WEAK AND STRONG DUALITY THEOREMS FOR FUZZY CONIC OPTIMIZATION PROBLEMS

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Abstract. The objective of this paper is to deal with the fuzzy conic programming problems. The aim here is to derive weak and strong duality theorems for a general fuzzy conic programming. Toward this end, The convexity-like concept of fuzzy mappings is introduced and then a specific ordering cone is established based on the parameterized representation of fuzzy numbers. Under this setting, duality theorems are extended from crisp conic optimization problems to fuzzy ones.

1. Introduction

Recently, convex fuzzy sets and convexity of fuzzy mappings have received considerable attention in dealing with mathematical treatment of fuzzy systems. In this regards, several types of convexity of fuzzy mappings and some important properties have been studied by many authors. Nanda et al. [14] and Syau [17] discussed the concept of convexity, invexity and B-preinvexity. Mishra et al. [13] introduced the concept of explicitly B-preinvex fuzzy mappings and Syau et al. [18] studied ϕ_1-convexity, Supp$-\phi_1-$convexity and Supp$-\phi_1-$qusiconvexity. Furthermore, the concept of pseudolinear and $\eta-$pseudolinear fuzzy mappings have been extended in [3, 9] by relaxing the definitions of pseudo-convex and pseudo-invex fuzzy mappings. Besides convex analysis which is a basic tool in the study of fuzzy convex optimization problems, weak and strong duality theorems for fuzzy optimization problems have also attracted a wide range of research. Rodder et al. [16] studied the duality of fuzzy linear programming problems and then this subject was extended by Bector et al. [1], Wu [20] and others [12, 15, 19]. Nowadays, Zhang et al. [22, 23] discussed the duality theory in fuzzy mathematical programming problems based on the convex fuzzy mappings.

Although, a lot of interesting explorations have been made in the study of duality theory in fuzzy optimization problems from different viewpoints, it seems that not much progress has been made in the aspect of extending the duality theorems for fuzzy conic optimization problems. Instead, here on the basis of the convexity-like concept of fuzzy mappings which is a more general than the convex one and an ordering cone defined by using parameterized representation of fuzzy numbers, the weak and strong duality theorems will be extended from crisp optimization

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problems to fuzzy ones. Notice that with the aid of the duality theorems and for the determination of solutions of the fuzzy primal problem [2, 8] we may consider the fuzzy dual problem that is possibly simpler than the fuzzy primal.

The present contribution is outlined as follows: Section 2 is devoted to give the definitions of fuzzy numbers and some results used later in the development of duality theorems in fuzzy environment. In Section 3 a boarder class of fuzzy optimization problems known as the conic convex programming is stated and primal and dual problems are established. Finally, Section 4 presents the fuzzy duality theorems and gives their detailed proofs based on convexity-like concept of fuzzy mappings.

2. **Preliminaries**

In the following, let a fuzzy number be defined as a fuzzy set $\widetilde{n} : \mathbb{R} \to [0,1]$ which is (see [4]): normal (i.e. there exist an element x_0 such that $\widetilde{n}(x_0) = 1$), fuzzy convex (i.e. $\widetilde{n}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{n}(x), \widetilde{n}(y)\}$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0,1]$), $\widetilde{n}$ is upper semi-continuous, $\mathtt{Supp}(\widetilde{n})$ is bounded, where $\mathtt{Supp}(\widetilde{n}) = \{x \in \mathbb{R};\ \widetilde{n}(x) > 0\}$ and its closure $\mathtt{cl}(\mathtt{Supp}(\widetilde{n}))$ is compact. Let $\mathbb{R}_F$ denote the family of fuzzy numbers. Also any real number $r \in \mathbb{R}$ can be regarded as a fuzzy number $\widetilde{r}$ defined by $\widetilde{r}(x) = 1$ if $x = r$, and $\widetilde{r}(x) = 0$ otherwise. Hence, the real number space $\mathbb{R}$ can be embedded in $\mathbb{R}_F$.

