FUZZY ORDER CONGRUENCES ON FUZZY POSETS

J. HAO, Q. LI AND L. GUO

Abstract. Fuzzy order congruences play an important role in studying the categorical properties of fuzzy posets. In this paper, the correspondence between the fuzzy order congruences and the fuzzy order-preserving maps is discussed. We focus on the characterization of fuzzy order congruences on the fuzzy poset in terms of the fuzzy preorders containing the fuzzy partial order. At last, fuzzy complete congruences on fuzzy complete lattices are discussed.

1. Introduction

Congruences are mostly studied in abstract algebra. In essence, they are equivalence relations compatible with the underlying algebraic structure and every congruence corresponds to a quotient structure that is made up of the equivalence classes. It is well known that in the classical setting, there exists a one-to-one correspondence between the set of congruences on an algebra A and the set of homomorphisms with domain X, which is the universe of algebra A.

On a poset, order congruences are those equivalence relations compatible with the order structure. Various kinds of order congruences have been proposed in the literature and each of them corresponds to a particular isotone map [7, 20, 21, 23, 26]. Particularly, order congruences proposed by Kortesi in [23] as the kernel of the isotone maps are the most general ones. In their work, the order congruences were characterized and the lattice properties of the set of order congruences on a poset were also investigated.

Much work has been devoted to the fuzzification of partially ordered sets. After Zadeh first introduced fuzzy orders in [35], more research in this field are developed [3, 5, 6, 15, 16, 22, 24, 25, 28, 29, 30, 37]. And since then, there are two directions in studying the fuzzy orders. One is to treat fuzzy orders as enriched categories over commutative unital quantale [16, 24, 29, 30]. Lai and Zhang [24] investigated the L-preordered sets as enriched categories over a commutative, unital quantale Ω. The other is from the viewpoint of fuzzy set theory [3, 15, 25, 37]. Fuzzy partially ordered set (X, e) was proposed by Fan in [15], where e is a reflexive, antisymmetric and transitive fuzzy relation over X. Fuzzy poset, sometimes called L-ordered set, was originally introduced in [3] in order to fuzzify the fundamental theorem of concept lattices. Particularly, a fuzzy poset (X, e) is called a fuzzy complete lattice

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[37] if every fuzzy subset has both join and meet that belong to X. In [25], Martinek discussed completely lattice L-ordered sets with and without L-equality.

In the literature, a lot of research work have been put on partially ordered algebras, such as partially ordered semigroups, groups, rings, S-posets and so on. Order congruences(partial order congruences) on partially ordered algebras play an important role in studying the categorical properties of these structures. In [33], Xie and Shi proposed the concept of pseudo order to characterize the order congruence on S-poset where the pseudo order is a preorder containing the partial order. Complete congruence is an equivalence relation compatible with the underlying complete lattice. In [17], Ganter and Wille discussed the correspondence between the complete congruence and formal context. The complete congruence is a particular kind of order congruence. Since complete lattice has strong connection with formal concept analysis, so the factorization of the concept lattice through a complete congruence generated by a special sub context is investigated in [17].

This paper is devoted to discuss the fuzzy order congruences on fuzzy posets. On a fuzzy poset, fuzzy order congruences are fuzzy equivalence relations which are compatible with the underlying fuzzy partial order. We discuss the fuzzy quotient poset and the corresponding properties. Then on a fuzzy poset, the fuzzy order congruence is characterized by the fuzzy preorders containing the fuzzy partial order. Some lattice properties of fuzzy preorders that contains the fuzzy partial order are also investigated. According to the fuzzy order congruences, we obtain some categorical properties of fuzzy posets. Since there is a closely relevant connection between fuzzy complete lattices and fuzzy concept analysis, so we apply this to a fuzzy complete lattice. We introduce the notion of fuzzy complete congruence on fuzzy complete lattice, which is more particular fuzzy order congruence and obtain some connection between the L-contexts and the fuzzy complete congruences.

The paper is arranged as follows. In Section 2, we recall some preliminaries on complete residuated lattices and fuzzy posets. The notion of fuzzy order congruence is proposed in Section 3. We discuss homomorphism theorem and characterize the fuzzy order congruence by the fuzzy preorders containing the fuzzy partial order. At the end of Section 3, we investigate some categorical properties of fuzzy posets. We construct new fuzzy preorders from the given ones and discuss their lattice properties in Section 4. In Section 5, we introduce and study fuzzy complete congruences on fuzzy complete lattices. At last it concludes the contribution and limitation of our work and discuss the future research.

