CATEGORIES OF FUZZY TOPOLOGY IN THE CONTEXT OF GRADED DITOPOLOGIES ON TEXTURES

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Abstract. This paper extends the notion of ditopology to the case where openness and closedness are given in terms of *a priori* unrelated grading functions. The resulting notion of graded ditopology is considered both in the setting of lattices and in that textures, the relation between the two approaches being discussed in detail. Interrelations between graded ditopologies and ditopologies on textures are also studied.

1. Introduction

In classical topology the notion of open set is usually taken as primitive with that of closed set being auxiliary. However, since the closed sets are easily obtained as the complements of open sets they often play an important, sometimes dominating role in topological arguments. Much the same situation holds for topologies on lattices, where the role of set complement is played by an order reversing involution. It is the case, however, that there may be no order reversing involution available, or that the presence of such an involution is otherwise irrelevant to the topic under consideration. To deal with such cases it is natural to consider a topological structure consisting of *a priori* unrelated families of open sets and of closed sets. This was the approach adapted from the beginning for the topological structures on textures, originally introduced as a point-based representation for fuzzy sets [2, 3]. These topological structures were given the name dichotomous topology, or ditopology for short. They consist of a family τ of open sets and a generally unrelated family κ of closed sets. Hence, both the open and the closed sets are regarded as primitive concepts for a ditopology.

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different. Indeed, a bitopology consists of two separate topological structures (complete with their open and closed sets) whose interrelations we wish to study, whereas a ditoplogy represents a single topological structure.

There is no reason why the notion of ditopology should be restricted to textures, and indeed in two recent papers it has been extended to completely distributive

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lattices. Specifically, in [26] Hutton dispaces are defined and the interrelation with ditopological texture spaces used to carry over to such spaces the textural notion of almost real dicompactness, while in [15] (quasi-) di-uniformities are related to Hutton (quasi-) uniformities in this more general setting.

It is the aim of this paper to apply this approach to the case where each set or lattice element is given a degree or grade of openness, see for example [12, 13, 14, 19, 20, 27]. Hence we will be considering *a priori* unrelated grading functions for openness and closedness, the result being named a graded ditopology. In § 2 this is applied to the lattice case, giving the two basic categories **$\mathfrak{LK}$-GDTOP** of $\mathbb{K}$-graded ditopologies on $\mathbb{L}$ and **$\mathfrak{LK}$-GDTOPS** of $\mathbb{K}$-graded ditopologies on $\mathbb{L}^X$. In § 3 these are related in a historical perspective with various notions of fuzzy topology occurring in the literature. Graded ditopologies in the context of textures are discussed in § 4, where it is shown that there is an equivalence between **$\mathfrak{KK}$-GDTOP** and the category **dfGDitop** of graded ditopological spaces. Finally, § 5 discusses the interplay between graded ditopologies and ditopologies in the textural setting and includes a characterization of graded ditopologies as a family of ditopologies.

We refer the reader to [9] for terms from lattice theory not defined here, while our general reference for category theory is [1].

For the benefit of the reader we conclude this introduction by recalling some basic concepts from the theory of ditopological texture spaces. Full details may be found in [2, 3, 4, 5].

Given a set S, a *texturing* of S is a set $\mathcal{S}$ of subsets of S which is a point separating (that is for any $s_1, s_2 \in S,\ s_1 \neq s_2$ there exists $A \in \mathcal{S}$ such that $s_1 \in A, s_2 \notin A$ or $s_1 \notin A, s_2 \in A$) completely distributive lattice with respect to inclusion which contains S, $\emptyset$, and for which meet coincides with intersection and finite joins with unions. The pair $(S, \mathcal{S})$ is then called a *texture*. If $\sigma : \mathcal{S} \to \mathcal{S}$ is an inclusion reversing involution then $(S, \mathcal{S}, \sigma)$ is called a *complemented texture*.

For $s \in S$, the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}, \quad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\},$$

where $\bigvee$ is the supremum in the lattice $(\mathcal{S}, \subseteq))$, are useful for defining textural concepts. We note the very useful fact that for $A, \mathcal{B} \in \mathcal{S}$, $A \not\subseteq B \iff \exists\, s \in S$, $A \not\subseteq Q_s$, $P_s \not\subseteq B$.

The sets P_s are clearly molecules (non-empty union irreducible elements) of $\mathcal{S}$ and $(S, \mathcal{S})$ is called *simple* if there are no other molecules in $\mathcal{S}$. On the other hand $(S, \mathcal{S})$ is called *plain* if $\mathcal{S}$ is closed under arbitrary unions, equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$. The following example describes various textures that will be of interest later.

Example 1.1. (1) Textures of the form $(X, \mathcal{P}(X))$, where X is a set are called *discrete textures*. For $x \in X$ we clearly have $P_x = \{x\}$, $Q_x = X \setminus \{x\}$, so the discrete textures are both plain and simple.

(2) If $(S, \mathcal{S})$, $(U, \mathcal{U})$ are textures their *product* $(S, \mathcal{S}) \times (U, \mathcal{U})$ is the texture $(S \times U, \mathcal{S} \otimes \mathcal{U})$, where the texturing $\mathcal{S} \otimes \mathcal{U}$ consists of arbitrary intersections of sets

of the form $(A \times U) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{U}$. Clearly $P_{(s,u)} = P_s \times P_u$ and $Q_{(s,u)} = (Q_s \times U) \cup (S \times Q_u)$. The notation $\overline{P}_{(s,u)}$, $\overline{Q}_{(s,u)}$ is reserved for the p-sets and q-sets of the product $(S, \mathcal{P}(S)) \times (U, \mathcal{U})$ which is involved in the definition of relations and corelations from $(S, \mathcal{S})$ to $(U, \mathcal{U})$, see [4].

(3) Let $\mathbb{L}$ be a completely distributive lattice and denote by $J(\mathbb{L})$ the set of non-zero join-irreducible elements in $\mathbb{L}$. For $a \in \mathbb{L}$ we set $\hat{a} = \{m \in J(\mathbb{L}) \mid m \leq a\}$ and $\mathcal{J}_{\mathbb{L}} = \{\hat{a} \mid a \in \mathbb{L}\}$. Then $\mathcal{J}_{\mathbb{L}}$ is isomorphic to $\mathbb{L}$ and $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}})$ is a simple texture. In case ${}^c : \mathbb{L} \to \mathbb{L}$ is an order reversing involution then $\lambda_{\mathbb{L}}(\hat{a}) = \widehat{a^c}$ is a complementation on $\mathcal{J}_{\mathbb{L}}$ and $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \lambda_{\mathbb{L}})$ is a complemented texture called the *Hutton texture of* $(\mathbb{L}, {}^c)$ in [3]. In the case of the lattice $\mathbb{L}^X$, where X is a non-empty set we have $\big(J(\mathbb{L}^X), \mathcal{J}_{\mathbb{L}^X}\big) = (X, \mathcal{P}(X)) \times \big(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}\big)$.

If $(S_k, \mathcal{S}_k)$, $k = 1, 2$ are textures, a *difunction* $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ is a special type of direlation between $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$, see [4] for details. There is an identity difunction (i_S, I_S) on $(S, \mathcal{S})$ and the composition of compatible difunctions is defined and associative. This gives the category of textures and difunctions which is denoted by **dfTex**. If we have complemented textures $(S_1, \mathcal{S}_1, \sigma_1)$, $(S_2, \mathcal{S}_2, \sigma_2)$ we may associate with $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ its *complement* $(f, F)' = (F', f') : (S_1, \mathcal{S}_1, \sigma_1) \to (S_2, \mathcal{S}_2, \sigma_1)$. In the case where $(f, F)' = (f, F)$, (f, F) is called *complemented*.

For $B \in \mathcal{S}_2$ the inverse image $f^{\leftarrow}B$ and inverse co-image $F^{\leftarrow}B$ of B in $\mathcal{S}_2$ are equal, which characterizes difunctions within the direlations. The mapping $B \mapsto f^{\leftarrow}B = F^{\leftarrow}B$ from $\mathcal{S}_2$ to $\mathcal{S}_1$ is known to preserve arbitrary joins and intersections [4], and conversely any mapping $\theta : \mathcal{S}_2 \to \mathcal{S}_1$ which preserves arbitrary joins and intersections gives rise to a difunction $(f^\theta, F^\theta) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ satisfying $(f^\theta)^{\leftarrow}B = \theta(B) = (F^\theta)^{\leftarrow}B$ for all $B \in \mathcal{S}_2$ [5, Proposition 4.1]. In case the textures are complemented then (f^θ, F^θ) is complemented if and only if θ preserves the complementations. The category of complemented textures and complemented difunctions is denoted by **cdfTex**.

As expected, for the identity difunction on $(S, \mathcal{S})$ we have $i_S^{\leftarrow}B = B = I_S^{\leftarrow}B$, $B \in \mathcal{S}$.

Finally we recall that a *dichotomous topology*, or *ditopology* for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

$$\begin{array}{ll} S, \emptyset \in \tau, & S, \emptyset \in \kappa, \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, & K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa, \\ G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau. & K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa. \end{array}$$

The elements of τ are called *open* and those of κ *closed*. We refer to τ as the *topology* and to κ as the *cotopology* of (τ, κ). We call $(S, \mathcal{S}, \tau, \kappa)$ a ditopological texture space. In case σ is a complementation on $(S, \mathcal{S})$ and $\kappa = \{\sigma(G) \mid G \in \tau\}$ then $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is called a *complemented ditopological texture space*.

The difunction $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is *bicontinuous* [5, Definition 2.2] if $G \in \tau_2 \implies F^{\leftarrow}G \in \tau_1$ and $K \in \kappa_2 \implies f^{\leftarrow}K \in \kappa_1$. The category of ditopological texture spaces and bicontinuous difunctions is denoted by **dfDitop**.

In the case of complemented ditopological spaces and bicontinuous complemented difunctions we obtain the (non-full) subcategory **cdfDitop**.

2. Graded Ditopologies and Graded Ditopological Spaces

Let $\mathfrak{L} = \mathbf{CLAT}$ and $\mathfrak{K} = \mathbf{CDLAT}$ denote the category of complete lattices and the category of completely distributive lattices respectively. Given $\mathbb{L} \in \mathrm{Ob}(\mathfrak{L})$ and $\mathbb{K} \in \mathrm{Ob}(\mathfrak{K})$ let $0_{\mathbb{L}},\ 1_{\mathbb{L}},\ 0_{\mathbb{K}},\ 1_{\mathbb{K}}$ denote the bottom and the top elements in the lattices $\mathbb{L}$ and $\mathbb{K}$ respectively; besides we assume that $0_{\mathbb{L}} \neq 1_{\mathbb{L}}$ and $0_{\mathbb{K}} \neq 1_{\mathbb{K}}$, that is all lattices in $\mathfrak{L}$ and $\mathfrak{K}$ contain at least two elements. As morphisms in the categories $\mathfrak{L}$ and $\mathfrak{K}$ we take all mappings $\theta : \mathbb{L}_1 \to \mathbb{L}_2,\ \mathbb{L}_1, \mathbb{L}_2 \in \mathrm{Ob}(\mathfrak{L})$ and $\psi : \mathbb{K}_1 \to \mathbb{K}_2,\ \mathbb{K}_1, \mathbb{K}_2 \in \mathrm{Ob}(\mathfrak{K})$ which preserve arbitrary joins and arbitrary meets, and hence, in particular, $\psi(0_{\mathbb{K}}) = 0_{\mathbb{K}},\ \psi(1_{\mathbb{K}}) = 1_{\mathbb{K}}$. Further, let $\mathfrak{L}_c = \mathbf{CLAT_c}$, $\mathfrak{K}_c = \mathbf{CDLAT_c}$ be subcategories of $\mathfrak{L}$ and $\mathfrak{K}$ respectively the lattices of which are endowed with an order reversing involution ${}^{c_{\mathbb{L}}} : \mathbb{L} \to \mathbb{L}$ and ${}^{c_{\mathbb{K}}} : \mathbb{K} \to \mathbb{K}$ respectively. Further, let $\mathfrak{S} = \mathbf{SET}$ be the category of sets.

