 145-174.
[10] J. A. Goguen, *The fuzzy tychonoff theorem*, J. Math. Anal. Appl., **43(3)** (1973), 734-742.
[11] D. Hofmann and P. L. Waszkiewicz, *Approximation in quantale-enriched categories*, Topology and its Applications, **158(8)** (2011), 963-977.
[12] P. T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982.
[13] G. M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Notes Series 64, Cambridge University Press, 1982.
[14] H. Lai and D. Zhang, *Complete and directed complete* Ω*-categories*, Theoret. Comput. Sci., **388(1-3)** (2007), 1-25.
[15] H. Lai and D. Zhang, *Many-valued complete distributivity*, arXiv:math/0603590v2 [math.CT], 12 May 2006.

[16] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano, **43(1)** (1973), 135-166.

[17] M. Liu, *On some related problems in Ω-categories and fuzzy domains*, Ph.D. Thesis, Shannxi normal university, Xi'an, China, 2013.

[18] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl., **56(3)** (1976), 623-633.

[19] S. E. Rodabaugh, *Powerset operator foundations for poslat fuzzy set theories and topologies*, in: U. Höhle, S. E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Boston, Dordrecht, London, (1999), 91-116.

[20] K. I. Rosenthal, *Quantales and Their Applications*, Pitman Research Notes in Mathematics Series, Longman, **234** (1990).

[21] I. Stubbe, *Categorical structures enriched in a quantaloid: Categories, distributors and functors*, Theory Appl. Categ., **14(1)** (2005), 1-45.

[22] I. Stubbe, *Categorical structures enriched in a quantaloid: Tensored and cotensored categories*, Theory Appl. Categ., **16(14)** (2006), 283-306.

[23] S. J. Vickers, *Topology via logic*, Cambridge University Press, Cambridge, 1989.

[24] K. R. Wagner, *Solving recursive domain equations with enriched categories*, Ph.D. Thesis, Carnegie Mellon University, Tech. Report CMU-CS-94-159, July, 1994.

[25] W. Yao, *An approach to fuzzy frames via fuzzy posets*, Fuzzy Sets and Systems, **166(1)** (2011), 75-89.

[26] W. Yao, *Quantitative domains via fuzzy sets: Part I: continuity of fuzzy directed-complete poset*, Fuzzy Sets and Systems, **161(7)** (2010), 983-987.

[27] D. Zhang, *An enriched category approach to many valued topology*, Fuzzy Sets and Systems, **158(4)** (2007), 349-366.

[28] Q. Y. Zhang, W. X. Xie and L. Fan, *Fuzzy complete lattices*, Fuzzy Sets and Systems, **160(16)** (2009), 2275-2291.

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