2. **Preliminaries**

A residuated lattice [4, 27, 31] $(L, *, \to, \vee, \wedge, 0, 1)$ is an algebra with 4 binary operators $*, \to, \vee, \wedge$ on L such that:
(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with the greatest element 1 and the least element 0;
(2) $(L, *, 1)$ is a commutative monoid and $*$ is isotonic at both arguments;
(3) $(*, \to)$ is an adjoint pair, i.e. $x * y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$.

A residuated lattice is said to be complete if the underlying lattice is complete.

In what follows, $*$ is sometimes called a generalized triangular norm and the implicator $\to$ is called the residuum of $*$. An implicator I is called left monotonic (respectively, right monotonic) if $I(\cdot, a)$ is decreasing for every $a \in L$ (respectively, $I(a, \cdot)$ is increasing). If I is both left monotonic and right monotonic, then it is called hybrid monotonic.

In the paper, L denotes a complete residuated lattice if not otherwise specified. Some basic properties of complete residuated lattices are collected from [4, 31].

(1) $1 \to a = a,\ a \le b \Leftrightarrow a \to b = 1$;
(2) $(a \to b) * (b \to c) \le a \to c$;
(3) $a \to (b \to c) = b \to (a \to c) = (a * b) \to c$;
(4) $a \le (b \to a * b),\ a * (a \to b) \le b$;
(5) $a * (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a * b_i)$;
(6) $a \to (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \to b_i),\ (\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I}(a_i \to b)$;
(7) $a \to (b \to a) = 1,\ a \to (b \to a * b) = 1$;
(8) $b \to c \le (a \to b) \to (a \to c),\ c \to b \le (b \to a) \to (c \to a)$.

When the operator $*$ is exactly the operator $\wedge$ of the residuated lattice, such structure is called a Heyting algebra. A complete Heyting algebra is a special case of a complete residuated lattice which is also called a frame.

In [18], Goguen introduced L-fuzzy set as a generalization of Zadeh's-fuzzy set with L being a complete residuated lattice. An L-fuzzy set A on U is a map $A : U \to L$ and all the L-fuzzy sets on U are denoted by L^U. For every $a \in L$, we use $\bar{a}$ to denote the constant L-fuzzy set on U. For $A, B \in L^U$, we denote $A \subseteq B$ if $A(x) \le B(x)$ for every $x \in U$.

Given two L-fuzzy sets A and B, new L-fuzzy sets can be induced as follows:

$$(A * B)(x) = A(x) * B(x);$$
$$(A \wedge B)(x) = A(x) \wedge B(x);$$
$$(A \vee B)(x) = A(x) \vee B(x);$$
$$(A \to B)(x) = A(x) \to B(x);$$
$$(\neg A)(x) = A(x) \to 0.$$

An L-fuzzy relation R on U is a map $R : U \times U \to L$.
(1) R is reflexive if $R(x, x) = 1$ for all $x \in U$;
(2) R is transitive if $\bigvee_{y\in U} R(x, y) * R(y, z) \le R(x, z)$ for all $x, z \in U$;
(3) R is symmetric if $R(x, y) = R(y, x)$ for all $x, y \in U$;
(4) R is antisymmetric if for all $x, y \in L,\ R(x, y) = R(y, x) = 1$ implies $x = y$.

In the paper, we use fuzzy set instead of L-fuzzy set for the consistency of the context.

Definition 2.1. [15] (1) Let X be a set, $e : X \times X \to L$ a fuzzy relation. e is called a fuzzy partial order on X if it is reflexive, antisymmetric and transitive. The pair (X, e) is called a fuzzy partially ordered set, fuzzy poset for short.
(2) Let $(X, e), (Y, e_Y)$ be two fuzzy posets. A map $f : (X, e) \to (Y, e_Y)$ is said to be fuzzy order-preserving if $e(x, y) \le e_Y(f(x), f(y))$ for all $x, y \in X$.

The fuzzy order used in the paper was independently introduced by Bělohlávek in [2, 3] and Fan, Zhang [15, 36]. It has been proved in [34] that they are equivalent with each other.

