APPROXIMATE FIXED POINT IN FUZZY NORMED SPACES FOR NONLINEAR MAPS

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Abstract. We define approximate fixed point in fuzzy norm spaces and prove the existence theorems, we also consider approximate pair constructive mapping and show its relation with approximate fuzzy fixed point.

1. Introduction

Chitra and Mordeson [8] defined fuzzy norm and thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [2], [3], [4].

Definition 1.1. Let U be a linear space on $\mathbf{R}$. A function $N : U \times \mathbf{R} \to [0,1]$ is called a fuzzy norm if for every $x, u \in U$ and $c \in \mathbf{R}$ the following properties are satisfied.

(F_{N1}) $N(x,t) = 0$, for every $t \in \mathbf{R}^- \cup \{0\}$,

(F_{N2}) $N(x,t) = 1$, if and only if $x = 0$ for every $t \in \mathbf{R}^+$,

(F_{N3}) $N(cx,t) = N(x, \frac{t}{|c|})$ for every $c \neq 0$ and t $\in \mathbf{R}^+$,

(F_{N4}) $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$, for every $s,t \in \mathbf{R}^+$,

(F_{N5}) the function $N(x,.)$ is nondecreasing on $\mathbf{R}$, and

$$\lim_{t\to\infty} N(x,t) = 1.$$

The pair (U,N) is called a fuzzy norm space. Sometimes, we need two additional conditions as follows :

(F_{N6}) $\forall t \in \mathbf{R}^+ \; N(x,t) > 0 \; \Rightarrow x = 0.$

(F_{N7}) The function $N(x,.)$ is continuous, for every $x \neq 0$, and is strictly increasing on the subset

$$\{t : \; 0 < N(x,t) < 1\}$$

is strictly increasing.
Let (U,N) be a fuzzy norm space. For all $\alpha \in (0,1)$, we define α norm on U as follows :

$$\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\} \; for \; every \; x \in U.$$

Received: March 2011; Revised: November 2011; Accepted: January 2012
Key words and phrases: Fuzzy norm space, F^z-approximate fixed point, Diameter F^z-approximate fixed point.

Then $\{\|x\|_\alpha : \ \alpha \in (0,1]\}$ is an ascending family of normed on U and are called $\alpha - norm$ on U corresponding to the fuzzy norm N on U. Here are a lemma and example which will be used in this paper.

Lemma 1.2. [1] *Let (U, N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7}. Define the function $N' : U \times \boldsymbol{R} \to [0,1]$ as follows:*

$$N'(x,t) = \begin{cases} \vee\{\alpha \in (0,1) : \|x\|_\alpha \le t\} & (x,t) \neq (0,0) \\ 0 & (x,t) = (0,0) \end{cases}$$

Then
a)N' is a fuzzy norm on U.
b) $N = N'$.

Lemma 1.3. [1] *Let (U, N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7}. and $\{x_n\} \subseteq U$, Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ if and only if*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$$

for every $\alpha \in (0,1)$.

Note that the sequence $\{x_n\} \subseteq U$ converges if there exists $x \in U$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1, for\ every\ t \in \boldsymbol{R}^+.$$

In this case x is called the limit of $\{x_n\}$.

Example 1.4. [1] Let V be a Real or Complex vector space and let N be defined on $V \times R$ as follows :

$$N(x,t) = \begin{cases} 1 & t > |x| \\ 0 & t \le |x| \end{cases}$$

for all $x \in V$ and $t \in R$. Then (N, V) is a fuzzy norm space and the function N satisfies condition F_{N6} and $\|x\|_\alpha = |x|$, for every $\alpha \in (0,1)$.

Remark 1.5. Let (U, N) be a fuzzy norm space and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ the set of all $\alpha-$norms on U. For two subsets A and B of U, we consider :

$$\delta(A,B) = \wedge\{\|x-y\|_\alpha : x \in A, y \in B,\ \alpha \in (0,1)\}.$$

Definition 1.6. Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$.