Let a set X and a lattice $L \in \mathrm{Ob}(\mathfrak{L})$ be given. Then the lattice structure from L can be pointwise extended to the L-powerset $\mathbb{L} = L^X$ and in the result we get also $\mathbb{L} \in \mathrm{Ob}(\mathfrak{L})$. Besides, if there is an order reversing involution on L, then this involution in a natural way is extended to the L-powerset $\mathbb{L} = L^X$ and in the result we obtain $\mathbb{L} \in \mathrm{Ob}(\mathfrak{L}_c)$.

Given an element $\alpha \in L$ by α^X we denote the element $\alpha^X \in L^X$ defined as the constant mapping $\alpha^X : X \to L$ with the value $\alpha^X(x) = \alpha$ for all $x \in X$. In particular 0_L^X and 1_L^X are the bottom and the top elements in $\mathbb{L} = L^X$ respectively.

Definition 2.1. Let $\mathbb{L} \in \mathrm{Ob}(\mathfrak{L})$ and $\mathbb{K} \in \mathrm{Ob}(\mathfrak{K})$ be given. By an $(\mathbb{L}, \mathbb{K})$-*graded ditopology* we call a pair of mappings $\otimes : \mathbb{L} \to \mathbb{K}$, $\wedge : \mathbb{L} \to \mathbb{K}$ such that

- (T_1) $\otimes(0_{\mathbb{L}}) = \otimes(1_{\mathbb{L}}) = 1_{\mathbb{K}}$;
- (T_2) $\otimes(a_1 \wedge a_2) \geq \otimes(a_1) \wedge \otimes(a_2)$ for all $a_1, a_2 \in \mathbb{L}$;
- (T_3) $\otimes\left(\bigvee_{i \in I} a_i\right) \geq \bigwedge_{i \in I} \otimes(a_i)$ for any family $\{a_i \mid i \in I\} \subseteq \mathbb{L}$.
- (C_1) $\wedge(0_{\mathbb{L}}) = \wedge(1_{\mathbb{L}}) = 1_{\mathbb{K}}$;
- (C_2) $\wedge(a_1 \vee a_2) \geq \wedge(a_1) \wedge \wedge(a_2)$ for all $a_1, a_2 \in \mathbb{L}$;
- (C_3) $\wedge\left(\bigwedge_{i \in I} a_i\right) \geq \bigwedge_{i \in I} \wedge(a_i)$ for any family $\{a_i \mid i \in I\} \subseteq \mathbb{L}$.

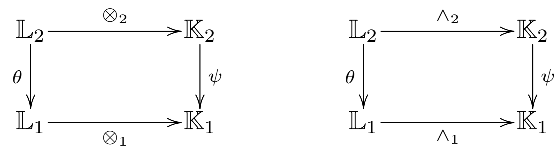
In this case the mapping $\otimes : \mathbb{L} \to \mathbb{K}$ is referred to as a $\mathbb{K}$-*graded topology on* $\mathbb{L}$ and the mapping $\wedge : \mathbb{L} \to \mathbb{K}$ as a $\mathbb{K}$-*graded cotopology on* $\mathbb{L}$. The tuple $(\mathbb{L}, \mathbb{K}, \otimes, \wedge)$ is referred to as a *graded ditopology*.

In order to define morphisms for the category $\mathfrak{L}\mathfrak{K}$-**GDTOP** of graded ditopologies let $(\mathbb{L}_1, \mathbb{K}_1, \otimes_1, \wedge_1)$ and $(\mathbb{L}_2, \mathbb{K}_2, \otimes_2, \wedge_2)$ be graded ditopologies.

Definition 2.2. By a morphism in the category $\mathfrak{L}\mathfrak{K}$-**GDTOP** we call a pair $(\theta, \psi) : (\mathbb{L}_1, \mathbb{K}_1, \otimes_1, \wedge_2) \to (\mathbb{L}_2, \mathbb{K}_2, \otimes_2, \wedge_2)$ such that $\theta : \mathbb{L}_1 \to \mathbb{L}_2$ and $\psi : \mathbb{K}_1 \to \mathbb{K}_2$ are morphisms in the category $\mathbf{CLAT}^{op}$ satisfying the following conditions:

(1) $\psi \circ \otimes_2 \leq \otimes_1 \circ \theta$;
(2) $\psi \circ \wedge_2 \leq \wedge_1 \circ \theta$,

see the diagrams:



Along with the category **𝔏𝔎-GDTOP** we consider its (non-full) subcategory **𝔏𝔎-GDTOPS** of graded ditopological spaces which is defined as follows.

Definition 2.3. The objects of the category **𝔏𝔎-GDTOPS** are tuples

$$(X, L, K, \otimes, \wedge)$$

where X is a set, $L \in \mathrm{Ob}(\mathfrak{L})$, $K \in \mathrm{Ob}(\mathfrak{K})$ and $(\mathbb{L}, \mathbb{K}, \otimes, \wedge)$ is a graded ditopology where $\mathbb{L} = L^X$ and $\mathbb{K} = K$. Explicitly this means that the mappings $\otimes : L^X \to K$, $\wedge : L^X \to K$ satisfy axioms similar to the ones in **𝔏𝔎-GDTOP**:

- (T_1) $\otimes(0_L^X) = \otimes(1_L^X) = 1_{\mathbb{K}}$;
- (T_2) $\otimes(U_1 \wedge U_2) \geq \otimes(U_1) \wedge \otimes(U_2)$ for all $U_1, U_2 \in L^X$;
- (T_3) $\otimes\left(\bigvee_{i \in I} U_i\right) \geq \bigwedge_{i \in \mathcal{J}} \otimes(U_i)$ for any family $\{U_i \mid i \in I\} \subseteq L^X$.
- (C_1) $\wedge(0_L^X) = \wedge(1_L^X) = 1_K$;
- (C_2) $\wedge(U_1 \vee U_2) \geq \wedge(U_1) \wedge \wedge(U_2)$ for all $U_1, U_2 \in L^X$;
- (C_3) $\wedge\left(\bigwedge_{i \in I} U_i\right) \geq \bigwedge_{i \in I} \wedge(U_i)$ for any family $\{U_i \mid i \in I\} \subseteq L^X$.

Definition 2.4. The morphisms in **𝔏𝔎-GDTOPS** are triples

$$(f, \varphi, \psi) : (X_1, L_1, \mathbb{K}_1, \underset{1}{\otimes}, \wedge_1) \to (X_2, L_2, \mathbb{K}_2, \underset{2}{\otimes}, \wedge_2),$$

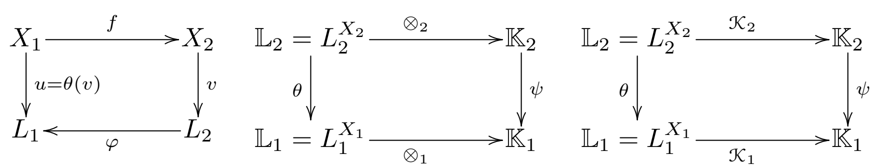
where $f : X_1 \to X_2$ is a morphism in **SET** and $\varphi : \mathbb{L}_1 \to \mathbb{L}_2$, $\psi : \mathbb{K}_1 \to \mathbb{K}_2$ are morphisms in $\mathbf{CLAT}^{op}$, such that

(1) $\psi \circ \otimes_2 \leq \otimes_1 \circ \theta$;
(2) $\psi \circ \wedge_2 \leq \wedge_1 \circ \theta$,

where

$$\theta = (f, \varphi)^{\leftarrow} : L_2^{X_2} \to L_1^{X_1},$$

that is $\theta(v) = \varphi \circ v \circ f$ for every $v \in \mathbb{L}_2^{X_2}$, cf [18] (see the diagrams below:)



Remark 2.5. In [21, 22] a category $\mathbf{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximate systems was introduced. One can show that this category is isomorphic to the subcategory $\mathfrak{L}\mathbb{K}^{id}$-**GDTOP** of the category **𝔏𝔎-GDTOP** where $\mathbb{K} \in \mathrm{Ob}(\mathfrak{K})$ is fixed and $\psi = id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ is the identity mapping. The superscript $\mathbb{M}$ in the above notation stands for a completely distributive lattice; in the present work the lattice $\mathbb{K}$ plays the role of the lattice $\mathbb{M}$.

Further, in [23] a variable-range **AS** counterpart of the category $\mathbf{AS}^{\mathbb{M}}$ whose objects are ($\mathbb{M}$-)approximate systems with variable ranges $\mathbb{M}$ was introduced. This category is closely related to category $\mathfrak{LK}$**-GDTOP** considered above.

In contrast with the approach accepted in this work, the definition of objects and morphisms in the categories $\mathbf{AS}^{\mathbb{M}}$ and **AS** and accordingly the method of research in these categories essentially relies on graded closure-type and interior-type operators on the lattices and not on the graded topological and co-topological type structures.

The detailed study of the categories considered in this paper and some other categories of graded ditopologies on textures in the framework of the theory of approximate-type systems [21, 22, 23] will be the subject of a forthcoming paper.

Remark 2.6. Some authors, following the variable-base approach developed by S.E. Rodabaugh (see e.g. [17, 18]), when working in the field of Fuzzy Topology as primary take the categories whose objects initially make a distinction between sets X and lattices $\mathbb{L}$ where fuzzy sets take their values. To put it in another way, in this case the objects are tuples like $(X, \mathbb{L}, \mathbb{M}, \mathcal{T})$ (cf our Definition 2.3 of the category $\mathfrak{LK}$**-GDTOPS**). Following this approach one can obtain objects of the form $(\mathbb{L}, \mathbb{M}, \mathcal{T})$, that is objects corresponding to the objects of our category $\mathfrak{LK}$**-GDTOP** (see Definition 2.4) by interpreting them as tuples $(\{\cdot\}, L, \mathbb{M}, \mathcal{T})$ above, where $\{\cdot\}$ is the one-point set.

Unlike the approach of S. E. Rodabaugh we prefer here to start with the category whose objects, referred to as graded topologies, initially do not distinguish between the set part X and the lattice part L of the object and to deal just with a lattice $\mathbb{L}$ (see Definition 2.1). Further, when needed, we split the lattice $\mathbb{L}$ into an L-powerset L^X of a set X, making thus a distinction between the set and the proper lattice parts. We accept here our approach since it is well coordinated both with the theory of approximate systems [21, 22, 23] and with the theory of textural ditopologies [2, 3, 4, 5, 6]: in both cases topological-type structures on lattices are taken as primary, and fuzzy topological-type spaces appear as secondary concepts.

