 APPROXIMATE FIXED POINT IN FUZZY NORMED SPACES  
FOR NONLINEAR MAPS  

S. A. M. MOHSENIALHOSSEINI, H. MAZAHERI AND M. A. DEHGHAN  

Abstract. We define approximate fixed point in fuzzy norm spaces and prove the existence theorems, we also consider approximate pair constructive mapping and show its relation with approximate fuzzy fixed point.  

1. Introduction  
Chitra and Mordeson [8] defined fuzzy norm and thereafter the concept of fuzzy norm space has been introduced and generalized in different ways by Bag and Samanta in [2], [3], [4].  

Definition 1.1. Let $U$ be a linear space on $\mathbb{R}$. A function $N : U \times \mathbb{R} \rightarrow [0,1]$ is called a fuzzy norm if for every $x,u \in U$ and $c \in \mathbb{R}$ the following properties are satisfied.  

$\begin{align*}  
(F_{N1}) \quad N(x,t) &= 0, \text{ for every } t \in \mathbb{R}^- \cup \{0\},  
(F_{N2}) \quad N(x,t) &= 1, \text{ if and only if } x = 0 \text{ for every } t \in \mathbb{R}^+,  
(F_{N3}) \quad N(cx,t) &= N(x,\frac{t}{|c|}) \text{ for every } c \neq 0 \text{ and } t \in \mathbb{R}^+,  
(F_{N4}) \quad N(x + u, s + t) &\geq \min\{N(x,s),N(u,t)\}, \text{ for every } s,t \in \mathbb{R}^+,  
(F_{N5}) \quad \text{the function } N(x,.) \text{ is nondecreasing on } \mathbb{R}, \text{ and}  
\lim_{t \to \infty} N(x,t) &= 1.  
\end{align*}$  

The pair $(U,N)$ is called a fuzzy norm space. Sometimes, we need two additional conditions as follows:  

$\begin{align*}  
(F_{N6}) \quad \forall t \in \mathbb{R}^+ \quad N(x,t) &> 0 \Rightarrow x = 0.  
(F_{N7}) \quad \text{The function } N(x,.) \text{ is continuous, for every } x \neq 0, \text{ and is strictly increasing on the subset }  
\{t : 0 < N(x,t) < 1\} \quad \text{is strictly increasing.}  
\end{align*}$  

Let $(U,N)$ be a fuzzy norm space. For all $\alpha \in (0,1)$, we define $\alpha$ norm on $U$ as follows:  

$\|x\|_{\alpha} = \{t > 0 : N(x,t) \geq \alpha\} \text{ for every } x \in U.$  

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Then \( \{\|x\|_\alpha : \alpha \in (0,1]\} \) is an ascending family of normed on \( U \) and are called \( \alpha \)−norm on \( U \) corresponding to the fuzzy norm \( N \) on \( U \). Here are a lemma and example which will be used in this paper.

**Lemma 1.2.** [1] Let \( (U,N) \) be a fuzzy norm space such that satisfy conditions \( F_{N6} \) and \( F_{N7} \). Define the function \( N' : U \times \mathbb{R} \to [0,1] \) as follows:

\[
N'(x,t) = \begin{cases} 
\lor\{\alpha \in (0,1) : \|x\|_\alpha \leq t\} & (x,t) \neq (0,0) \\
0 & (x,t) = (0,0) 
\end{cases}
\]

Then

a) \( N' \) is a fuzzy norm on \( U \).

b) \( N = N' \).

**Lemma 1.3.** [1] Let \( (U,N) \) be a fuzzy norm space such that satisfy conditions \( F_{N6} \) and \( F_{N7} \) and \( \{x_n\} \subseteq U \), Then \( \lim_{n \to \infty} N(x_n - x, t) = 1 \) if and only if

\[
\lim_{n \to \infty} \|x_n - x\|_\alpha = 0
\]

for every \( \alpha \in (0,1) \).

Note that the sequence \( \{x_n\} \subseteq U \) converges if there exists \( x \in U \) such that

\[
\lim_{n \to \infty} N(x_n - x, t) = 1, \text{for every } t \in \mathbb{R}^+.
\]

In this case \( x \) is called the limit of \( \{x_n\} \).

**Example 1.4.** [1] Let \( V \) be a Real or Complex vector space and let \( N \) be defined on \( V \times \mathbb{R} \) as follows:

\[
N(x,t) = \begin{cases} 
1 & t > |x| \\
0 & t \leq |x|
\end{cases}
\]

for all \( x \in V \) and \( t \in \mathbb{R} \). Then \( (N,V) \) is a fuzzy norm space and the function \( N \) satisfies condition \( F_{N6} \) and \( \|x\|_\alpha = |x| \), for every \( \alpha \in (0,1) \).