The α-level set of a fuzzy number $\widetilde{n} \in \mathbb{R}_F$, denoted by, $[\widetilde{n}]_\alpha$ is defined as a closed and bounded interval $[\underline{n}(\alpha), \overline{n}(\alpha)] = \{x \in \mathbb{R};\ \widetilde{n}(x) \geq \alpha\}$ if $0 < \alpha \leq 1$, and $[\underline{n}(\alpha), \overline{n}(\alpha)] = \mathtt{cl}(\mathtt{Supp}(\widetilde{n}))$ if $\alpha = 0$, where $\underline{n}(\alpha)$ and $\overline{n}(\alpha)$ denote respectively the left- and right-hand end points of $[\widetilde{n}]_\alpha$. As follows, a fuzzy number $\widetilde{n}$ can be identified by a parameterized triples $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \alpha \in [0,1]\}$. For $\widetilde{n}, \widetilde{m} \in \mathbb{R}_F$, represented respectively by $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \alpha \in [0,1]\}$ and $\{(\underline{m}(\alpha), \overline{m}(\alpha), \alpha) : \alpha \in [0,1]\}$, and $c \in \mathbb{R}$, the fuzzy operations can be defined by (see [4, 7])

$$\widetilde{n} + \widetilde{m} = \{(\underline{n}(\alpha) + \underline{m}(\alpha), \overline{n}(\alpha) + \overline{m}(\alpha), \alpha) :\ \alpha \in [0,1]\},$$
$$c\widetilde{n} = \{(c\underline{n}(\alpha), c\overline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\}, \quad c \geq 0.$$

As is well known [6, 7], the space of fuzzy numbers does not constitute any linear space under operations mentioned above, that is, addition, subtraction and scalar multiplication derived from the usual extension principle.
Goetschel and Voxman [7] have embedded the space of fuzzy numbers in a topological vector space in their own manner different from the usual way.
In [7] the negative scalar multiplication of a fuzzy number $\widetilde{n} \in \mathbb{R}_F$ and a negative real number $c < 0$ is defined as

$$c\widetilde{n} = \{(c\overline{n}(\alpha), c\underline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\},$$

where $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) :\ \alpha \in [0,1]\}$ being the parametric representation of $\widetilde{n}$.

From now on, the attention is restricted to the case of *bounded real-valued functions* $\underline{n}(\alpha)$, $\overline{n}(\alpha)$ on $[0,1]$. In view of this restriction, Goetschel and Voxman [7] pointed out that the family of parametric representations of members of $\mathbb{R}_F$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\aleph = \{ \ \{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\} \ ; \ \underline{n}, \ \overline{n} : [0,1] \to \mathbb{R} \ \}.$$

Let the latter vector space be metricized by the metric

$$d(\widetilde{n}, \widetilde{m}) = \sup_{\alpha \in [0,1]} \max\{ \ |\underline{n}(\alpha) - \underline{m}(\alpha)| \ , \ |\overline{n}(\alpha) - \overline{m}(\alpha)|\}.$$

Follows from [13] the vector space $\aleph$ together with the metric d form a metric space $(\aleph, d)$. Let $\aleph_+ = \{ \ \{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\} \ ; \ \underline{n}, \ \overline{n} : [0,1] \to \mathbb{R}^+ \ \}$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers.

Definition 2.1. [21] (i) Let C be a nonempty subset of a real linear space X. The set C is called a *cone* if $x \in C$ and $\lambda \geq 0$ then $\lambda x \in C$.

(ii) A cone C is said to be *convex* if $\lambda x + (1 - \lambda)y \in C$ for $0 < \lambda < 1$ and any $x, y \in C$. A convex cone which characterizes the partial ordering on a real linear space is called an *ordering cone* or a *positive cone.*

(iii) Let X be the real linear space with an ordering cone C. The cone $C^* = \{l \in X^*; \ l(x) \geq 0, \ \forall x \in C\}$, is called the *dual cone* for C, where X^* denotes the dual space of X.

For the parameterized representation of fuzzy numbers, there exist some approaches [5, 7] that can be used to order (partially) the elements of $\mathbb{R}_F$. For instance, we can adopt the following approach:

Definition 2.2. For $\widetilde{n}, \widetilde{m} \in \mathbb{R}_F$ represented respectively by $\{(\underline{n}(\alpha), \overline{n}(\alpha), \alpha) : \ \alpha \in [0,1]\}$ and $\{(\underline{m}(\alpha), \overline{m}(\alpha), \alpha) : \ \alpha \in [0,1]\}$, a *partial order relation* on $\mathbb{R}_F$ defines $\widetilde{n} \preceq \widetilde{m}$ if and only if $\underline{n}(\alpha) \leq \underline{m}(\alpha)$ and $\overline{n}(\alpha) \leq \overline{m}(\alpha), \ \forall \alpha \in [0,1]$ or if and only if $\widetilde{m} - \widetilde{n} \in \aleph_+$. Further, $\widetilde{n} \prec \widetilde{m}$ if and only if $\widetilde{m} - \widetilde{n} \in \aleph_+ \backslash \{\widetilde{0}\}$ and $\widetilde{n} \approx \widetilde{m}$ if and only if $\widetilde{n} - \widetilde{m} \in \aleph_+$ and $\widetilde{m} - \widetilde{n} \in \aleph_+$.