3. Categories of Fuzzy Topology as Subcategories of the Categories $\mathfrak{LK}$-GDTOP and $\mathfrak{LK}$-GDTOPS

For historical reasons we consider here how various categories related to fuzzy topology can be described as subcategories of the categories $\mathfrak{LK}$**-GDTOP** and $\mathfrak{LK}$**-GDTOPS**. To make the description of different categories of this type better coordinated we use the notation $\mathbf{TOP}(\mathfrak{S}_0, \mathfrak{L}_0, \mathfrak{K}_0)$, where $\mathfrak{S}_0$ stands for the class of sets on which fuzzy topologies are defined, $\mathfrak{L}_0$ denotes the class of lattices L taken for the construction of powersets L^X of L-subsets of sets X, and $\mathfrak{K}_0$ stands for the class of lattices in which the fuzzy topological structures can take their values. Further, let $\mathbf{2}$ denote the two point lattice $\{0, 1\}$ and let $\mathbf{1}$ denote a one-point set $\{0\}$. Besides when a class of lattices consists of a single representative, say $\mathfrak{L}_0 = \{L\}$ we write just L instead of $\{L\}$.

(1) Let L be a fixed frame with an order reversing involution $^c : L \to L$. In 1973 J. A. Goguen [10] introduced the category of L-topological spaces

which in the above notation is presented as $\mathbf{TOP}(\mathfrak{S}, L, \mathbf{2})$. This category can be characterized as the (non-full) subcategory $L^{id}\mathbf{2}$-$\mathbf{GDTOPS_c}$ of our category $\mathfrak{L}\mathfrak{K}$-**GDTOPS**, where $\varphi = id_L : L \to L$ is the identity mapping and $\wedge(A) = \otimes(A^c)$ for every $A \in L^X$. In particular

- if $L = [0,1]$ is the unit interval viewed as a lattice with its natural order and involution $\alpha^c = 1 - \alpha$ we obtain the category $\mathbf{TOP}(\mathfrak{S}, [0,1], \mathbf{2})$ of Chang fuzzy topological spaces [7];
- if $L = \{0,1\}$ is the two-point lattice, we obtain the category **TOP** of ordinary (that is crisp) topological space interpreted as the category $\mathbf{TOP}(\mathfrak{S}, \mathbf{2}, \mathbf{2})$.

(2) In 1980 B. Hutton introduced the category of fuzzy topology [11], which in our notation is presented as $\mathbf{TOP}(\mathbf{1}, \mathfrak{L}_c, \mathbf{2})$. This category can be characterized as the full subcategory $\mathfrak{L}\mathbf{2}$-$\mathbf{GDTOP_c}$ of the category $\mathfrak{L}\mathfrak{K}$-$\mathbf{GDTOP_c}$, where $\wedge(a) = \otimes(a^c)$ for every $a \in \mathbb{L}$.

(3) Developing Hutton's ideas, S. E. Rodabaugh introduced the category in which both sets and lattices in the base of the powersets are allowed to be changed [16, 17]. In our notation this category can be presented as $\mathbf{TOP}(\mathfrak{S}, \mathfrak{L}, \mathbf{2})$. This category is isomorphic to the full subcategory $\mathfrak{L}\mathbf{2}$-$\mathbf{GDTOPS_c}$ of the category $\mathfrak{L}\mathfrak{K}$-$\mathbf{GDTOPS_c}$ where $f : X_1 \to X_2$ is a morphism in $\mathfrak{S}$, $\mathbb{K} = \{0,1\}$ is the two-point lattice and $\wedge(A) = \otimes(A^c)$ for every $A \in L^X$.

(4) An alternative view on the subject of Fuzzy Topology was proposed by U. Höhle [12]. In our notation Höhle's category is described as $\mathbf{TOP}(\mathfrak{S}, \mathbf{2}, \mathbb{K})$. Thus in contrast with the categories considered above, in U.Höhle's approach the objects are powersets of ordinary sets, endowed with fuzzy topological structures. Later the same category (with $L = [0,1]$) was rediscovered by Mingsheng Ying under the name of a fuzzifying topology [27]. The Höhle-Ying category is isomorphic to our category $\mathbf{2}\mathbb{K}$-$\mathbf{GDTOPS_c}$.

(5) In the approach proposed by T. Kubiak [13] and A.Šostak [19, 20] (independently), see also [14], both the L-powersets and the topological structure are fuzzy. In the above notations this category can be described as $\mathbf{TOP}(\mathfrak{S}, L, \mathbb{K})$. This category is isomorphic to the (non-full) subcategory $L^{id}\mathbb{K}^{id}$-$\mathbf{GDTOPS_c}$ of the category $\mathfrak{L}\mathfrak{K}$-$\mathbf{GDTOPS_c}$ of graded ditopological spaces where $\psi = id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ and $\varphi = id_L : L \to L$ are the identity morphisms.

4. Graded Ditopologies on Textures

In this section we consider graded ditopologies in a textural setting.

Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. Then $\mathcal{S} \in \mathrm{Ob}\,\mathfrak{K} \subseteq \mathrm{Ob}\,\mathfrak{L}$, $\mathcal{V} \in \mathfrak{K}$ and we may consider a $(\mathcal{S}, \mathcal{V})$-graded ditopology $(\mathcal{T}, \mathcal{C})$. This leads at once to the following definition.

Definition 4.1. Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and consider $\mathcal{T}, \mathcal{C} : \mathcal{S} \to \mathcal{V}$ satisfying

(T_1): $\mathcal{T}(S) = \mathcal{T}(\emptyset) = V$,

(T_2): $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)\ \forall A_1, A_2 \in \mathcal{S}$,

(T_3): $\bigcap_{j\in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j\in J} A_j)\ \forall A_j \in \mathcal{S},\ j \in J,$

and

(C_1): $\mathcal{C}(S) = \mathcal{C}(\emptyset) = V$,
(C_2): $\mathcal{C}(A_1) \cap \mathcal{C}(A_2) \subseteq \mathcal{C}(A_1 \cup A_2)\ \forall A_1, A_2 \in \mathcal{S}$,
(C_3): $\bigcap_{j\in J} \mathcal{C}(A_j) \subseteq \mathcal{C}(\bigcap_{j\in J} A_j)\ \forall A_j \in \mathcal{S},\ j \in J.$

Then $\mathcal{T}$ is called a $(V, \mathcal{V})$-*graded topology*, $\mathcal{C}$ a $(V, \mathcal{V})$-*graded cotopology* and $(\mathcal{T}, \mathcal{C})$ a $(V, \mathcal{V})$-*graded ditopology* on $(S, \mathcal{S})$. We refer to $(S, \mathcal{S}, \mathcal{T}, \mathcal{C}, V, \mathcal{V})$ as a *graded ditopological texture space.*

If $(S, \mathcal{S}, \sigma)$ is a complemented texture and $(\mathcal{T}, \mathcal{C})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, it is easy to show that $(\mathcal{C} \circ \sigma, \mathcal{T} \circ \sigma)$ is also a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. We call $(\mathcal{T}, \mathcal{C})$ *complemented* if $(\mathcal{T}, \mathcal{C}) = (\mathcal{C} \circ \sigma, \mathcal{T} \circ \sigma)$.

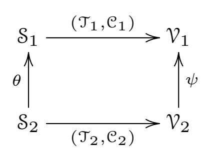
Now consider graded ditopological texture spaces $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{C}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$. Since $(\mathcal{S}_k, \mathcal{V}_k, \mathcal{T}_k, \mathcal{C}_k)$, $k = 1, 2$ are graded ditopolgies we may consider a morphism $(\theta, \psi) : (\mathcal{S}_1, \mathcal{V}_1, \mathcal{T}_1, \mathcal{C}_1) \to (\mathcal{S}_2, \mathcal{V}_2, \mathcal{T}_2, \mathcal{C}_2)$. Hence by Definition 2.2, θ, ψ are mappings $\theta : \mathcal{S}_2 \to \mathcal{S}_1$, $\psi : \mathcal{V}_2 \to \mathcal{V}_1$ preserving arbitrary joins and intersections and satisfying the conditions $\psi \circ \mathcal{T}_2 \le \mathcal{T}_1 \circ \theta$, $\psi \circ \mathcal{C}_2 \le \mathcal{C}_1 \circ \theta$ which in this context mean

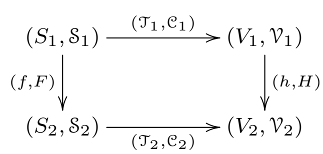
$$\psi(\mathcal{T}_2(A)) \subseteq \mathcal{T}_1(\theta(A))\ \forall A \in \mathcal{S}_2,\ \psi(\mathcal{C}_2(A)) \subseteq \mathcal{C}_1(\theta(A))\ \forall A \in \mathcal{S}_2.$$

By [5, Proposition 4.1] there exist difunctions $(f, F) = (f^\theta, F^\theta) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) = (f^\psi, F^\psi) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ characterized by

$$f^{\leftarrow} A = \theta(A) = F^{\leftarrow} A\ \forall A \in \mathcal{S}_2,\ h^{\leftarrow} B = \psi(B) = H^{\leftarrow} B\ \forall B \in \mathcal{V}_2 \tag{1}$$

See the diagrams below:





This leads us to the following definition.

Definition 4.2. Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{C}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded ditopological texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. For the pair $((f, F), (h, H))$ we call (f, F) *continuous with respect to* (h, H) if

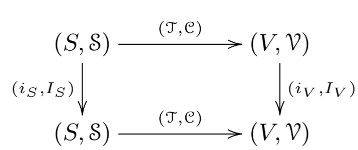
$$H^{\leftarrow} \mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow} A)\ \forall A \in \mathcal{S}_2, \tag{2}$$

and *cocontinuous with respect to* (h, H) if

$$h^{\leftarrow} \mathcal{C}_2(A) \subseteq \mathcal{C}_1(f^{\leftarrow} A)\ \forall A \in \mathcal{S}_2. \tag{3}$$

The difunction (f, F) will be called *bicontinuous with respect to* (h, H) if it is both continuous and cocontinuous with respect to (h, H).

Remark 4.3. If we specialize to $(S_1, \mathcal{S}_1) = (S_2, \mathcal{S}_2) = (S, \mathcal{S})$, $(V_1, \mathcal{V}_1) = (V_2, \mathcal{V}_2) = (V, \mathcal{V})$ and the identity difunctions, as in the diagram below,



we have $I_V^{\leftarrow}\mathcal{T}(A) = \mathcal{T}(A) = \mathcal{T}(I_S^{\leftarrow}A)$, $i_V^{\leftarrow}\mathcal{C}(A) = \mathcal{C}(A) = \mathcal{C}(i_S^{\leftarrow}A)$ for all $A \in \mathcal{S}$, so (i_S, I_S) is bicontinuous with respect to (i_V, I_V).