**Remark 1.5.** Let \( (U,N) \) be a fuzzy norm space and \( \{\|\cdot\|_\alpha : \alpha \in (0,1]\} \) the set of all \( \alpha \)-norms on \( U \). For two subsets \( A \) and \( B \) of \( U \), we consider :

\[
\delta(A,B) = \lor\{\|x - y\|_\alpha : x \in A, y \in B, \alpha \in (0,1]\}.
\]

**Definition 1.6.** Let \( (U,N) \) be a fuzzy norm space which satisfies conditions \( F_{N6} \) and \( F_{N7} \) and \( \{\|\cdot\|_\alpha : \alpha \in (0,1]\} \) be the set of \( \alpha \)-norms defined on \( U \). Suppose \( A \) and \( B \) are nonempty subsets of \( U \) and \( T : A \cup B \to U \).

For a given \( \epsilon > 0 \), \( x \in A \cup B \) is said to be a \( F^\varepsilon \)-approximate fixed point for \( T \) if for some \( \alpha \in (0,1) \)

\[
\|x - Tx\|_\alpha \leq \delta(A,B) + \epsilon.
\]

**Remark 1.7.** In the rest of the paper for a given \( \epsilon > 0 \), we will denote the set of all \( F^\varepsilon \)-approximate fixed points of \( T \), by

\[
F^\varepsilon_T(A,B) = \{x \in A \cup B : \|x - Tx\|_\alpha \leq \delta(A,B) + \epsilon, \text{ for some } \alpha \in (0,1]\}.
\]
2. $F^2$-Approximate Fixed Point Maps

**Proposition 2.1.** Let $(U, N)$ be a fuzzy norm space which satisfies conditions $F_{N6}$ and $F_{N7}$ and $\{\|\cdot\|_\alpha: \alpha \in (0,1)\}$ be the set of $\alpha$-norms defined on $U$. Suppose $A, B$ be nonempty subsets of $U$ and $T: A \cup B \to U$. If for $x \in A \cup B$ and $\alpha \in (0,1)$

$$\lim_{n \to \infty} \|T^n x - T^{n+1} x\|_\alpha = \delta(A, B),$$

then there exists a $F^2$-approximate fixed point in $A \cup B$.

**Proof.** Suppose $x \in A \cup B$ and $\epsilon > 0$. Since

$$\lim_{n \to \infty} \|T^n x - T^{n+1} x\|_\alpha = \delta(A, B).$$

$$\exists N_0 > 0 \text{ s.t. } \forall n \geq N_0: \|T^n x - T^{n+1} x\|_\alpha < \delta(A, B) + \epsilon.$$ 

If $n = N_0$, then for some $\alpha \in (0,1)$

$$\|T^{N_0} x - T(T^{N_0} x)\|_\alpha < \delta(A, B) + \epsilon.$$ 

Therefore, $T^{N_0} x \in F^2(A, B)$ and $F^2(A, B) \neq \emptyset$. Hence there exists a $F^2$-approximate fixed point in $A \cup B$. 

**Proposition 2.2.** Let $(U, N)$ be a fuzzy norm space such that satisfy conditions $F_{N6}$ and $F_{N7}$ and $\{\|\cdot\|_\alpha: \alpha \in (0,1)\}$ be the set of $\alpha$-norms defined on $U$. Suppose $A$ and $B$ are nonempty subsets of $U$ and $T: A \cup B \to U$. If for $x, y \in A \cup B$

$$\|T x - T y\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - T x\|_\alpha + \|y - T y\|_\alpha) + k\delta(A, B), \forall \alpha \in (0,1)$$

where $m, l, k \geq 0$ and $m + 2l + k < 1$. Then there exists a $F^2$-approximate fixed point in $A \cup B$.