It should be noted that if the above partial ordering $\preceq$ is considered, then $\aleph_+$ is obviously a convex and ordering cone in $\mathbb{R}_F$.

Definition 2.3. Let S be a nonempty convex subset of the real linear space $\mathbb{R}^n$ and let $\mathbb{R}_F$ be the partially ordered fuzzy numbers space with the ordering cone $\aleph_+$. (i) A fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is called *convex,* if $\lambda \widetilde{f}(x) + (1-\lambda)\widetilde{f}(y) - \widetilde{f}(\lambda x + (1-\lambda)y) \in \aleph_+$, for all $x, y \in S$ and $\lambda \in [0,1]$.

(ii) A fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is called *convex-like,* if the fuzzy set $\widetilde{f}(S) + \aleph_+$ is convex.

Lemma 2.4. *Any convex fuzzy mapping $\widetilde{f} : S \to \mathbb{R}_F$ is also convex-like.*

Proof. It should be shown that the fuzzy set $\widetilde{f}(S) + \aleph_+$ is a convex set. For this purpose, choose arbitrary elements $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{f}(S) + \aleph_+$ and an arbitrary number $\lambda \in [0,1]$. Then, there are elements $x_1, x_2 \in S$ and $\widetilde{h}_1, \widetilde{h}_2 \in \aleph_+$ with $\widetilde{y}_1 \approx \widetilde{f}(x_1) + \widetilde{h}_1$, and $\widetilde{y}_2 \approx \widetilde{f}(x_2) + \widetilde{h}_2$. Thus, one obtains

$$\begin{aligned}&\lambda\widetilde{y}_1 + (1-\lambda)\widetilde{y}_2 \approx \lambda\widetilde{f}(x_1) + (1-\lambda)\widetilde{f}(x_2) + \lambda\widetilde{h}_1 + (1-\lambda)\widetilde{h}_2\\ &\in \{\widetilde{f}(\lambda x_1 + (1-\lambda)x_2);\ x_1, x_2 \in S,\ \lambda \in [0,1]\} + \aleph_+ + \lambda\aleph_+ + (1-\lambda)\aleph_+\\ &\text{(from convexity of } \widetilde{f})\\ &\approx \{\widetilde{f}(\lambda x_1 + (1-\lambda)x_2);\ x_1, x_2 \in S,\ \lambda \in [0,1]\} + \aleph_+,\end{aligned}$$

that is, $\lambda\widetilde{y}_1 + (1-\lambda)\widetilde{y}_2 \in \widetilde{f}(S) + \aleph_+$. Hence, the fuzzy set $\widetilde{f}(S) + \aleph_+$ is convex and the fuzzy mapping $\widetilde{f}$ is convex-like. □

Assumption 2.5. Before proceeding to present formally a fuzzy conic convex optimization problem, a number of assumptions are necessary.

A1. Let S be a nonempty subspace of the real linear space $\mathbb{R}$.

A2. Let $(\mathbb{R}_F, d)$ be a partially ordered fuzzy space with the ordering cone $\aleph_+$.

A3. Let $\widetilde{f} : S \to \mathbb{R}_F$ be a given fuzzy objective functional.

A4. Let $\widetilde{g} : S \to \mathbb{R}_F^n$ be a given fuzzy constraint mapping. Without loss of generality, n is taken to be 1.

A5. Let the composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ be convex-like with respect to the product cone $\aleph_+^2$ in $\mathbb{R}_F^2$.

A6. Let the constraints set be given as $S_c = \{x \in S;\ \widetilde{g}(x) \in -\aleph_+\}$ which is assumed to be nonempty.

Remark that if these three things: the set S, fuzzy functional $\widetilde{f}$ and fuzzy mapping $\widetilde{g}$ are convex, then, composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ is convex-like with respect to $\aleph_+^2$ in $\mathbb{R}_F^2$.