For a given $\epsilon > 0$, $x \in A \cup B$ is said to be a F^z-approximate fixed point for T if for some $\alpha \in (0,1)$

$$\|x - Tx\|_\alpha \le \delta(A,B) + \epsilon.$$

Remark 1.7. In the rest of the paper for a given $\epsilon > 0$, we will denote the set of all F^z-approximate fixed points of T, by

$$F^z_T(A,B) = \{x \in A \cup B : \|x - Tx\|_\alpha \le \delta(A,B) + \epsilon,\ for\ some\ \alpha \in (0,1)\}.$$

2. F^z-Approximate Fixed Point Maps

Proposition 2.1. *Let (U,N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A,B be nonempty subsets of U and $T : A \cup B \to U$. If for $x \in A \cup B$ and $\alpha \in (0,1)$*

$$Lim_{n\to\infty}\|T^n x - T^{n+1}x\|_\alpha = \delta(A,B),$$

then there exists a F^z-approximate fixed point in $A \cup B$.

Proof. Suppose $x \in A \cup B$ and $\epsilon > 0$. Since

$$Lim_{n\to\infty}\|T^n x - T^{n+1}x\|_\alpha = \delta(A,B).$$

$$\exists N_0 > 0 \ s.t. \ \forall n \geq N_0 : \ \|T^n x - T^{n+1}x\|_\alpha < \delta(A,B) + \epsilon.$$

If $n = N_0$, then for some $\alpha \in (0,1)$

$$\|T^{N_0}x - T(T^{N_0}x)\|_\alpha < \delta(A,B) + \epsilon.$$

Therefore, $T^{N_0}x \in F^z_T(A,B)$ and $F^z_T(A,B) \neq \emptyset$. Hence there exists a F^z-approximate fixed point in $A \cup B$. □

Proposition 2.2. *Let (U,N) be a fuzzy norm space such that satisfy conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. If for $x,y \in A \cup B$*

$$\|Tx - Ty\|_\alpha \leq m(\|x-y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Ty\|_\alpha) + k\delta(A,B), \ \forall \alpha \in (0,1)$$

where $m,l,k \geq 0$ and $m + 2l + k < 1$. Then there exists a F^z-approximate fixed point in $A \cup B$.

Proof. If $x \in A \cup B$, then for some $\alpha \in (0,1)$ we have:

$$\|Tx - T^2x\|_\alpha \leq m(\|x-y\|_\alpha) + l(\|x - Tx\|_\alpha + \|Tx - T^2x\|_\alpha) + k\delta(A,B).$$

Therefore

$$\|Tx - T^2x\|_\alpha \leq \frac{m+l}{1-l}\|x - Tx\|_\alpha + \frac{k}{1-l}\delta(A,B).$$

Now if $p = \frac{m+l}{1-l}$, Then

$$\|Tx - T^2x\|_\alpha \leq p\|x - Tx\|_\alpha + (1-p)\delta(A,B)$$

and $\|T^2x - T^3x\|_\alpha \leq p^2\|x - Tx\|_\alpha + (1-p^2)\delta(A,B)$, hence,

$$\|T^n x - T^{n+1}x\|_\alpha \leq p^n\|x - Tx\|_\alpha + (1-p^n)\delta(A,B).$$

since $p < 1$, for $\alpha \in (0,1)$, we have

$$\|T^n x - T^{n+1}x\|_\alpha \to \delta(A,B) \ as \ n \to \infty.$$

Therefore, by Proposition 2.1, there exists a F^z-approximate fixed point in $A \cup B$. □

Definition 2.3. Let (U, N) be a fuzzy norm space which satisfy conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. For some $\alpha \in (0,1)$ we define diameter $F_T^z(A, B)$ as follows :

$$diam(F_T^z(A,B)) = \vee\{\|x-y\|_\alpha : x, y \in F_T^z(A,B)\}.$$

Proposition 2.4. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$. If there exists $k \in [0,1]$ such that $\|Tx - Ty\|_\alpha \leq k\|x-y\|_\alpha$ for some $\alpha \in (0,1)$, then*

$$diam(F_T^z(A,B)) \leq \frac{2\epsilon}{1-k} + \frac{2\delta(A,B)}{1-k}.$$

Proof. If $x, y \in A \cup B$, then for some $\alpha \in (0,1)$ we have

$$\begin{aligned} \|x-y\|_\alpha &\leq \|x-Tx\|_\alpha + \|Tx-Ty\|_\alpha + \|Ty-y\|_\alpha \\ &\leq \epsilon_1 + k\|x-y\|_\alpha + 2\delta(A,B) + \epsilon_2. \end{aligned}$$