**Proof.** If $x \in A \cup B$, then for some $\alpha \in (0,1)$ we have:

$$\|T x - T^2 x\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - T x\|_\alpha + \|T x - T^2 x\|_\alpha) + k\delta(A, B).$$

Therefore

$$\|T x - T^2 x\|_\alpha \leq \frac{m + l}{1 - l} \|x - T x\|_\alpha + \frac{k}{1 - l} \delta(A, B).$$

Now if $p = \frac{m + l}{1 - l}$, Then

$$\|T x - T^2 x\|_\alpha \leq p \|x - T x\|_\alpha + (1 - p) \delta(A, B)$$

and $\|T^2 x - T^3 x\|_\alpha \leq p^2 \|x - T x\|_\alpha + (1 - p^2) \delta(A, B)$, hence,

$$\|T^n x - T^{n+1} x\|_\alpha \leq p^n \|x - T x\|_\alpha + (1 - p^n) \delta(A, B).$$

since $p < 1$, for $\alpha \in (0,1)$, we have

$$\|T^n x - T^{n+1} x\|_\alpha \to \delta(A, B) \text{ as } n \to \infty.$$ 

Therefore, by Proposition 2.1, there exists a $F^2$-approximate fixed point in $A \cup B$. 

$\square$
Definition 2.3. Let \((U, N)\) be a fuzzy norm space which satisfy conditions \(F_{N_6}\) and \(F_{N_7}\) and let \(\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}\) be the set of \(\alpha\)-norms defined on \(U\). Suppose \(A\) and \(B\) are nonempty subsets of \(U\) and \(T : A \cup B \rightarrow U\). For some \(\alpha \in (0, 1)\) we define diameter \(F_T^\alpha(A, B)\) as follows:

\[
\text{diam}(F_T^\alpha(A, B)) = \vee \{\|x - y\|_\alpha : x, y \in F_T^\alpha(A, B)\}.
\]

Proposition 2.4. Let \((U, N)\) be a fuzzy norm space which satisfies conditions \(F_{N_6}\) and \(F_{N_7}\) and let \(\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}\) be the set of \(\alpha\)-norms defined on \(U\). Suppose \(A\) and \(B\) are nonempty subsets of \(U\) and \(T : A \cup B \rightarrow U\). If there exists \(k \in [0, 1]\) such that \(\|Tx - Ty\|_\alpha \leq k\|x - y\|_\alpha\) for some \(\alpha \in (0, 1)\), then

\[
\text{diam}(F_T^\alpha(A, B)) \leq \frac{2\epsilon}{1 - k} + \frac{2\delta(A, B)}{1 - k}.
\]

Proof. If \(x, y \in A \cup B\), then for some \(\alpha \in (0, 1)\) we have

\[
\|x - y\|_\alpha \leq \|x - Tx\|_\alpha + \|Tx - Ty\|_\alpha + \|Ty - y\|_\alpha \\
\leq \epsilon_1 + k\|x - y\|_\alpha + 2\delta(A, B) + \epsilon_2.
\]

By taking \(\epsilon = \text{Max}\{\epsilon_1, \epsilon_2\}\), we have

\[
\|x - y\|_\alpha \leq \frac{2\epsilon}{1 - k} + \frac{2\delta(A, B)}{1 - k}.
\]

Therefore \(\text{diam}(F_T^\alpha(A, B)) \leq \frac{2\epsilon}{1 - k} + \frac{2\delta(A, B)}{1 - k}\). \(\square\)

Definition 2.5. Let \((U, N)\) be a fuzzy norm space which satisfies conditions \(F_{N_6}\) and \(F_{N_7}\) and let \(\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}\) be the set of \(\alpha\)-norms defined on \(U\). Suppose \(A\) and \(B\) are nonempty subsets of \(U\) and \(T : A \cup B \rightarrow U, S : A \cup B \rightarrow U\). A point \((x, y)\) in \(A \times B\) is said a \(F^2\)-approximate fixed point for \((T, S)\), if for some \(\alpha \in (0, 1)\) there exists \(\epsilon > 0\) such that

\[
\|(Tx, Sy)\|_\alpha \leq \delta(A, B) + \epsilon.
\]

Put:

\[
F_{(T, S)}(A, B) = \{(x, y) \in A \times B : \|x - Tx\|_\alpha \leq \delta(A, B) + \epsilon \text{ for some } \alpha \in (0, 1)\}.
\]

Proposition 2.6. Let \((U, N)\) be a fuzzy norm space which satisfies conditions \(F_{N_6}\) and \(F_{N_7}\) and let \(\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}\) be the set of \(\alpha\)-norms defined on \(U\). Suppose \(A\) and \(B\) be nonempty subsets of \(U\) and \(T : A \cup B \rightarrow U\) and \(S : A \cup B \rightarrow U\). If for \((x, y) \in A \times B\) and some \(\alpha \in (0, 1)\)

\[
\text{Lim}_{n \to \infty} \|T^n x - S^n y\|_\alpha = \delta(A, B),
\]

then there exists a \(F^2\)-approximate fixed point in \(A \times B\).