3. Fuzzy Conic Convex Programming

Now it is time to establish the general fuzzy conic convex programming, the so-called fuzzy constrained optimization(FCO) problem.

Problem FCO:

$$\begin{array}{ll}\text{minimize} & \widetilde{f}(x)\\ \text{subject to} & \widetilde{g}(x) \in -\aleph_+,\\ & x \in S.\end{array}$$

In optimization theory, Problem FCO is also called *primal* problem.

In the remainder of this section two theorems which are known [11, 21] as *Eidelheit separation theorem* and *separation theorem*, must be taken into account because of furnishing the proofs of the main results in this study.

Theorem 3.1. *(Eidelheit separation) Let S and W be nonempty convex subsets of a real topological linear space X with $int(S) \neq \emptyset$. Then, $int(S) \cap W = \emptyset$ if and only if there are a continuous linear functional $l \in X^* \backslash \{0_{X^*}\}$ and a real number ζ with*

$$l(s) \leq \zeta \leq l(w), \quad \forall s \in S,\ \forall w \in W,\ \textit{and } l(s) < \zeta, \quad \forall s \in int(S).$$

Theorem 3.2. *(Separation) Let S be a nonempty convex and closed subset of a real locally convex space X. Then, $x \in X \backslash S$ if and only if there is a continuous linear functional $l \in X^* \backslash \{0_{X^*}\}$ with $l(x) < \inf_{s \in S} l(s)$.*

Recall that the ordering cone $\aleph_+$ is closed. As a first result, next lemma states Problem FCO is equivalent to the following fuzzy unconstrained optimization(FUCO) problem.

Problem FUCO:

$$\min_{x \in S} \quad \sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x))$$

where $\aleph_+^*$ denotes the dual cone of $\aleph_+$.

Lemma 3.3. *Suppose the assumptions A1-A6 are fulfilled. Then $\bar{x}$ is a minimal solution of the Problem FCO if and only if $\bar{x}$ is a minimal solution of Problem FUCO. In this case the extremal values of both optimization problems are fuzzy equal.*

Proof. It is firstly assumed that $\bar{x} \in S_c$ is a minimal solution of the Problem FCO, that is, $\widetilde{f}$ attains its minimum at $\bar{x}$ on S_c. For every $x \in S$ with $\widetilde{g}(x) \in -\aleph_+$, it holds that

$$l(\widetilde{g}(x)) \leq 0, \quad \forall l \in \aleph_+^*,$$

therefore, it is guaranteed that $\sup_{l \in \aleph_+^*} l(\widetilde{g}(x)) = 0$.
Using Theorem 3.2 and the fact that $\aleph_+$ is convex and closed, for any $x \in S$ with $\widetilde{g}(x) \notin -\aleph_+$ there exists an $\bar{l} \in \aleph_+^* \backslash \{0_{\mathbb{R}}\}$ such that $\bar{l}(\widetilde{g}(x)) > 0$, resulting in $\sup_{l \in \aleph_+^*} l(\widetilde{g}(x)) = \infty$.
Furthermore, one can easily verify that for every $x \in S$

$$\begin{aligned}
\sup_{l \in \aleph_+^*} \widetilde{f}(\bar{x}) + l(\widetilde{g}(\bar{x})) &\approx \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(\bar{x}) + \widetilde{l}(\widetilde{g}(\bar{x})) \\
&\approx \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(\bar{x})) \\
&\approx \widetilde{f}(\bar{x}) \\
&\preceq \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(x)) \\
&\preceq \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(x) + \widetilde{l}(\widetilde{g}(x)) \approx \sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x)).
\end{aligned}$$

Hence, $\bar{x} \in S_c$ is also a minimal solution of Problem FUCO.
To prove the converse assertion, suppose that $\bar{x} = Arg \min_{x \in S} \{\sup_{l \in \aleph_+^*} \widetilde{f}(x) + l(\widetilde{g}(x))\}$. If $\widetilde{g}(\bar{x}) \notin -\aleph_+$ then, by the same arguments as used above one gets $\sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) = \infty$, that obviously contradicts the assumption of the solvability of Problem FUCO. In this case, it results in $\sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) = 0$.