On the other hand, considering three graded ditopologies and bicontinuous difunctions between them, as in the diagram below,

$$\begin{array}{ccc}
(S_1, \mathcal{S}_1) & \xrightarrow{(\mathcal{T}_1, \mathcal{C}_1)} & (V_1, \mathcal{V}_1) \\
{\scriptstyle (f,F)}\big\downarrow & & \big\downarrow{\scriptstyle (h,H)} \\
(S_2, \mathcal{S}_2) & \xrightarrow{(\mathcal{T}_2, \mathcal{C}_2)} & (V_2, \mathcal{V}_2) \\
{\scriptstyle (g,G)}\big\downarrow & & \big\downarrow{\scriptstyle (h,K)} \\
(S_3, \mathcal{S}_3) & \xrightarrow{(\mathcal{T}_3, \mathcal{C}_3)} & (V_3, \mathcal{V}_3)
\end{array}$$

we see that $K^{\leftarrow}\mathcal{T}_3(A) \subseteq \mathcal{T}_2(G^{\leftarrow}A)$ for all $A \in \mathcal{S}_3$, whence

$$H^{\leftarrow}(K^{\leftarrow}\mathcal{T}_3(A)) \subseteq H^{\leftarrow}\mathcal{T}_2(G^{\leftarrow}A) \subseteq \mathcal{T}_1(F^{\leftarrow}(G^{\leftarrow}A))$$

since $G^{\leftarrow}A \in \mathcal{S}_2$. Hence $(K \circ H)^{\leftarrow}\mathcal{T}_3(A) \subseteq \mathcal{T}_1((G \circ H)^{\leftarrow}A$ for all $A \in \mathcal{S}_3$, that is $G \circ F$ is continuous with respect to $K \circ H$. Likewise it is cocontinuous, and hence bicontinuous. Hence bicontinuity is preserved under the composition $((g, G), (k, K)) \circ ((f, F), (h, H)) = ((g \circ f, G \circ F), (k \circ h, K \circ H))$.

Remark 4.3 ensures that the class of graded ditopological texture spaces and relatively bicontinuous difunction pairs between them form a category that we will denote by **dfGDitop**. With $(S, \mathcal{S}, \mathcal{T}, \mathcal{C}, V, \mathcal{V}) \in \mathrm{Ob}\,$**dfGDitop** we may associate the graded ditopology $(\mathcal{S}, \mathcal{V}, \mathcal{T}, \mathcal{C}) \in \mathrm{Ob}\,\mathfrak{K}\mathfrak{K}$**-GDTOP** and with the **dfGDitop** morphism $((f, F), (h, H))$ the pair (θ, ψ) defined by (4.2). It is now clear that (θ, ψ) is a $\mathfrak{K}\mathfrak{K}$**-GDTOP** morphism and that

$$\mathfrak{E}_g((S_1, \mathcal{S}_1, \mathcal{T}_1, \mathcal{C}_1, V_1, \mathcal{V}_1) \xrightarrow{((f,F),(h,H))} (S_2, \mathcal{S}_2, \mathcal{T}_2, \mathcal{C}_2, V_2, \mathcal{V}_2))$$
$$= (\mathcal{S}_1, \mathcal{V}_1, \mathcal{T}_1, \mathcal{C}_1) \xrightarrow{(\theta,\psi)} (\mathcal{S}_2, \mathcal{V}_2, \mathcal{T}_2, \mathcal{C}_2)$$

is a functor $\mathfrak{E}_g :$ **dfGDitop** $\to \mathfrak{K}\mathfrak{K}$**-GDTOP**.

In the opposite direction take $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C}) \in \mathrm{Ob}\,\mathfrak{K}\mathfrak{K}$**-GDTOP**. Corresponding to $\mathbb{L}, \mathbb{K}$ we have the simple textures $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}})$, $(J(\mathbb{K}), \mathcal{J}_{\mathbb{K}})$ (cf. [3]) We define $\widehat{\mathcal{T}}, \widehat{\mathcal{C}} : \mathcal{J}_{\mathbb{L}} \to \mathcal{J}_{\mathbb{K}}$ by $\widehat{\mathcal{T}}(\hat{a}) = \widehat{\mathcal{T}(a)}$, $\widehat{\mathcal{C}}(\hat{a}) = \widehat{\mathcal{C}(a)}$, $\forall\, a \in \mathbb{L}$. Using the isomorphism $a \mapsto \hat{a}$ between $\mathbb{L}$ and $\mathcal{J}_{\mathbb{L}}$, $\mathbb{K}$ and $\mathcal{J}_{\mathbb{K}}$ it is easy to check that $(\widehat{\mathcal{T}}, \widehat{\mathcal{C}})$ is a $(J(\mathbb{K}), \mathcal{J}_{\mathbb{K}})$-graded ditopology on $(M_{\mathbb{L}}, \mathcal{J}_{\mathbb{L}})$. A mapping $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ preserving arbitrary meets and joins gives rise to the mapping $\widehat{\theta} : \mathcal{J}_{\mathbb{L}_2} \to \mathcal{J}_{\mathbb{L}_1}$, $\widehat{\theta}(\widehat{\alpha}) = \widehat{\theta(\alpha)}$, which also preserves arbitrary meets and joins, and hence by [5, Proposition 4.1] we obtain the difunction $(f^{\widehat{\theta}}, F^{\widehat{\theta}}) : (J(\mathbb{L}_1), \mathcal{J}_{\mathbb{L}_1}) \to (J(\mathbb{L}_2), \mathcal{J}_{\mathbb{L}_2})$, which is characterized by

$$\big(f^{\widehat{\theta}}\big)^{\leftarrow}\widehat{\alpha} = \widehat{\theta(\alpha)} = \big(F^{\widehat{\theta}}\big)^{\leftarrow}\widehat{\alpha}.$$

In this way a $\mathfrak{K}\mathfrak{K}$**-GDTOP** morphism

$$(\theta, \psi) : (\mathbb{L}_1, \mathbb{K}_1, \mathcal{T}_1, \mathcal{C}_1) \to (\mathbb{L}_2, \mathbb{K}_2, \mathcal{T}_2, \mathcal{C}_2)$$

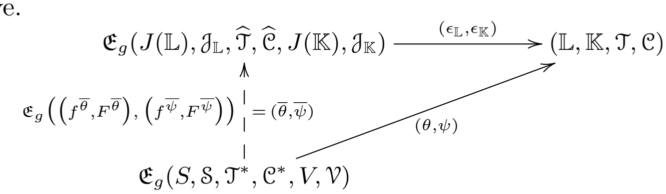
produces

$$\big((f^{\widehat{\theta}}, F^{\widehat{\theta}}), (f^{\widehat{\psi}}, F^{\widehat{\psi}})\big) : (J(\mathbb{L}_1), \mathcal{J}_{\mathbb{L}_1}, \widehat{\mathcal{T}_1}, \widehat{\mathcal{C}_1}, J(\mathbb{K}_1), \mathcal{J}_{\mathbb{K}_1}) \\ \to (J(\mathbb{L}_2), \mathcal{J}_{\mathbb{L}_2}, \widehat{\mathcal{T}_2}, \widehat{\mathcal{C}_2}, J(\mathbb{K}_2), \mathcal{J}_{\mathbb{K}_2}),$$

which is easily seen to be a **dfGDitop**-morphism. In this way we obtain a functor $\mathfrak{H}_g : \mathfrak{K}\mathfrak{K}$**-GDTOP** $\to$ **dfGDitop**.

Theorem 4.4. *The functor $\mathfrak{E}_g$ is a co-adjoint and $\mathfrak{H}_g$ the corresponding adjoint.*

Proof. The argument follows the same lines as for the functors $\mathfrak{E}$, $\mathfrak{H}$ in [26]. Take $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C}) \in \mathrm{Ob}\, \mathfrak{K}\mathfrak{K}$**-GDTOP**. The complete lattice isomorphisms $\epsilon_{\mathbb{L}} : \mathbb{L} \to J(\mathbb{L})$, $\epsilon_{\mathbb{K}} : \mathbb{K} \to M_{\mathbb{K}}$ given by $a \mapsto \hat{a}$ give rise to the $\mathfrak{K}\mathfrak{K}$**-GDTOP**-isomorphism $(\epsilon_{\mathbb{L}}, \epsilon_{\mathbb{K}}) : \mathfrak{E}_g(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_{\mathbb{K}}) \to (\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C})$. We claim that the $\mathfrak{E}_g$-costructured arrow $\big((J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_{\mathbb{K}}), (\epsilon_{\mathbb{L}}, \epsilon_{\mathbb{K}})\big)$ with codomain $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C})$ is $\mathfrak{E}_g$-universal for $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C})$. Take $(S, \mathcal{S}, \mathcal{T}^*, \mathcal{C}^*, V, \mathcal{V}) \in \mathrm{Ob}\,$**dfGDitop** and a $\mathfrak{K}\mathfrak{K}$**-GDTOP**-morphism $(\theta, \psi) : \mathcal{E}_g(S, \mathcal{S}, \mathcal{T}^*, \mathcal{C}^*, V, \mathcal{V}) \to (\mathbb{L}, \mathbb{K}, \tau, \kappa)$. Define $\overline{\theta} : \mathcal{J}_{\mathbb{L}} \to \mathcal{S}$, $\overline{\psi} : \mathcal{J}_{\mathbb{K}} \to \mathcal{V}$ by $\overline{\theta}(\hat{a}) = \theta(a)$, $a \in \mathbb{L}$ and $\overline{\psi}(\hat{b}) = \psi(b)$, $b \in \mathbb{K}$, respectively. Then $\overline{\theta} \circ \epsilon_{\mathbb{L}} = \theta$, $\overline{\psi} \circ \epsilon_{\mathbb{K}} = \psi$, whence $\overline{\theta}$, $\overline{\psi}$ preserve arbitrary joins and intersections. It follows easily that $\big((f^{\overline{\theta}}, F^{\overline{\theta}}), (f^{\overline{\psi}}, F^{\overline{\psi}})\big)$ is the (clearly unique) **dfGDitop**-morphism from $(S, \mathcal{S}, \mathcal{T}^*, \mathcal{T}^*, V, \mathcal{V})$ to $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_b K)$ making the diagram below commutative.



This establishes our claim and we deduce that $\mathfrak{E}_g$ is a co-adjoint. Moreover, since $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_{\mathbb{K}}) = \mathfrak{H}_g(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C})$ we see that $\mathfrak{H}_g$ is the corresponding adjoint (see [1]). □

From the above we see that $\epsilon = ((\epsilon_{\mathbb{L}}, \epsilon_{\mathbb{K}})) : \mathfrak{E}_g \circ \mathfrak{H}_g \to \mathrm{id}_{\mathfrak{K}\mathfrak{H}\textbf{-GDTOP}}$ is a natural isomorphism, and the corresponding natural transformation $\eta = ((\eta_S, \eta_V)) : \mathrm{id}_{\textbf{dfGDitop}} \to \mathfrak{H} \circ \mathfrak{E}$ is given by $\eta_S = (f^{\delta_S}, F^{\delta_S})$, $\eta_V = (f^{\delta_V}, F^{\delta_V})$ where $\delta_S : \mathcal{J}_{\mathcal{S}} \to \mathcal{S}$ is the bijection $\delta_S(\widehat{A}) = A$, $A \in \mathcal{S}$ and $\delta_V : \mathcal{J}_{\mathcal{V}} \to \mathcal{V}$ is the bijection $\delta_V(\widehat{A}) = A$, $A \in \mathcal{V}$. It follows that η is also a natural isomorphism, whence $\mathfrak{H}_g$ and $\mathfrak{E}_g$ are equivalences between **dfGDitop** and $\mathfrak{K}\mathfrak{K}$**-GDTOP**.