Proof. For \(\epsilon > 0\) and \(\alpha \in (0, 1)\), suppose \((x, y) \in A \times B\). Since

\[
\text{Lim}_{n \to \infty} \|T^n x - S^n y\|_\alpha = \delta(A, B),
\]

\[
\exists N_0 > 0 \text{ s.t. } \forall n \geq N_0 : \|T^n x - S^n y\|_\alpha < \delta(A, B) + \epsilon.
\]
If $n = N_0$, then for some $\alpha \in (0, 1)$
\[ \|T(T^{N_0-1}x - S(S^{N_0-1}y))\|_\alpha < \delta(A, B) + \epsilon. \]

Now by taking $x_0 = T^{N_0-1}(x)$ and $y_0 = S^{N_0-1}(y)$, we have
\[ \|T(x_0) - S(y_0)\|_\alpha < \delta(A, B) + \epsilon, \]
hence $F^*_z(A, B) \neq \emptyset$. Therefore, there exists a $F^z$-approximate fixed point in $A \times B$. \hfill \Box

**Proposition 2.7.** Let $(U, N)$ be a fuzzy norm space which satisfies conditions $F_{N_6}$ and $F_{N_7}$ and let $\{\|\|_\alpha : \alpha \in (0, 1)\}$ be the set of $\alpha$-norms defined on $U$. Suppose $A$ and $B$ be nonempty subsets of $U$ and $T : A \cup B \to U$ and $S : A \cup B \to U$. For any $(x, y) \in A \times B$ and for some $\alpha \in (0, 1)$ :
\[ \|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A, B), \]
where $m, l, k \geq 0$ and $m + 2l + k < 1$. If $x$ is a $F^z$-approximate fixed point for $T$ and $y$ a $F^z$-approximate fixed point for $S$, then there exists a $F^z$-approximate fixed point in $A \times B$.

**Proof.** If $(x, y) \in A \times B$, then for some $\alpha \in (0, 1)$ we have :
\[ \|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + l(\|x - Tx\|_\alpha + \|y - Sy\|_\alpha) + k\delta(A, B). \]
Therefore
\[ \|Tx - S(Tx)\|_\alpha \leq \frac{m + l}{1 - l} \|x - Tx\|_\alpha + \frac{k}{1 - l}\delta(A, B). \]

Now if $p = \frac{m + l}{1 - l}$, then
\[ \|Tx - S(Tx)\|_\alpha \leq p \|x - Tx\|_\alpha + (1 - p)\delta(A, B) \quad (*) \]
and
\[ \|Sy - T(Sy)\|_\alpha \leq p \|x - Tx\|_\alpha + (1 - p)\delta(A, B) \quad (**) \]
If $x$ is a $F^z$-approximate fixed point for $T$, then there exists an $\epsilon > 0$ and by $(*)$ for some $\alpha \in (0, 1)$:
\[ \|Tx - S(Tx)\|_\alpha \leq p \|x - Tx\|_\alpha + (1 - p)\delta(A, B) \]
\[ \leq p(\delta(A, B) + \epsilon) + (1 - p)\delta(A, B) \]
\[ = \delta(A, B) + p\epsilon \]
\[ < \delta(A, B) + \epsilon. \]

Hence $(x, Tx) \in F^*_z(T, S)(A, B)$, and if $y$ is a $F^z$-approximate fixed point for $S$, then there exists an $\epsilon > 0$ and by $(**)$ for some $\alpha \in (0, 1)$:
\[ \|Ty - T(Sy)\|_\alpha \leq p \|y - Sy\|_\alpha + (1 - p)\delta(A, B) \]
\[ \leq p(\delta(A, B) + \epsilon) + (1 - p)\delta(A, B) \]
\[ = \delta(A, B) + p\epsilon \]
\[ < \delta(A, B) + \epsilon. \]
Hence \((y, Sy) \in F_{(T, S)}^z(A, B)\). Therefore there exists a \(F^z\)-approximate fixed point in \(A \times B\). \(\square\)