Example 2.2. (1) Let e be a fuzzy partial order on X. The dual fuzzy relation $e^{-1} \in L^{X\times X}$ defined by $e^{-1}(x,y) = e(y,x)$ of e, for all $x, y \in X$, is also a fuzzy partial order on X. Moreover, the symmetrization $e^s = e \wedge e^{-1}$ of e is a fuzzy partial order on X.
(2) Let (X,e) be a fuzzy poset and $X_1 \subseteq X$, then $e_{X_1} \in L^{X_1\times X_1}$ defined by $e_{X_1}(x,y) = e(x,y)$, for every $x, y \in X_1$, is a fuzzy partial order on X_1. And (X_1, e_{X_1}) is called a sub fuzzy poset of (X,e).
(3) Let $\{(X,e_i), i \in I\}$ be a family of fuzzy posets. Define $e(x,y) = \bigwedge_{i\in I} e_i(x_i,y_i)$ for every $(x,y) = ((x_i)_{i\in I}, (y_i)_{i\in I})$, then it is a fuzzy partial order on the product of $\{(X,e_i), i \in I\}$.
(4) Let $L = \{0,1\}$ be the truth values. Then the fuzzy partial orders on a set X are just the classical partial orders on X.

Let (X,e) be a fuzzy poset, we define $\leq = \{(x,y)|e(x,y) = 1\}$. Then $\leq$ is a classical partial order on X and $(X,\leq)$ is called the underlying partially ordered set of (X,e). We denote it by X_0. If we take $L = \{0,1\}$, then the fuzzy poset based on L is a classical poset. So fuzzy posets are generalizations of posets. For a classical partial order $\leq$, a fuzzy partial order $e_{\leq}$ is defined as $e_{\leq}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$
Now we review some special fuzzy subsets of a fuzzy poset, named fuzzy lower set and fuzzy upper set.

Definition 2.3. [32, 37] Let (X,e) be a fuzzy poset. $A \in L^X$ is called a fuzzy lower set if $A(x) * e(y,x) \leq A(y)$ for all $x, y \in X$; $B \in L^X$ is called a fuzzy upper set if $B(x) * e(x,y) \leq B(y)$ for all $x, y \in X$.

Next, we recall some preliminaries of fuzzy complete lattices [32, 37].

Definition 2.4. [32, 37] Let (X,e) be a fuzzy poset, $x_0, x_1 \in X$ and $A \in L^X$.
(1) x_0 is a join of $A \Longleftrightarrow \forall x \in X, e(x_0,y) = \bigwedge_{x\in X} A(x) \to e(x,y)$.
(2) x_1 is a meet of $A \Longleftrightarrow \forall x \in X, e(y,x_1) = \bigwedge_{x\in X} A(x) \to e(y,x)$.

We denote the join and meet of $A \in L^X$ by $\bigsqcup A$ and $\bigsqcap A$ respectively. The join and meet of a fuzzy subset, if they exist, then they are unique.

Let (X,e) be a fuzzy poset, for every $A \in L^X$, $\downarrow A$ and $\uparrow A$ are defined as follows [32, 37]:

$$\downarrow A(x) = \bigvee_{y\in X} A(y) * e(x,y); \quad \uparrow A(x) = \bigvee_{y\in X} A(y) * e(y,x).$$

Then $\bigsqcup A = \bigsqcup \downarrow A$ and $\bigsqcap A = \bigsqcap \uparrow A$ [32, 37] if they exist.

Let (X,e) and (Y,e_Y) be fuzzy posets, $f : (X,e) \to (Y,e_Y)$, we define fuzzy forward powerset operators as follows:

$$\widetilde{f}_*^{\to}(A)(y) = \bigvee_{x\in X} A(x) * e_Y(y, f(x)), \ \widetilde{f}^{*\to}(A)(y) = \bigvee_{x\in X} A(x) * e_Y(f(x), y).$$

Definition 2.5. [37] Let (X,e) be a fuzzy poset.
(1) (X,e) is called a fuzzy complete lattice if $\bigsqcup A$ and $\bigsqcap A$ exist for every fuzzy subset $A \in L^X$.

(2) Let (X, e) and (Y, e_Y) be fuzzy complete lattices. A map $f : (X, e) \to (Y, e_Y)$ is said to be fuzzy join-preserving if it satisfies $f(\bigsqcup A) = \bigsqcup \widetilde{f_*}^{\to}(A)$ for every $A \in L^X$; and f is said to be fuzzy meet-preserving if it satisfies $f(\bigsqcap A) = \bigsqcap \widetilde{f}^{*\to}(A)$ for every $A \in L^X$. f is called a fuzzy complete lattice homomorphism if it is fuzzy join-preserving and fuzzy meet-preserving.