The situation for $\mathfrak{K}\mathfrak{K}$**-GDTOPS** is similar, but now we have the graded ditopological texture space $(J(\mathbb{L}^X), \mathcal{J}_{\mathbb{L}^X}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_{\mathbb{K}})$ and the difunction $\big(f^{\widehat{\theta}}, F^{\widehat{\theta}}\big) : \big(J(\mathbb{L}_1^{X_1}), \mathcal{J}_{\mathbb{L}_1^{X_1}}\big) \to \big(J(\mathbb{L}_2^{X_2}), \mathcal{J}_{\mathbb{L}_2^{X_2}}\big)$ which is more specialized. It is well known that $(J(\mathbb{L}^X), \mathcal{J}_{\mathbb{L}^X}) = (X \times J(\mathbb{L}), \mathcal{P}(X) \otimes \mathcal{J}_{\mathbb{L}})$, where the elements of the product texturing are arbitrary intersections of sets of the form $(A \times J(\mathbb{L})) \cup (X \times \hat{a})$ for $A \subseteq X$ and $a \in \mathbb{L}$. Let us determine the inverse image under $f^{\widehat{\theta}}$ of $(A \times M_{\mathbb{L}_2}) \cup (X_2 \times \hat{a})$, $A \subseteq X_2$, $a \in \mathbb{L}_2$. By Definition 2.4 we have $\theta(v) = \varphi \circ v \circ \rho$, $v \in \mathbb{L}_2^{X_2}$, $\rho : X_1 \to X_2$ and $\varphi : \mathbb{L}_2 \to \mathbb{L}_1$. If we define v by $v(x) = \begin{cases} a & x \notin A \\ 1_{\mathbb{L}_2} & x \in A \end{cases}$ for $x \in X_2$, then

$\widehat{v} = (A \times J(\mathbb{L}_2)) \cup (X_2 \times \hat{a})$, so the required inverse image is

$$\widehat{\theta} = \widehat{\varphi \circ v \circ \rho} = (\rho^{-1}(A) \times J(\mathbb{L}_1)) \cup (X_1 \times \widehat{\varphi(a)}).$$

Corresponding to the point function $\rho : X_1 \to X_2$ we have the difunction $(f_\rho, F_\rho) : (X_1, \mathcal{P}(X_1)) \to (X_2, \mathcal{P}(X_2))$ characterized by $f_\rho^{\leftarrow} A = \rho^{-1}(A) = F_\rho^{\leftarrow} A$ for all $A \subseteq X_2$ (see [4]). On the other hand, corresponding to $\varphi : \mathbb{L}_2 \to \mathbb{L}_1$ we have $\widehat{\varphi} : \mathcal{J}_{\mathbb{L}_2} \to \mathcal{J}_{\mathbb{L}_1}$ which, like φ, preserves arbitrary meets and joins, hence the difunction $(f^{\widehat{\varphi}}, F^{\widehat{\varphi}}) : (J(\mathbb{L}_1), \mathcal{J}_{\mathbb{L}_1}) \to (J(\mathbb{L}_2), \mathcal{J}_{\mathbb{L}_2})$ which is characterized by $(f^{\widehat{\varphi}})^{\leftarrow} \hat{a} = \widehat{\varphi(a)} = (F^{\widehat{\varphi}})^{\leftarrow} \hat{a}$ for all $a \in L_2$. By [6, Lemma 3.3] we may form the product $(f_\rho, F_\rho) \times (f^{\widehat{\varphi}}, F^{\widehat{\varphi}}) = (f_\rho \times f^{\widehat{\varphi}}, F_\rho \times F^{\widehat{\varphi}})$ which is a difunction from $(X_1 \times J(\mathbb{L}_1), \mathcal{P}(X_1) \otimes \mathcal{J}_{\mathbb{L}_1})$ to $(X_2 \times J(\mathbb{L}_2), \mathcal{P}(X_2) \otimes \mathcal{J}_{\mathbb{L}_2})$. A straightforward but slightly tedious argument which we omit shows that

$$\begin{aligned} (f_\rho \times f^{\widehat{\varphi}})^{\leftarrow}((A \times J(\mathbb{L}_2)) \cup (X_2 \times \hat{a})) &= ((f_\rho^{\leftarrow} A) \times J(\mathbb{L}_1)) \cup (X_1 \times (f^{\widehat{\varphi}})^{\leftarrow} \hat{a}) \\ &= (\rho^{-1}(A) \times J(\mathbb{L}_1)) \cup (X_1 \times \widehat{\varphi(a)}) \\ &= (f^{\widehat{\theta}})^{\leftarrow}((A \times J(\mathbb{L}_2)) \cup (X_2 \times \hat{a})). \end{aligned}$$

Since the inverse images preserve arbitrary intersections we deduce $(f_\rho \times f^{\widehat{\varphi}})^{\leftarrow} B = (f^{\widehat{\theta}})^{\leftarrow} B$ for all $B \in \mathcal{P}(X)_2 \otimes \mathcal{J}_{\mathbb{L}_2}$, whence $f_\rho \times f^{\widehat{\varphi}} = f^{\widehat{\theta}}$ and so $(f_\rho, F_\rho) \times (f^{\widehat{\varphi}}, F^{\widehat{\varphi}}) = \left(f^{\widehat{\theta}}, F^{\widehat{\theta}}\right)$ by [4, Proposition 2.27]. Finally we note that since $(X_k, \mathcal{P}(X_k))$, $k = 1, 2$ are plain any difunction between them must have the form (f_ρ, F_ρ) for some point function $\rho : X_1 \to X_2$, so the difunction $\left(f^{\widehat{\theta}}, F^{\widehat{\theta}}\right)$ is characterized as being a product of two difunctions. This leads us to consider the category **dfGDitops** whose objects are graded ditopological texture spaces of the form $(X \times S, \mathcal{P}(X) \otimes \mathcal{S}, \mathcal{T}, \mathcal{C}, V, \mathcal{V})$ and with morphisms $((f, F) \times (g, G), (h, H)$, $(f, F) : (X_1, \mathcal{P}(X_1)) \to (X_2, \mathcal{P}(X_2))$, $(g, G) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, where $(f, F) \times (g, G)$ is bicontinuous with respect to (h, H). It is clear that the functors $\mathfrak{E}_g$, $\mathfrak{H}_g$ specialize to an equivalence between the categories **$\mathfrak{K}\mathfrak{K}$-GDTOPS** and **dfGDitops**, and we omit the details.

To conclude this section we mention briefly the case where the graded cotopology is obtained from the graded topology by applying an order reversing involution $^c : \mathbb{L} \to \mathbb{L}$. Specifically this means $\mathcal{C}(a) = \mathcal{T}(a^c)$ for all $a \in \mathbb{L}$. When passing to the texture $(J(\mathbb{L}), \mathcal{J}_{\mathbb{L}})$ we obtain the complementation $\lambda : \mathcal{J}_{\mathbb{L}} \to \mathcal{J}_{\mathbb{L}}$ defined by $\lambda(\hat{a}) = \widehat{a^c}$, and then $\widehat{\mathcal{C}}(\hat{a}) = \widehat{T}(\widehat{a^c}) = \widehat{\mathcal{T}}(\lambda(\hat{a}))$ which gives $\widehat{\mathcal{C}} = \widehat{\mathcal{T}} \circ \lambda$. Hence $(\widehat{\mathcal{T}}, \widehat{\mathcal{C}})$ is complemented. If for the morphism (θ, ψ) the mapping θ preserves the involutions as well as arbitrary meets and joins then $(f^{\widehat{\theta}}, F^{\widehat{\theta}})$ is a complemented difunction [4, Definition 2.18]. This characterizes the objects and morphisms for the category **cdfGDitop** corresponding to **$\mathfrak{K}\mathfrak{K}$-GDTOP$_c$**, and it is again clear that we obtain an equivalence.

5. Graded Ditopologies and Ditopologies

The following example shows how a ditopology on a texture $(S, \mathcal{S})$ can be used to produce a graded ditopology on $(S, \mathcal{S})$.

Example 5.1. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ the discrete texture on a singleton. To be specific we will take $(V, \mathcal{V}) = (\mathbf{1}, \mathcal{P}(\mathbf{1}))$. Define

$\tau^g : \mathcal{S} \to \mathcal{P}(\mathbf{1})$ by $\tau^g(A) = \mathbf{1} \iff A \in \tau$. Then τ^g is a $(V,\mathcal{V})$-graded topology on $(S,\mathcal{S})$. Likewise, κ^g defined by $\kappa^g(A) = \mathbf{1} \iff A \in \kappa$ is a $(V,\mathcal{V})$-graded cotopology on $(S,\mathcal{S})$. We call (τ^g,κ^g) the graded ditopology on $(S,\mathcal{S})$ corresponding to (τ,κ).

Noting that the only difunction on $(\mathbf{1},\mathcal{P}(\mathbf{1}))$ is the identity (i,I) and that (f,F) is (τ_1,κ_1)-(τ_2,κ_2) continuous $\iff$ $(A \in \tau_2 \implies F^{\leftarrow}A \in \tau_2)$ $\iff$ $I^{\leftarrow}\tau_2^g(A) \subseteq \tau_1^g(F^{\leftarrow}A)$ $\iff$ (f,F) is continuous with respect to (i,I) by (4.2), with a similar result for cocontinuity by (4.3), we may define a functor $\mathfrak{G} : \mathbf{dfDitop} \to \mathbf{dfGDitop}$ by

$$\mathfrak{G}((S_1,\mathcal{S}_1,\tau_1,\kappa_1) \xrightarrow{(f,F)} (S_2,\mathcal{S}_2,\tau_2,\kappa_2))$$
$$= (S_1,\mathcal{S}_1,\tau_1^g,\kappa_1^g,\mathbf{1},\mathcal{P}(\mathbf{1})) \xrightarrow{((f,F),(i,I))} (S_2,\mathcal{S}_2,\tau_2^g,\kappa_2^g,\mathbf{1},\mathcal{P}(\mathbf{1})),$$

and the following is then immediate.

Proposition 5.2. *Functor $\mathfrak{G}$ is an embedding of the category* **dfDitop** *as a full subcategory* $\mathbf{dfGDitop}_{(\mathbf{1},\mathcal{P}(\mathbf{1}))}$ *of* **dfGDitop**.

We now wish to consider the general case of an arbitrary texture $(V,\mathcal{V})$. Our first task is to show that a $(V,\mathcal{V})$-graded ditopology on $(S,\mathcal{S})$ gives rise to a family of ditopologies.