Now it can be shown that for all $x \in S_c$

$$\begin{aligned}\widetilde{f}(\bar{x}) \approx \widetilde{f}(\bar{x}) + \sup_{l \in \aleph_+^*} l(\widetilde{g}(\bar{x})) &\approx \widetilde{f}(\bar{x}) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(\bar{x})) \\ &\approx \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(\bar{x}) + \widetilde{l}(\widetilde{g}(\bar{x})) \\ &\preceq \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{f}(x) + \widetilde{l}(\widetilde{g}(x)) \\ &\approx \widetilde{f}(x) + \sup_{\widetilde{l} \in \aleph_+^*} \widetilde{l}(\widetilde{g}(x)) \approx \widetilde{f}(x).\end{aligned}$$

This simply means that $\bar{x} \in S_c$ is a minimizer of $\widetilde{f}$ on S_c and the axiom is satisfied. □

Now it is time to associate another problem to the primal Problem FCO. This new problem is produced by a two-stage process from Problem FUCO. The first stage is carried out by exchanging **min** and **sup** and in the second stage **min** and **sup** are then replaced by **inf** and **max**, respectively. The resulted optimization problem, the so-called *dual* problem reads:

Problem FDUCO:

$$\max_{l \in \aleph_+^*} \quad \inf_{x \in S} \widetilde{f}(x) + l(\widetilde{g}(x)).$$

The equivalency of Problem FCO and Problem FUCO demonstrated by Lemma 3.3 may be investigated again between Problem FDUCO and the following one.

Problem FDCO:

$$\begin{aligned}&\text{maximize} && \widetilde{\lambda} \\ &\text{subject to} && \widetilde{f}(x) + l(\widetilde{g}(x)) \succeq \widetilde{\lambda}, \quad \forall x \in S, \\ & && \widetilde{\lambda} \in \mathbb{R}_F,\ l \in \aleph_+^*.\end{aligned}$$

Analogous to the pervious case, the equivalence may be stated as: if $\bar{l} \in \aleph_+^*$ is a maximal solution of the dual Problem FDUCO with the maximal value $\bar{\widetilde{\lambda}}$, then $(\bar{\widetilde{\lambda}}, \bar{l})$ is a maximal solution of Problem FDCO. Conversely, for every maximal solution $(\bar{\widetilde{\lambda}}, \bar{l})$ of the Problem FDCO, $\bar{l}$ is a maximal solution of Problem FDUCO with the maximal value $\bar{\widetilde{\lambda}}$.

4. **Weak and Strong Duality Theorems**

The objective of this section is to derive the relationships between the primal Problem FCO and the dual Problem FDCO by means of the so-called *weak duality theorem* and *strong duality theorem*. The major result that will be shown later says that the primal and dual problems are equivalent.

Theorem 4.1. *(Weak duality) Suppose that assumptions A1-A6 are satisfied. Let* $\hat{x} \in S_c$ *and* $\widehat{l} \in \aleph_+^*$ *with the maximal value* $\widetilde{\widehat{\lambda}} \approx \inf_{x \in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x))$ *be respectively the feasible solutions of primal Problem FCO and dual Problem FDCO. Then the objective value of dual Problem FDCO is always fuzzy less than or fuzzy equal to the objective value of primal Problem FCO, that is:*

$$\inf_{x \in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x)) \preceq \widetilde{f}(\hat{x}).$$

Proof. Using the fact that $\widetilde{g}(\hat{x}) \in -\aleph_+$, one completes the proof in a straightforward manner by showing that for any $\hat{x} \in S_c$ and $\widehat{l} \in \aleph_+^*$

$$\inf_{x \in S} \widetilde{f}(x) + \widehat{l}(\widetilde{g}(x)) \preceq \widetilde{f}(\hat{x}) + \widehat{l}(\widetilde{g}(\hat{x})) \preceq \widetilde{f}(\hat{x}).$$

This holds true, since $\widetilde{g}(\hat{x}) \in -\aleph_+$ leads to $\widehat{l}(\widetilde{g}(x)) \leq 0$. □

Now, in order to investigate the strong duality theorem, it needs beforehand to provide a sufficient level of detail in describing the concepts used within the study. (See [10])

Definition 4.2. (i) A fuzzy subset $\widetilde{A} \subset \widetilde{B} = \{(x, \mu_{\widetilde{B}}(x));\ x \in \mathbb{R},\ 0 \leq \mu_{\widetilde{B}}(x) \leq 1\}$ is a *fuzzy neighborhood* of $x_0 \in \widetilde{B}$ if and only if $\mu_{\widetilde{A}}(x_1) - \epsilon < \mu_{\widetilde{B}}(x_0) < \mu_{\widetilde{A}}(x_2) + \epsilon$, for all $x_1, x_2 \in \widetilde{B}$ and $0 < \epsilon < 1$.