**Proposition 2.8.** Let \((U, N)\) be a fuzzy norm space which satisfies conditions \(F_{N6}\) and \(F_{N7}\) and let \(\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}\) be the set of \(\alpha\)-norms defined on \(U\). Suppose \(A\) and \(B\) be nonempty subsets of \(U\) and \(T: A \cup B \to U\) is a continuous map. For any \((x, y) \in A \times B\) and some \(\alpha \in (0, 1)\)

\[
\|Tx - Sy\|_\alpha \leq m(\|x - y\|_\alpha) + k\delta(A, B),
\]

where \(m, k \geq 0\) and \(m + k = 1\) and the sequences \(\{x_n\}, \{y_n\}\) be defined as follows:

\[
x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n, \quad n \geq 0.
\]

If \(\{x_n\}\) has a convergent subsequence in \(A\), then there exists a point \(x_k \in A\) such that

\[
\|x_k - Tx_k\|_\alpha = \delta(A, B) \quad \alpha \in (0, 1).
\]

**Proof.** We have

\[
\begin{align*}
\|x_{n+1} - y_{n+1}\|_\alpha &= \|Tx_n - Sy_n\|_\alpha \\
&\leq m\|x_n - y_n\|_\alpha + (k)\delta(A, B) \\
&\leq ... \\
&\leq m^{n+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^n)(k)\delta(A, B)
\end{align*}
\]

If \(\{x_{n_p}\}_{p \geq 1}\) be convergent to \(x_k \in A\), then

\[
\|x_{n+1} - y_{n+1}\|_\alpha \leq m^{n+1}\|x_0 - y_0\|_\alpha + (1 + m + m^2 + ... + m^{n_p})(k)\delta(A, B).
\]

Since \(T\) is continuous, we have,

\[
\|x_{n_p+1} - Tx_{n_p}\| \to \frac{k}{1 - m}\delta(A, B) \quad \alpha \in (0, 1).
\]

Therefore, \(\|x_k - Tx_k\| = \delta(A, B)\). \(\square\)

**Example 2.9.** Let us consider the linear space \(U = C[0, 1]\), of all continuous real valued functions on \([0, 1]\) with the usual linear operations. Consider two norms on \(C[0, 1]\) defined by

\[
\|x\|_0 = \left\{ \int_0^1 (x(t))^2 dt \right\}^{\frac{1}{2}}
\]

\[
\|x\|_1 = \sup_{0 \leq \varepsilon \leq 1} |x(t)|.
\]

Define two fuzzy norms \(N_1\) and \(N_2\) by

\[
N_1(x, t) = \begin{cases} 
1 & t > \|x\|_0 \\
0 & t \leq \|x\|_0
\end{cases}
\]

\[
N_2(x, t) = \begin{cases} 
1 & t > \|x\|_1 \\
0 & t \leq \|x\|_1
\end{cases}
\]
and

\[
N_2(x, t) = \begin{cases} 
1 & t > \|x\|_1 \\
\frac{1}{2} & \|x\|_0 \leq t \leq \|x\|_1 \\
0 & t \leq \|x\|_0
\end{cases}
\]

For the fuzzy norm \(N_1\), its \(\alpha\)-norm \(\|\cdot\|_\alpha^1\) is given by \(\|\cdot\|_\alpha^1 = \|x\|_0 \forall \alpha \in (0, 1)\).

For the fuzzy norm \(N_2\), its \(\alpha\)-norm \(\|\cdot\|_\alpha^2\) is given by

\[
\|x\|_\alpha = \begin{cases} 
\|x\|_0 & 0 < \alpha \leq \frac{1}{2} \\
\|x\|_1 & \frac{1}{2} < \alpha < 1
\end{cases}
\]

Suppose \(A\) and \(B\) be nonempty subsets of \(U\) and \(T : A \cup B \to U\). We consider \(T(x) = x + 1, \forall x \in A \cup B\). It can easily be seen that for \(\frac{1}{2} < \alpha < 1\), and \(x, y \in X\).

\[
\|Tx - Ty\|_\alpha \leq \frac{1}{2}\|x - y\|_\alpha
\]

However, \(T\) has no any fixed point. But by the Proposition 2.2, for some \(\epsilon > 0\), \(T\) has a \(F^2\)-approximate fixed point in \(A \times B\). That is, there exists \(x_0 \in U\) such that

\[
\|T(x_0) - x_0\|_\alpha \leq \epsilon.
\]

References


S. A. M. Mohsenialhosseini, Faculty of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

E-mail address: amah@vru.ac.ir

H. Mazaheri*, Faculty of Mathematics, Yazd University, Yazd, Iran

E-mail address: hmazaheri@yazduni.ac.ir

M. A. Dehghan, Faculty of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

E-mail address: dehghan@vru.ac.ir

*Corresponding author