By taking $\epsilon = Max\{\epsilon_1, \epsilon_2\}$, we have

$$\|x-y\|_\alpha \leq \frac{2\epsilon}{1-k} + \frac{2\delta(A,B)}{1-k}.$$

Therefore $diam(F_T^z(A,B)) \leq \frac{2\epsilon}{1-k} + \frac{2\delta(A,B)}{1-k}$. □

Definition 2.5. Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U and $T : A \cup B \to U$, $S : A \cup B \to U$. A point (x, y) in $A \times B$ is said a F^z-approximate fixed point for (T, S), if for some $\alpha \in (0,1)$ there exists $\epsilon > 0$ such that

$$\|(Tx, Sy)\|_\alpha \leq \delta(A,B) + \epsilon.$$

Put :

$$F_{(T,S)}^z(A,B) = \{(x,y) \in A \times B : \|x - Tx)\|_\alpha \leq \delta(A,B) + \epsilon \ for\ some\ \alpha \in (0,1)\}.$$

Proposition 2.6. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0,1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$ and $S : A \cup B \to U$. If for $(x, y) \in A \times B$ and some $\alpha \in (0,1)$*

$$Lim_{n\to\infty}\|T^n x - S^n y\|_\alpha = \delta(A,B),$$

then there exists a F^z-approximate fixed point in $A \times B$.

Proof. For $\epsilon > 0$ and $\alpha \in (0,1)$, suppose $(x, y) \in A \times B$. Since

$$Lim_{n\to\infty}\|T^n x - S^n y\|_\alpha = \delta(A,B),$$

$$\exists N_0 > 0 \ s.t.\ \forall n \geq N_0 : \ \|T^n x - S^n y\|_\alpha < \delta(A,B) + \epsilon.$$

If $n = N_0$, then for some $\alpha \in (0, 1)$

$$\|T(T^{N_0-1}x - S(S^{N_0-1}y)\|_\alpha < \delta(A, B) + \epsilon.$$

Now by takeing $x_0 = T^{N_0-1}(x)$ and $y_0 = S^{N_0-1}(y)$, we have

$$\|T(x_0) - S(y_0)\|_\alpha < \delta(A, B) + \epsilon,$$

hence $F^z_{(T,S)}(A, B) \neq \emptyset$. Therefore, there exists a F^z-approximate fixed point in $A \times B$. □

Proposition 2.7. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$ and $S : A \cup B \to U$. For any $(x, y) \in A \times B$ and for some $\alpha \in (0, 1)$:*

$$\|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A, B),$$

where $m, l, k \geq 0$ and $m + 2l + k < 1$. If x is a F^z-approximate fixed point for T and y a F^z-approximate fixed point for S, then there exists a F^z-approximate fixed point in $A \times B$.

Proof. If $(x, y) \in A \times B$, then for some $\alpha \in (0, 1)$ we have :

$$\|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A, B).$$

Therefore

$$\|Tx - S(Tx)\|_\alpha \leq \frac{m + l}{1 - l}\|x - Tx\|_\alpha + \frac{k}{1 - l}\delta(A, B).$$

Now if $p = \frac{m+l}{1-l}$, then

$$\|Tx - S(Tx)\|_\alpha \leq p\|x - Tx\|_\alpha + (1 - p)\delta(A, B) \ (*)$$

and

$$\|Sy - T(Sy)\|_\alpha \leq p\|x - Tx\|_\alpha + (1 - p)\delta(A, B) \ (**).$$

If x is a F^z-approximate fixed point for T, then there exists an $\epsilon > 0$ and by $(*)$ for some $\alpha \in (0, 1)$:

$$\begin{aligned}\|Tx - S(Tx)\|_\alpha &\leq p\|x - Tx\|_\alpha + (1 - p)\delta(A, B)\\ &\leq p(\delta(A, B) + \epsilon) + (1 - p)\delta(A, B)\\ &= \delta(A, B) + p\epsilon\\ &< \delta(A, B) + \epsilon.\end{aligned}$$