Definition 5.3. Let $(\mathcal{T},\mathcal{C})$ be a $(V,\mathcal{V})$-graded ditopology on $(S,\mathcal{S})$, and take $v \in V$. We define

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\},\ \mathcal{C}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{C}(A)\}.$$

Proposition 5.4. *Let $(\mathcal{T},\mathcal{C})$ be a $(V,\mathcal{V})$-graded ditopology on $(S,\mathcal{S})$.*

(1) *$(\mathcal{T}^v,\mathcal{C}^v)$ is a ditopology on $(S,\mathcal{S})$ for each $v \in V$.*

(2) *For each $v \in V$ we have*

$$\mathcal{T}^v = \bigcap\{\mathcal{T}^{v'} \mid P_v \not\subseteq Q_{v'}\} \text{ and } \mathcal{C}^v = \bigcap\{\mathcal{C}^{v'} \mid P_v \not\subseteq Q_{v'}\}.$$

(3) *For each $A \in \mathcal{S}$ we have*

$$\mathcal{T}(A) = \bigvee\{P_v \mid A \in \mathcal{T}^v\} = \bigcap\{Q_v \mid A \notin \mathcal{T}^v\},$$
$$\mathcal{C}(A) = \bigvee\{P_v \mid A \in \mathcal{C}^v\} = \bigcap\{Q_v \mid A \notin \mathcal{C}^v\}.$$

(4) *If σ is a complementation on $(S,\mathcal{S})$ then $(\mathcal{T}\circ\sigma)^v = \sigma(\mathcal{T}^v)$ and $(\mathcal{C}\circ\sigma)^v = \sigma(\mathcal{C}^v)$ for all $v \in V$. In particular, $(\mathcal{T},\mathcal{C})$ is a complemented $(V,\mathcal{V})$-graded ditopology if and only if $\{(\mathcal{T}^v,\mathcal{C}^v)\}_{v\in V}$ is a family of complemented ditopologies on $(S,\mathcal{S})$.*

Proof. (1). We prove that $\mathcal{T}^v$, $v \in V$, is an $\mathcal{S}$-topology, the proof that $\mathcal{C}^v$ is an $\mathcal{S}$-cotopology being similar.

By T_1, $\mathcal{T}(S), \mathcal{T}(\emptyset) = V$, so $P_v \subseteq V$ gives $S,\ \emptyset \in \mathcal{T}^v$.

If $A_1, A_2 \in \mathcal{T}^v$ then $P_v \subseteq \mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ by T_2, so $A_1 \cap A_2 \in \mathcal{T}^v$.

Similarly, T_2 shows that if $A_j \in \mathcal{T}^v$ then $\bigvee_{j\in J} A_j \in \mathcal{T}^v$, which completes the proof that $\mathcal{T}^v$ is an $\mathcal{S}$-topology.

(2) Take $A \in \mathcal{T}^v$ and $P_v \not\subseteq Q_{v'}$. Then $P_{v'} \subseteq P_v \subseteq \mathcal{T}(A)$, which gives $A \in \mathcal{T}^{v'}$. Hence, $\mathcal{T}^v \subseteq \bigcap\{\mathcal{T}^{v'} \mid P_v \not\subseteq Q_{v'}\}$.

Now take $A \in \mathcal{S}$ with $A \notin \mathcal{T}^v$. Then $P_v \not\subseteq \mathcal{T}(A)$, so there exists $v' \in V$ with $P_v \not\subseteq Q_{v'}$ and $P_{v'} \not\subseteq \mathcal{T}(A)$. Now $A \notin \mathcal{T}^{v'}$, and we have established that $\bigcap\{\mathcal{T}^{v'} \mid P_v \not\subseteq Q_{v'}\} \subseteq \mathcal{T}^v$.

The proof for $\mathcal{C}^v$ is similar, and is omitted.

(3) Again, we prove only the equalities for $\mathcal{T}(A)$.

Since $A \in \mathcal{T}^v$ implies $P_v \subseteq \mathcal{T}(A)$ we certainly have $\bigvee\{P_v \mid A \in \mathcal{T}^v\} \subseteq \mathcal{T}(A)$. If the opposite inclusion does not hold we have $v \in V$ with $\mathcal{T}(A) \not\subseteq Q_v$ and $P_v \not\subseteq \bigvee\{P_v \mid A \in \mathcal{T}^v\}$. But now $P_v \subseteq \mathcal{T}(A)$, so $A \in \mathcal{T}^v$ and we have a contradiction. This proves the first equality.

Since $A \notin \mathcal{T}^v \implies P_v \not\subseteq \mathcal{T}(A) \implies \mathcal{T}(A) \subseteq Q_v$ we have $\mathcal{T}(A) \subseteq \bigcap\{Q_v \mid A \notin \mathcal{T}^v\}$. If the opposite inclusion does not hold we have $v \in V$ with $\bigcap\{Q_v \mid A \notin \mathcal{T}^v\} \not\subseteq Q_v$ and $P_v \not\subseteq \mathcal{T}(A)$. But now $A \notin \mathcal{T}^v$, and we have a contradiction. This proves the second equality.

(4) For $A \in \mathcal{S}$ we have

$$A \in (\mathcal{T} \circ \sigma)^v \iff P_v \subseteq \mathcal{T}(\sigma(A)) \iff \sigma(A) \in \mathcal{T}^v \iff A \in \sigma(\mathcal{T}^v),$$

which proves $(\mathcal{T} \circ \sigma)^v = \sigma(\mathcal{T}^v)$. Likewise, $(\mathcal{C} \circ \sigma)^v = \sigma(\mathcal{C}^v)$.

If $(\mathcal{T}, \mathcal{C})$ is complemented then $\mathcal{C} = \mathcal{T} \circ \sigma$ so $\sigma(\mathcal{T}^v) = (\mathcal{T} \circ \sigma)^v = \mathcal{C}^v$, which shows that $(\mathcal{T}^v, \mathcal{C}^v)$ is complemented.

Conversely, suppose that all the ditopologies $(\mathcal{T}^v, \mathcal{C}^v)$ are complemented, but $\mathcal{T} \neq \mathcal{C} \circ \sigma$. Then for some $A \in \mathcal{S}$ we have $\mathcal{T}(A) \neq \mathcal{C}(\sigma(A))$. If $\mathcal{T}(A) \not\subseteq \mathcal{C}(\sigma(A))$ there exists $v \in V$ with $\mathcal{T}(A) \not\subseteq Q_v$ and $P_v \not\subseteq \mathcal{C}(\sigma(A))$. Hence $A \in \mathcal{T}^v$ and $\sigma(A) \notin \mathcal{C}^v$, which implies that$A \notin \sigma(\mathcal{C}^v) = \mathcal{T}^v$, since $(\mathcal{T}^v, \mathcal{C}^v)$ is complemented, this is a contradiction. Likewise, $\mathcal{C}(\sigma(A)) \not\subseteq \mathcal{T}(A)$ leads to a contradiction. □

We now consider the converse problem of obtaining a graded ditopology from a family of ditopologies. Proposition 5.4 (2) suggests the following definition:

Definition 5.5. Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and for any $v \in V$ let (τ_v, κ_v) be a ditopology on $(S, \mathcal{S})$. The family (τ_v, κ_v), $v \in V$, will be said to be $\mathcal{V}$-*compatible* if it satisfies the equalities

$$\tau_v = \bigcap\{\tau_{v'} \mid P_v \not\subseteq Q_{v'}\} \text{ and } \kappa_v = \bigcap\{\kappa_{v'} \mid P_v \not\subseteq Q_{v'}\}.$$

We see from Proposition 5.4 (2) that the family $(\mathcal{T}^v, \mathcal{C}^v)$, $v \in V$, of ditopologies obtained from a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$ is $\mathcal{V}$-compatible. For a general $\mathcal{V}$-compatible family of ditopologies we would like to use equalities analogous to Proposition 5.4 (3) to define a corresponding $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, but first we need the following result.

Lemma 5.6. *For textures $(S, \mathcal{S})$, $(V, \mathcal{V})$, let (τ_v, κ_v) be a $\mathcal{V}$-compatible family of ditopologies on $(S, \mathcal{S})$. Then, for all $A \in \mathcal{S}$,*

$$\bigvee\{P_v \mid A \in \tau_v\} = \bigcap\{Q_v \mid A \notin \tau_v\}, \ \bigvee\{P_v \mid A \in \kappa_v\} = \bigcap\{Q_v \mid A \notin \kappa_v\}.$$

Proof. Again, we content ourselves with establishing the first equality.

If $\bigvee\{P_v \mid A \in \tau_v\} \not\subseteq \bigcap\{Q_v \mid A \notin \tau_v\}$ then for some $v \in V$ with $A \in \tau_v$ we have $P_v \not\subseteq \bigcap\{Q_v \mid A \notin \tau_v\}$, hence we have $v' \in V$ with $P_v \not\subseteq Q_{v'}$ and $A \notin \tau_{v'}$. By compatibility we have $\tau_v \subseteq \tau_{v'}$, so $A \in \tau_{v'}$, which is a contradiction.

If the opposite inclusion is false we have $v \in V$ with $\bigcap\{Q_v \mid A \notin \tau_v\} \not\subseteq Q_v$ and $P_v \not\subseteq \bigvee\{P_v \mid A \in \tau_v\}$, which leads at once to the contradiction $A \in \tau_v$ and $A \notin \tau_v$. □

Now we may give:

Proposition 5.7. *For textures $(S, \mathcal{S})$, $(V, \mathcal{V})$, let (τ_v, κ_v) be a $\mathcal{V}$-compatible family of ditopologies on $(S, \mathcal{S})$, and define mappings $\mathcal{T}, \mathcal{C} : \mathcal{S} \to \mathcal{V}$ by*

$$\mathcal{T}(A) = \bigvee\{P_v \mid A \in \tau_v\} = \bigcap\{Q_v \mid A \notin \tau_v\}, \tag{4}$$

$$\mathcal{C}(A) = \bigvee\{P_v \mid A \in \kappa_v\} = \bigcap\{Q_v \mid A \notin \kappa_v\} \tag{5}$$

for all $A \in \mathcal{S}$. Then $(\mathcal{T}, \mathcal{C})$ is a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$ which satisfies $(\mathcal{T}^v, \mathcal{C}^v) = (\tau_v, \kappa_v)$ for all $v \in V$.

Proof. The mappings $\mathcal{T}$ and $\mathcal{C}$ are well defined by Lemma 5.6, and certainly maps from $\mathcal{S}$ to $\mathcal{V}$. We prove that $\mathcal{T}$ is a $(V, \mathcal{V})$-graded topology satisfying $\mathcal{T}^v = \tau_v$ for all $v \in V$, the proof for $\mathcal{C}$ being similar.

T_1 For $A = S$, $\emptyset$, $A \in \tau_v$ for all $v \in V$, whence $\mathcal{T}(A) = \bigvee\{P_v \mid v \in V\} = V$.

T_2 Suppose we have $A_1, A_2 \in \mathcal{S}$ with $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \not\subseteq \mathcal{T}(A_1 \cap A_2)$, and take $v \in V$ with $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \not\subseteq Q_v$ and $P_v \not\subseteq \mathcal{T}(A_1 \cap A_2)$. Then by the second equality in (4) we obtain $A_1, A_2 \in \tau_v$. Since τ_v is an $\mathcal{S}$-topology this gives $A_1 \cap A_2 \in \tau_v$, whence we have the contradiction $P_v \subseteq \mathcal{T}(A_1 \cap A_2)$ by the first equality in (4).

T_3 Similar to T_2.

It remains to verify that $\mathcal{T}^v = \tau_v$, for all $v \in V$. Suppose first that $\mathcal{T}^v \not\subseteq \tau_v$ for some $v \in V$, and take $A \in \mathcal{T}^v \setminus \tau_v$. Since $A \in \mathcal{T}^v$ we have $P_v \subseteq \mathcal{T}(A)$. On the other hand $A \notin \tau_v = \bigcap\{\tau_{v'} \mid P_v \not\subseteq Q_{v'}\}$ by $\mathcal{V}$-compatibility, so we have $v' \in V$ with $P_v \not\subseteq Q_{v'}$ and $A \notin \tau_{v'}$. By the second equality in (4) we have $\mathcal{T}(A) \subseteq Q_{v'}$, which gives the contradiction $P_v \not\subseteq \mathcal{T}(A)$.