Let (X, e) be a fuzzy complete lattice. Then the underlying poset X_0 is a complete lattice. For every $A \subseteq X$, $\bigvee A = \bigsqcup \chi_A$, where χ_A is the characteristic function of A.

Proposition 2.6. *Let $(X, e), (Y, e_Y)$ be fuzzy complete lattices and $f : (X, e) \to (Y, e_Y)$ a fuzzy complete lattice homomorphism. Then $f : X_0 \to Y_0$ is a complete lattice homomorphism.*

Proof. Since $\widetilde{f_*}^{\to}(\chi_A)(y) = \bigvee_{x \in A} \chi_A(x) \wedge e_Y(y, f(x)) = \bigvee_{x \in A} e_Y(y, f(x)) = \downarrow \chi_{f(A)}(y)$ for every $y \in Y$, we have $f(\bigsqcup \chi_A) = \bigsqcup \widetilde{f_*}^{\to}(\chi_A) = \bigsqcup \downarrow \chi_{f(A)} = \bigsqcup \chi_{f(A)}$, i.e. f preserves arbitrary joins. $\square$

Proposition 2.7. [37] *Let (X, e) be a fuzzy poset. Then*
(1) (X, e) is a fuzzy complete lattices if and only if $\bigsqcup \phi$ exist for all $\phi \in L^X$;
(2) (X, e) is a fuzzy complete lattices if and only if $\bigsqcap \phi$ exist for all $\phi \in L^X$.

In [3], theory of formal context is generalized to L-context, where L is a complete residuated lattice. The author devoted much of their effort to study the properties of L-context. In the following, we recall some basic concepts in the fuzzy formal concept analysis.

Definition 2.8. [3] A formal L-context is a triple (X, Y, I), where $I : X \times Y \to L$. The elements of X and Y are called objects and attributes respectively, and the degree $I(x, y)$ is interpreted as the truth degree of which the object $x \in X$ has the attribute $y \in Y$.

Definition 2.9. [3] Let (X, Y, I) be an L-context. For $A \in L^X, B \in L^Y$, $A^{\uparrow}, B_{\downarrow}$ are defined as follows:

$$\begin{aligned} A^{\uparrow}(y) &= \bigwedge_{x \in X} A(x) \to I(x, y), \forall y \in Y; \\ B^{\downarrow}(x) &= \bigwedge_{y \in Y} B(y) \to I(x, y), \forall x \in X. \end{aligned}$$

$A^{\uparrow}$ is the fuzzy set of all attributes common to all objects from A and $B^{\downarrow}$ is the fuzzy set of all objects common to all attributes from B. A pair $\langle A, B\rangle$ is called a formal L-concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$, where $A \in L^X, B \in L^Y$. The set $\mathscr{B}_I = \{\langle A, B\rangle \in L^X \times L^Y | A^{\uparrow} = B, B^{\downarrow} = A\}$ is called the L-concept lattice. The fuzzy partial order E on $\mathscr{B}_I$ is defined by

$$E(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = \widetilde{e}(A_1, A_2) = \bigwedge_{x \in X} A_1(x) \to A_2(x) \quad (\text{or} = \widetilde{e}(B_2, B_1)).$$

For every $\mathscr{U} \in L^{\mathscr{B}_I}$,

$$\bigsqcup \mathscr{U} = \langle (\bigwedge_{\langle A,B\rangle \in \mathscr{B}_I} \mathscr{U}(\langle A, B\rangle) \to B)^{\downarrow}, \bigwedge_{\langle A,B\rangle \in \mathscr{B}_I} \mathscr{U}(\langle A, B\rangle) \to B\rangle; \tag{1}$$

$$\bigsqcap \mathscr{U} = \langle \bigwedge_{\langle A,B\rangle \in \mathscr{B}_I} \mathscr{U}(\langle A, B\rangle) \to A, (\bigwedge_{\langle A,B\rangle \in \mathscr{B}_I} \mathscr{U}(\langle A, B\rangle) \to A)^{\uparrow}\rangle. \tag{2}$$

A reflexive, symmetric and transitive fuzzy relation R on a set X is called a fuzzy equivalence relation. Zadeh introduced fuzzy equivalence relations[35] as a generalization of equivalence relations. Since then researchers[12, 13, 14] have devoted much effort to study fuzzy equivalence relations in order to measure the degree of indistinguishability or similarity between the objects of a given set. And also fuzzy equivalence relations have been applied to different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc. As fuzzy equivalence relations appeared, they have been given various names, such as similarity relations[35], indistinguishability operators [12, 13, 14], T-equivalences [10, 11], many-valued equivalence relations. In [8], the authors investigate various properties of equivalence classes of fuzzy equivalence relations over a complete residuated lattice systematically. The set of fuzzy equivalence relations on given set X forms a complete lattice. The meet coincides with the fuzzy sets intersection of the fuzzy relations, but the join does not coincide with the ordinary fuzzy sets union.