(ii) Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets such that $\widetilde{A} \subseteq \widetilde{B}$. Then, the fuzzy set $\widetilde{A}$ is said to be *fuzzy interior* to $\widetilde{B}$ if and only if $\widetilde{A}$ is a fuzzy neighborhood of $\widetilde{B}$.

(iii) The *interior of* $\widetilde{B}$ is the union of all fuzzy interior subsets of $\widetilde{B}$, that is, the interior of $\widetilde{B}$ denoted by $int(\widetilde{B})$ is the largest open fuzzy set contained in $\widetilde{B}$.

Definition 4.3. Recall the assumption A6 which states $S_c = \{x \in S;\ \widetilde{g}(x) \in -\aleph_+\}$. It is called that *the generalized Slater condition* holds true if there exists a vector $\hat{x} \in S_c$ such that $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$.

Theorem 4.4. *(Strong duality) Suppose that the assumptions A1-A6 are fulfilled. Furthermore, assume that the primal Problem FCO is solvable and the generalized Slater condition is satisfied. Then the dual Problem FDCO is also solvable and there is no gap between the primal Problem FCO and the dual Problem FDCO, i.e. the extremal values of the two problems are fuzzy equal.*

Proof. The procedure of the proof starts with constructing a fuzzy set as follows:

$$\widetilde{M} = \{(\widetilde{f}(x) + \widetilde{F},\ \widetilde{g}(x) + \widetilde{G}) \in \mathbb{R}_F^2;\ x \in S,\ \widetilde{F} \succeq \widetilde{0},\ \widetilde{G} \in \aleph_+\} = (\widetilde{f}, \widetilde{g})(S) + \aleph_+^2.$$

As follows from assumptions A1-A6, one can observe that the composite fuzzy mapping $(\widetilde{f}, \widetilde{g}) : S \to \mathbb{R}_F^2$ is convex-like and therefore the set $\widetilde{M}$ is convex. Since $int(\aleph_+) \neq \emptyset$, this implies that the set $\widetilde{M}$ has a nonempty interior $int(\widetilde{M})$ as well. With respect to the solvability of the primal Problem FCO, there exists a vector $\bar{x} \in S_c$ such that $\widetilde{f}(\bar{x}) \preceq \widetilde{f}(x),\ \forall x \in S_c$. This shows that $(\widetilde{f}(\bar{x}), \widetilde{0}) \notin int(\widetilde{M})$, and therefore $int(\widetilde{M}) \cap \{(\widetilde{f}(\bar{x}), \widetilde{0})\} = \emptyset$. Regarding the latter attainment and

Theorem 3.1, there exist a real number ζ and a composite continuous linear functional $(l_1, l_2) \in \mathbb{R}_F^{*2}$ with $(l_1, l_2) \neq (0,0)$ so that $(l_1, l_2) \circ (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \geq \zeta \geq (l_1, l_2) \circ (\widetilde{f}(\bar{x}), \widetilde{0})$, $\forall (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \in int(\widetilde{M})$, or

$$l_1(\widetilde{\widetilde{f}}) + l_2(\widetilde{\widetilde{g}}) \geq \zeta \geq l_1(\widetilde{f}(\bar{x})) + 0, \quad \forall (\widetilde{\widetilde{f}}, \widetilde{\widetilde{g}}) \in int(\widetilde{M}). \tag{1}$$

Recall that any convex subset of a normed space whose interior set is nonempty, is contained in the closure of that its interior. According to this assertion one may conclude from the inequality (1)

$$l_1(\widetilde{f}(x) + \widetilde{F}) + l_2(\widetilde{g}(x) + \widetilde{G}) \geq \zeta \geq l_1(\widetilde{f}(\bar{x})), \quad \forall x \in S,\ \widetilde{F} \succeq \widetilde{0},\ \widetilde{G} \in \aleph_+ . \tag{2}$$