Hence $(x, Tx) \in F^z_{(T,S)}(A, B)$, and
if y is a F^z-approximate fixed point for S, then there exists an $\epsilon > 0$ and by $(**)$ for some $\alpha \in (0, 1)$:

$$\begin{aligned}\|Ty - T(Sy)\|_\alpha &\leq p\|y - Sy\|_\alpha + (1 - p)\delta(A, B)\\ &\leq p(\delta(A, B) + \epsilon) + (1 - p)\delta(A, B)\\ &= \delta(A, B) + p\epsilon\\ &< \delta(A, B) + \epsilon.\end{aligned}$$

Hence $(y, Sy) \in F^z_{(T,S)}(A, B)$. Therefore there exists a F^z-approximate fixed point in $A \times B$. □

Proposition 2.8. *Let (U, N) be a fuzzy norm space which satisfies conditions F_{N6} and F_{N7} and let $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ be the set of $\alpha-$norms defined on U. Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$. is a continuous map. For any $(x, y) \in A \times B$ and some $\alpha \in (0, 1)$*

$$\|Tx - Sy\|_\alpha \le m(\|x - y\|_\alpha) + k\delta(A, B),$$

where $m, k \ge 0$ and $m + k = 1$ and the sequences $\{x_n\}, \{y_n\}$ be defined as follows :

$$x_{n+1} = Sy_n,\ y_{n+1} = Tx_n\ n \ge 0\ .$$

If $\{x_n\}$ has a convergent subsequence in A, then there exists a point $x_k \in A$ such that

$$\|x_k - Tx_k\|_\alpha = \delta(A, B)\ \alpha \in (0, 1).$$

Proof. We have

$$\begin{aligned} \|x_{n+1} - y_{n+1}\|_\alpha &= \|Tx_n - Sy_n\|_\alpha \\ &\le m\|x_n - y_n\|_\alpha + (k)\delta(A, B) \\ &\le ... \\ &\le m^{n+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^n)(k)\delta(A, B) \end{aligned}$$

If $\{x_{n_p}\}_{p\ge1}$ be convergent to $x_k \in A$, then

$$\|x_{n+1} - y_{n+1}\|_\alpha \le m^{n_p+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^{n_p})(k)\delta(A, B).$$

Since T is continuous, we have,

$$\|x_{n_p+1} - Tx_{n_p}\| \to \frac{k}{1-m}\delta(A, B)\ \alpha \in (0, 1).$$

Therefore, $\|x_k - Tx_k\| = \delta(A, B)$. □

Example 2.9. Let us consider the linear space $U = C[0, 1]$, of all continuous real valued functions on [0,1] with the usual linear operations.
Consider two norms on $C[0, 1]$ defined by

$$\|x\|_0 = \{\int_{x_0}^{1} (x(t))^2 dt\}^{\frac{1}{2}}$$

$$\|x\|_1 = \sup_{0\le t\le 1} |x(t)|.$$

Define two fuzzy norms N_1 and N_2 by

$$N_1(x, t) = \begin{cases} 1 & t > \|x\|_0 \\ \\ 0 & t \le \|x\|_0 \end{cases}$$

and

$$N_2(x,t) = \begin{cases} 1 & t > \|x\|_1 \\ \frac{1}{2} & \|x\|_0 \leq t \leq \|x\|_1 \\ 0 & t \leq \|x\|_0 \end{cases}$$

For the fuzzy norm N_1, its $\alpha-$norm $\|.\|_\alpha^1$ is given by $\|.\|_\alpha^1 = \|x\|_0 \ \forall \alpha \in (0,1)$.

For the fuzzy norm N_2, its $\alpha-$norm $\|.\|_\alpha^2$ is given by

$$\|x\|_\alpha = \begin{cases} \|x\|_0 & 0 < \alpha \leq \frac{1}{2} \\ \|x\|_1 & \frac{1}{2} < \alpha < 1 \end{cases}$$

Suppose A and B be nonempty subsets of U and $T : A \cup B \to U$. We consider $T(x) = x + 1$, $\forall x \in A \cup B$. It can easily be seen that for $\frac{1}{2} < \alpha < 1$, and $x, y \in X$.

$$\|Tx - Ty\|_\alpha \leq \frac{1}{2}\|x - y\|_\alpha$$

However, T has no any fixed point. But by the Proposition 2.2, for some $\epsilon > 0$, T has a F^z-approximate fixed point in $A \times B$. That is, there exists $x_0 \in U$ such that

$$\|T(x_0) - x_0\|_\alpha \leq \epsilon.$$

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