Secondly, suppose $\tau_v \not\subseteq \mathcal{T}^v$ for some $v \in V$, and take $A \in \tau_v \setminus \mathcal{T}^v$. Since $A \notin \mathcal{T}^v$ we have $P_v \not\subseteq \mathcal{T}(A)$ and so we have $v' \in V$ with $P_v \not\subseteq Q_{v'}$ and $P_{v'} \not\subseteq \mathcal{T}(A)$, whence $A \notin \tau_{v'}$ by the first equality in (4). On the other hand $A \in \tau_v$, and $\mathcal{V}$-compatibility gives $\tau_v \subseteq \tau_{v'}$, so we have the contradiction $A \in \tau_{v'}$. □

Example 5.8. (1) Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ a texture. The family (τ_v, κ_v), where $(\tau_v, \kappa_v) = (\tau, \kappa)$ for all $v \in V$, is clearly $\mathcal{V}$-compatible, hence defines a $(V, \mathcal{V})$-graded ditopology $(\mathcal{T}, \mathcal{C})$ on $(S, \mathcal{S})$ with $(\mathcal{T}^v, \mathcal{C}^v) = (\tau, \kappa)$ for all $v \in V$. From (5.7), (5.7) we easily obtain

$$\mathcal{T}(A) = \begin{cases} V, & A \in \tau, \\ \emptyset, & A \notin \tau, \end{cases} \text{ and } \mathcal{C}(A) = \begin{cases} V, & A \in \kappa, \\ \emptyset, & A \notin \kappa. \end{cases}$$

In the special case where V is a singleton, in particular for the case $(V, \mathcal{V}) = (\mathbf{1}, \mathcal{P}(\mathbf{1}))$, we obtain a $(\mathbf{1}, \mathcal{P}(\mathbf{1}))$- graded ditopology $(\mathcal{T}, \mathcal{C})$ for which $(\mathcal{T}^0, \mathcal{C}^0) = (\tau, \kappa)$. Hence, the ditopologies on $(S, \mathcal{S})$ are in one-to-one correspondence with the $(\mathbf{1}, \mathcal{P}(\mathbf{1}))$-graded ditopologies on $(S, \mathcal{S})$. This is the same correspondence noted in Example 5.1 and Proposition 5.2.

(2) Let $(V, \mathcal{V}) = (V, \mathcal{P}(V))$ be the discrete texture on V, and for $v \in V$ let (τ_v, κ_v) be an arbitrary ditopology on $(S, \mathcal{S})$. Since for the discrete texture on V, $P_v \not\subseteq Q_{v'}$ holds if and only if $v = v'$, we see that the family (τ_v, κ_v), $v \in V$ is again $\mathcal{V}$-compatible. The corresponding $(V, \mathcal{V})$-graded ditopology $(\mathcal{T}, \mathcal{C})$ is given by

$$\mathcal{T}(A) = \{v \mid A \in \tau_v\} \text{ and } \mathcal{C}(A) = \{v \mid A \in \kappa_v\},$$

and satisfies $(\mathcal{T}^v, \mathcal{C}^v) = (\tau_v, \kappa_v)$ for each $v \in V$.

Proposition 5.9. *For $k = 1, 2$ let $(\mathcal{T}_k, \mathcal{C}_k)$ be $(V_k, \mathcal{V}_k)$-graded ditopologies on $(S_k, \mathcal{S}_k)$, and let $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ be difunctions. Then the following are equivalent:*

(1) *(f, F) is bicontinuous with respect to (h, H).*
(2) *(f, F) is $(\mathcal{T}_1^{v_1}, \mathcal{C}_1^{v_1})$-$(\mathcal{T}_2^{v_2}, \mathcal{C}_2^{v_2})$ bicontinuous for all $v_1 \in V_1$, $v_2 \in V_2$ satisfying $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$.*
(3) *(f, F) is $(\mathcal{T}_1^{v_1}, \mathcal{C}_1^{v_1})$-$(\mathcal{T}_2^{v_2}, \mathcal{C}_2^{v_2})$ bicontinuous for all $v_1 \in V_1$, $v_2 \in V_2$ satisfying $H^{\leftarrow} P_{v_2} \not\subseteq Q_{v_1}$.*

Naturally, we may replace H by h in (2) and (3).

Proof. We give the proof for continuity, that for cocontinuity being similar.

(1) $\Longrightarrow$ (2) Take $A \in \mathcal{T}_2^{v_2}$ and v_1, v_2 with $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$. Then $P_{v_2} \subseteq \mathcal{T}_2(A)$, so

$$P_{v_1} \subseteq H^{\leftarrow} P_{v_2} \subseteq H^{\leftarrow} \mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow} A)$$

by continuity. Hence $F^{\leftarrow} A \in \mathcal{T}_1^{v_1}$, showing that $(f, F) : (S_1, \mathcal{S}_1, \mathcal{T}_1^{v_1}, \mathcal{C}_1^{v_1}) \to (S_2, \mathcal{S}_2, \mathcal{T}_2^{v_2}, \mathcal{C}_2^{v_2})$ is continuous.

(2) $\Longrightarrow$ (3) Immediate since $H^{\leftarrow} P_{v_2} \not\subseteq Q_{v_1}$ implies $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$.

(3) $\Longrightarrow$ (1) Suppose that (f, F) satisfies (3), but it is not continuous with respect to (h, H). Then there exists $A \in \mathcal{S}_2$ for which $H^{\leftarrow} \mathcal{T}_2(A) \not\subseteq \mathcal{T}_1(F^{\leftarrow} A)$, so we may take $v_1 \in V_1$ satisfying $H^{\leftarrow} \mathcal{T}_2(A) \not\subseteq Q_{v_1}$ and $P_{v_1} \not\subseteq \mathcal{T}_1(F^{\leftarrow} A)$.

From the definition of $H^{\leftarrow} \mathcal{T}_2(A)$ there exists $v_2 \in V_2$ satisfying $\overline{P}_{(v_1, v_2)} \not\subseteq H$ and $\mathcal{T}_2(A) \not\subseteq Q_{v_2}$. By [4, Lemma 2.4 (1)] we have $H^{\leftarrow} \not\subseteq \overline{Q}_{(v_2, v_1)}$, and since $H^{\leftarrow}$ is a relation this is equivalent to $(H^{\leftarrow})^{\rightarrow} P_{v_2} \not\subseteq Q_{v_1}$ by [4, Lemma 2.6 (1)]. But by definition $H^{\leftarrow} P_{v_2} = (H^{\leftarrow})^{\rightarrow} P_{v_2}$, so $H^{\leftarrow} P_{v_2} \not\subseteq Q_{v_1}$ and by (3), (f, F) is $\mathcal{T}_1^{v_1}$-$\mathcal{T}_2^{v_2}$ continuous.

From $\mathcal{T}_2(A) \not\subseteq Q_{v_2}$ we have $P_{v_2} \subseteq \mathcal{T}_2(A)$, and so $A \in \mathcal{T}_2^{v_2}$. The continuity of (f, F) now gives $F^{\leftarrow} A \in \mathcal{T}_1^{v_1}$, so $P_{v_1} \subseteq \mathcal{T}_1(F^{\leftarrow} A)$, which is a contradiction. This establishes that (f, F) is continuous with respect to (h, H). □

We now see that instead of graded ditopological spaces $(S, \mathcal{S}, \mathcal{T}, \mathcal{C}, V, \mathcal{V})$ we may consider $(S, \mathcal{S})$, $(V, \mathcal{V})$ together with a $\mathcal{V}$-compatible family $\{(\tau_v, \kappa_v)\}_{v \in V}$ of ditopologies on $(S, \mathcal{S})$, the bicontinuity of (f, F) with respect to (h, H) being given as in Proposition 5.9 (2) or (3). Again we obtain a category, which we will denote by **A**.

We note that **dfGDitop** may be regarded as a concrete category over **dfTex** $\times$ **dfTex**, the forgetful functor $\mathfrak{U}$ taking $(S,\mathcal{S},\mathcal{T},\mathcal{C},V,\mathcal{V})$ to $((S,\mathcal{S}),(V,\mathcal{V}))$ and the pair $((f,F),(h,H))$ to itself. Likewise **A** is a concrete category over **dfTex** $\times$ **dfTex** with forgetful functor $\mathfrak{V}$ taking $(S,\mathcal{S},\{(\tau_v,\kappa_v)\}_{v\in V},V,\mathcal{V})$ to $((S,\mathcal{S}),(V,\mathcal{V}))$ and $((f,F),(h,H))$ to itself.

Proposition 5.10. *The categories* **dfGDitop** *and* **A** *are concretely isomorphic.*

Proof. Bearing in mind Propositions 5.4, 5.7 and 5.9 we see that we may set up a functor $\mathfrak{F}:\mathbf{dfGDitop}\to\mathbf{A}$ by setting

$$\mathfrak{F}(S,\mathcal{S},\mathcal{T},\mathcal{C},V,\mathcal{V})=(S,\mathcal{S},\{(\mathcal{T}^v,\mathcal{C}^v)\}_{v\in V},V,\mathcal{V})$$

and $\mathfrak{F}((f,F),(h,H))=((f,F),(h,H))$, also a functor $\mathfrak{G}:\mathbf{A}\to\mathbf{dfGDitop}$ by $\mathfrak{G}(S,\mathcal{S},\{(\tau_v,\kappa_v)\}_{v\in V},V,\mathcal{V})=(S,\mathcal{S},\mathcal{T},\mathcal{C},V,\mathcal{V})$, where $(\mathcal{T},\mathcal{C})$ is given by (5.7), (5.7), and $\mathfrak{G}((f,F),(h,H))=((f,F),(h,H))$. Moreover we clearly have $\mathfrak{F}\circ\mathfrak{G}=id_{\mathbf{A}}$ and $\mathfrak{G}\circ\mathfrak{F}=id_{\mathbf{dfGDitop}}$, so $\mathfrak{F}$ is an isomorphism. Finally $\mathfrak{V}\circ\mathfrak{F}=\mathfrak{V}$, so $\mathfrak{F}$ is a concrete functor and hence a concrete isomorphism. □

In view of this result it is appropriate to extend the definition of **dfGDitop** to include the representation of $(V,\mathcal{V})$-graded ditopologies as $\mathcal{V}$-compatible families of ditopologies indexed over V. That is, we will not distinguish the category **A** from the category **dfGDitop**.

If we consider only $(V,\mathcal{V})$-graded ditopologies for a fixed texture $(V,\mathcal{V})$, one possible natural choice for (h,H) is the identity difunction (i_V,I_V). This leads to the (generally not full) subcategory of **dfGDitop** whose objects have the form $(S,\mathcal{S},\tau,\kappa,V,\mathcal{V})$, and the morphisms are $((f,F),(i_V,I_V))$. We will denote this subcategory by $\mathbf{dfGDitop}_{(V,\mathcal{V})}$.

Since $(V,\mathcal{V})$ is understood, it will be sufficient to denote the objects by $(S,\mathcal{S},\mathcal{T},\mathcal{C})$, and the morphisms by (f,F). By Definition 4.1 we see that, $(f,F):(S_1,\mathcal{S}_1,\mathcal{T}_1,\mathcal{C}_1)\to(S_2,\mathcal{S}_2,\mathcal{T}_2,\mathcal{C}_2)$ is bicontinuous, and hence a morphism, if and only it satisfies $\mathcal{T}_2(A)\subseteq\mathcal{T}_1(F^{\leftarrow}A)$ and $\mathcal{C}_2(A)\subseteq\mathcal{C}_1(f^{\leftarrow}A)$ for all $A\in\mathcal{S}_2$.