In the case that $x = \bar{x}$ and $\widetilde{F} \approx \widetilde{0}$, it follows immediately from the inequality (2) that

$$l_2(\widetilde{G}) \geq -l_2(\widetilde{g}(\bar{x})), \quad \forall \widetilde{G} \in \aleph_+ . \tag{3}$$

Needless to say that $l_2 \in \aleph_+^*$. Further, imposing $\widetilde{G} \approx \widetilde{0}$ in inequality (3) results in $l_2(\widetilde{g}(\bar{x})) \geq 0$. On the other hand, from $\widetilde{g}(\bar{x}) \in -\aleph_+$ and $l_2 \in \aleph_+^*$ one gets $l_2(\widetilde{g}(\bar{x})) \leq 0$, which leads to $l_2(\widetilde{g}(\bar{x})) = 0$.

Take $x = \bar{x}$ and $\widetilde{G} \approx \widetilde{0}$. In this regard, the inequality (2) becomes $l_1(\widetilde{F}) \geq 0,\ \forall \widetilde{F} \succeq \widetilde{0}$, which implies that $l_1 \geq 0$. The aim is to show that $l_1 > 0$. Suppose it is not the case, that is, $l_1 = 0$, then the inequality (2) with $\widetilde{G} \approx \widetilde{0}$ becomes $l_2(\widetilde{g}(x)) \geq 0,\ \forall x \in S$. In virtue of the latter inequality and the generalized Slater condition, there is one $\hat{x} \in S$ so that $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$. Thus $l_2(\widetilde{g}(\hat{x})) = 0$. This gives rise to $l_2 = 0$. By contradiction, let $l_2 \neq 0$, that is, there is one $\widetilde{G} \in \mathbb{R}_F$ such that $l_2(\widetilde{G}) > 0$. Then, it is not hard to see that

$$l_2(\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x})) > 0, \quad \forall \lambda \in [0,1]. \tag{4}$$

In this case, $\widetilde{g}(\hat{x}) \in -int(\aleph_+)$ provides the existence of a $\bar{\lambda} \in (0,1)$ such that

$$\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x}) \in -int(\aleph_+), \quad \forall \lambda \in [0, \bar{\lambda}].$$

This implies that $l_2(\lambda\widetilde{G} + (1-\lambda)\widetilde{g}(\hat{x})) \leq 0,\ \forall \lambda \in [0, \bar{\lambda}]$, which is contradict the inequality (4).

Note that the above argument shows that if $l_1 = 0$, this may leads to $l_2 = 0$. This contradicts $(l_1, l_2) \neq (0,0)$. Thereby, it must be $l_1 \neq 0$ and therefore $l_1 > 0$.

Now, by taking $\widetilde{F} \approx \widetilde{0}$ and $\widetilde{G} \approx \widetilde{0}$, one may obtain from the inequality (2) that

$$l_1(\widetilde{f}(x)) + l_2(\widetilde{g}(x)) \geq l_1(\widetilde{f}(\bar{x})), \quad \forall x \in S,$$

equivalently

$$\widetilde{f}(x) + {l_1}^{-1} l_2(\widetilde{g}(x)) \geq \widetilde{f}(\bar{x}), \quad \forall x \in S.$$

The subsequence results follow directly from $\bar{l} = {l_1}^{-1} l_2 \in \aleph_+^*$ with $\bar{l}(\widetilde{g}(\bar{x})) = 0$. That is, to say,

$$\widetilde{f}(x) + \bar{l}(\widetilde{g}(x)) \succeq \widetilde{f}(\bar{x}) + \bar{l}(\widetilde{g}(\bar{x})), \quad \forall x \in S.$$

Hence,

$$\widetilde{f}(\bar{x}) \approx \widetilde{f}(\bar{x}) + \bar{l}(\widetilde{g}(\bar{x})) \approx \inf_{x\in S} \widetilde{f}(x) + \bar{l}(\widetilde{g}(x)),$$

where by Theorem 4.1, $\bar{l} \in \aleph_+^*$ is a maximal solution of the dual Problem FDCO. It is obvious from the above fuzzy equality that the extremal values of the primal Problem FCO and the dual Problem FDCO are fuzzy equal. □

5. Conclusion

The aim of the paper is to present an extension of the classical duality theorems known as weak and strong duality theorems for conic programming problems to those arising for fuzzy ones. The key ideas are the convexity-like concept of fuzzy mappings and the treatment of the ordering cone $\aleph_+$ established upon the parameterized representation of fuzzy numbers.

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