We have already considered the special case $\mathbf{dfGDitop}_{(\mathbf{1},\mathcal{P}(\mathbf{1}))}$ in Example 5.1. By Proposition 5.2 this is a full subcategory of **dfGDitop**.

In general construction in Examples 5.8 (1) associates a $(V,\mathcal{V})$-graded ditopology $(\mathcal{T},\mathcal{C})$ with a given ditopology (τ,κ) on $(S,\mathcal{S})$. Setting $\mathfrak{G}(S,\mathcal{S},\tau,\kappa)=(S,\mathcal{S},\mathcal{T},\mathcal{C})$ and $\mathfrak{G}(f,F)=(f,F)$ clearly gives us a functor from category **dfDitop** to the category $\mathbf{dfGDitop}_{(V,\mathcal{V})}$.

Theorem 5.11. *The functor* $\mathfrak{G}:\mathbf{dfDitop}\to\mathbf{dfGDitop}_{(V,\mathcal{V})}$ *is an adjoint. In particular case* $(V,\mathcal{V})=(\mathbf{1},\mathcal{P}(\mathbf{1}))$*,* $\mathfrak{G}$ *is an isomorphism.*

Proof. Take $(S,\mathcal{S},\mathcal{T},\mathcal{C})\in\operatorname{Ob}\mathbf{dfGDitop}_{(V,\mathcal{V})}$. We claim that

$$\Big((i_S,I_S),\ \Big(S,\mathcal{S},\ \bigcap\nolimits_{v\in V}\mathcal{T}^v,\ \bigcap\nolimits_{v\in V}\mathcal{C}^v\Big)\Big)$$

is a $\mathfrak{G}$-structured arrow with domain $(S,\mathcal{S},\mathcal{T},\mathcal{C})$. For this purpose it will suffice to show that

$$(S,\mathcal{S},\mathcal{T},\mathcal{C}) \xrightarrow{(i_S,I_S)} \mathfrak{G}\Big(S,\mathcal{S},\ \bigcap\nolimits_{v\in V}\mathcal{T}^v,\ \bigcap\nolimits_{v\in V}\mathcal{C}^v\Big)$$

is bicontinuous with respect to (i_V,I_V). If we set

$$\mathfrak{G}\Big(S,\mathcal{S},\ \bigcap\nolimits_{v\in V}\mathcal{T}^v,\ \bigcap\nolimits_{v\in V}\mathcal{C}^v\Big) = (S,\mathcal{S},\mathcal{T}^*,\mathcal{C}^*),$$

we have to show that $\mathcal{T}^*(A)\subseteq\mathcal{T}(A)$ and $\mathcal{C}^*(A)\subseteq\mathcal{C}(A)$ for all $A\in\mathcal{S}$. If $\mathcal{T}^*(A)\not\subseteq\mathcal{T}(A)$ for some A, then we have $v\in V$ with $\mathcal{T}^*(A)\not\subseteq Q_v$ and $P_v\not\subseteq\mathcal{T}(A)$. Now

$$\mathcal{T}^*(A)\not\subseteq Q_v \implies \mathcal{T}^*(A)\neq\emptyset \implies \mathcal{T}^*(A)=V \implies A\in\mathcal{T}^v \implies P_v\subseteq\mathcal{T}(A)$$

which is a contradiction. Hence $\tau^*(A)\subseteq\tau(A)$, and likewise $\kappa^*(A)\subseteq\kappa(A)$ for all $A\in\mathcal{S}$, so our claim is established.

To complete the proof we have to show that this $\mathfrak{G}$-structured arrow is $\mathfrak{G}$-universal for $(S,\mathcal{S},\tau,\kappa)$. Hence, for an arbitrary $\mathfrak{G}$-structured arrow $((f,F),\ (T,\mathcal{T},\tau,\kappa))$ with domain $(S,\mathcal{S},\mathcal{T},\mathcal{C})$ we have to show the existence of a unique **dfDitop** morphism

$$(g,G):\Big(S,\mathcal{S},\ \bigcap\nolimits_{v\in V}\mathcal{T}^v,\ \bigcap\nolimits_{v\in V}\mathcal{C}^v\Big)\to(T,\mathcal{T},\tau,\kappa)$$

for which the following diagram is commutative.

$$\begin{array}{ccc}
(S,\mathcal{S},\mathcal{T},\mathcal{C}) & \xrightarrow{(i_S,I_S)} & \mathfrak{G}\Big(S,\mathcal{S},\ \bigcap_{v\in V}\mathcal{T}^v,\ \bigcap_{v\in V}\mathcal{C}^v\Big) \\
 & \searrow^{(f,F)} & \Big\downarrow{\scriptstyle \mathfrak{G}(g,G)} \\
 & & \mathfrak{G}(T,\mathcal{T},\tau,\kappa)
\end{array}$$

Clearly the only possible choice for (g,G) is (f,F), so it remains only to show that $(f,F):\Big(S,\mathcal{S},\ \bigcap_{v\in V}\mathcal{T}^v,\ \bigcap_{v\in V}\mathcal{C}^v\Big)\to(T,\mathcal{T},\tau,\kappa)$ is bicontinuous. Denote $\mathfrak{G}(T,\mathcal{T},\tau,\kappa)$ by $(T,\mathcal{T},\overline{\mathcal{T}},\overline{\mathcal{C}})$. Then the continuity of $(f,F):(S,\mathcal{S},\mathcal{T},\mathcal{C})\to(T,\mathcal{T},\overline{\mathcal{T}},\overline{\mathcal{C}})$ with respect to (i_V,I_V) gives

$$\overline{\mathcal{T}}(A)\subseteq\mathcal{T}(F^{\leftarrow}A),\ \overline{\mathcal{C}}(A)\subseteq\mathcal{C}(f^{\leftarrow}A) \tag{6}$$

for all $A\in\mathcal{T}$. Now take $G\in\tau$ From (6) we deduce that $F^{\leftarrow}G\notin\bigcap_{v\in V}\mathcal{T}^v$. Then $F^{\leftarrow}G\notin\mathcal{T}^v$ for some $v\in V$, whence $P_v\not\subseteq\mathcal{T}(F^{\leftarrow}G)$, and hence we deduce $P_v\not\subseteq\overline{\mathcal{T}}(G)$. On the other hand $G\in\tau$ gives $\overline{\mathcal{T}}(G)=V$, which is a contradiction. Hence (f,F) is continuous, and likewise it is cocontinuous, so $\mathfrak{G}$ is an adjoint.

Since $\mathfrak{G}$: **dfDitop** $\to$ **dfGDitop**$_{(V,\mathcal{V})}$ is an adjoint, we know from [1, Theorem 19.1] that functor $\mathcal{F}$: **dfGDitop**$_{(V,\mathcal{V})}$ $\to$ **dfDitop** defined by $\mathfrak{F}(S,\mathcal{S},\mathcal{T},\mathcal{C})=\Big(S,\mathcal{S},\ \bigcap_{v\in V}\mathcal{T}^v,\ \bigcap_{v\in V}\mathcal{C}^v\Big)$ for all $(S,\mathcal{S},\mathcal{T},\mathcal{C})\in$ Ob **dfGDitop**$_{(V,\mathcal{V})}$ is a co-adjoint functor.

In case V has a single element, and specifically when $(V,\mathcal{V})=(\mathbf{1},\mathcal{P}(\mathbf{1}))$, we have $\mathfrak{F}(S,\mathcal{S},\mathcal{T},\mathcal{C})=(S,\mathcal{S},\mathcal{T}^0,\mathcal{C}^0)$, and it is trivial to verify that $\mathfrak{F}\circ\mathfrak{G}=id_{\mathbf{dfDitop}}$ and

$\mathfrak{G} \circ \mathfrak{F} = id_{\mathbf{dfGDitop}_{(\mathbf{1},\mathcal{P}(\mathbf{1}))}}$. Hence $\mathfrak{G}$ is an isomorphism in this case, so **dfDitop** is isomorphic to $\mathbf{dfGDitop}_{(\mathbf{1},\mathcal{P}(\mathbf{1}))}$. This result is also an immediate consequence of Proposition 5.2. □

Remark 5.12. The approach described in this section is "pointed" only to the extent that our family of ditopologies is indexed over V. If we are prepared to consider a family indexed over the generally larger set of all join-irreducible elements we are in an essentially point-free situation. Hence for $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C}) \in \mathrm{Ob}\, \mathfrak{K}\mathfrak{K}$-**GDTOP** we may consider $\mathfrak{E}_g(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C}) = (J(\mathbb{L}), \mathcal{J}_{\mathbb{L}}, \widehat{\mathcal{T}}, \widehat{\mathcal{C}}, J(\mathbb{K}), \mathcal{J}_{\mathbb{K}})$ and then translate back to $(\mathbb{L}, \mathbb{K}, \mathcal{T}, \mathcal{C})$ using the complete lattice isomorphisms between $J(\mathbb{L})$ and $\mathbb{L}$, $\mathcal{J}(\mathbb{K})$ and $\mathbb{K}$. Here we note that under this correspondence P_m, Q_m, $m \in \wedge_{\mathbb{K}}$ will correspond respectively to m, $q_m = \bigvee\{b \in \mathbb{K} \mid m \not\leq b\}$ (see [26]). Hence from the ditopology $(\widehat{\mathcal{T}}^m, \widehat{\mathcal{C}}^m)$ on $(J(\mathbb{L}), \mathcal{J}(\mathbb{L}))$ we obtain at once the ditopology (τ^m, κ^m), $\tau^m = \{a \in \mathbb{L} \mid m \leq \mathcal{T}(a)\}, \kappa^m = \{a \in \mathbb{L} \mid m \leq \mathcal{C}(a)\}$ on $\mathbb{L}$, i.e. the Hutton bispace $(\mathbb{L}, \tau^m, \kappa^m)$ in the terminology of [15, 26]. The remaining concepts and results of this section can likewise be expressed in this setting and we leave details to the interested reader.

6. **Conclusions**

We end with some comments on possible applications of this work.

As mentioned earlier the main advantage of the use of a ditopology in place of a topology, in whatever setting, is that it does away with the need for a complement of any form. Hence, as we have seen it enables the extension of the very useful notion of Hutton space to that of a complement free (graded) Hutton dispace. It is anticipated that these will find many practical applications, particularly to computer science, in the case where the setting up of an appropriate order reversing involution is difficult or impossible but there are natural notions of "open" and "closed" freely available.

On the other hand textures can be used to represent many types of special "fuzzy" structure that play important roles in practical applications. Here, for example, intuitionistic (fuzzy) sets spring to mind [2].

Let us recall from Example 1.1 (3) that the Hutton texture of a completely distributive lattice is simple. Not all textures are simple, and indeed not all plain textures are simple. Plain textures are studied in detail in [25] where the theory of plain (ditopological) texture spaces is shown to be mathematically much more straightforward than in the general case. Naturally the work described in this paper applies also to plain textures, and will greatly benefit from this simpler theory. Moreover, any partially ordered set gives rise to such a texture in a simple way [8], see [24] for an important special case. The ubiquitous nature of partially ordered sets in every walk of life should guarantee the wide application of the resulting theory.

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