Example 2.10. Let $X = \{x_1, x_2, x_3, x_4\}$ and $L = \{0, 0.5, 1\}$.
(1) We define $e : X \times X \to L$ as follows:

$$e = (e_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 \\ 1 & 0.5 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Obviously, we can see that (X, e) is a fuzzy poset.
(2)We define $R : X \times X \to L$ as follows:

$$R = (R_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0.5 & 1 & 1 \\ 0 & 0.5 & 1 & 1 \end{pmatrix}$$

According to the definition, R is a fuzzy equivalence relation.

Let R be a fuzzy equivalence relation on a set X. For every $a \in X, R_a$ is called a fuzzy equivalence class determined by R, where $R_a(y) = R(a, y)$ for every $y \in X$. The set $X/R = \{R_a, a \in X\}$ is called the factor set of X with respect to R.

Lemma 2.11. [8] *Let R be a fuzzy equivalence relation on a set X. Then for every $x, y \in X$, the following is true:*
(1) $R(x, y) = \bigvee_{z \in X} R(x, z) * R(y, z)$;
(2) $R(x, y) = 1 \Leftrightarrow R_x = R_y$.

For every $p \in L, p \neq 0$, the p cut relation is defined by $R_p = \{(x, y) | R(x, y) \geq p\}$. Obviously, R_p is a classical equivalence relation on X. More about order congruences on posets can be referred to [23].

3. Fuzzy Order Congruences

Let (X, e) be a fuzzy poset, R a fuzzy equivalence relation on X. For all $x, y \in X$, we define

$$R_e(x, y) = e \circ R(x, y) = \bigvee_{z \in X} e(x, z) * R(z, y).$$

Obviously, R_e is reflexive. However, R_e is not transitive in general. The transitive closure [19] of R_e is defined by $R_e^T = \bigcup_{i=1}^{\infty} R_e^n$, where $R_e^n = R_e^{n-1} \circ R_e, n \geq 2$.

It is easy to verify that the following two properties hold:
(i) $e(x,y) \leq R_e^T(x,y)$;
(ii) $R(x,y) \leq R_e^T(x,y) \wedge (R_e^T)^{-1}(x,y)$.

Definition 3.1. Let (X,e) be a fuzzy poset, R a fuzzy equivalence relation on X. R is called a fuzzy order congruence if $R(x,y) = R_e^T(x,y) \wedge (R_e^T)^{-1}(x,y)$ for all $x,y \in X$.

A suitable fuzzy partial order $e_{X/R}$ on the fuzzy quotient set X/R is defined by

$$e_{X/R}(R_x, R_y) = R_e^T(x,y) \quad \text{for all } x,y \in X. \tag{3}$$

Obviously, $e_{X/R} : X/R \times X/R \to L$ is a map. In the following, we prove that it is a fuzzy partial order on X/R. First, $e_{X/R}(R_x, R_x) = R_e^T(x,x) = 1$, so $e_{X/R}$ is reflexive. Second, $e_{X/R}(R_x, R_y) = e_{X/R}(R_y, R_x) = 1$ implies $R_e^T(x,y) = R_e^T(y,x) = 1$, i.e. $R(x,y) = 1$. From the Lemma 2.11, we have $R_x = R_y$. Thus $e_{X/R}$ is antisymmetric. Third, $e_{X/R}(R_x, R_y) * e_{X/R}(R_y, R_z) = R_e^T(x,y) * R_e^T(y,z) \leq R_e^T(x,z) = e_{X/R}(R_x, R_z)$ which follows that $e_{X/R}$ is transitive.

Proposition 3.2. *Let (X,e) be a fuzzy poset, R a fuzzy order congruence on (X,e). Then the fuzzy quotient map $\eta_R : (X,e) \to (X/R, e_{X/R}), x \mapsto R_x$ is fuzzy order-preserving.*

Example 3.3. Let X, e, R be defined as in Example 2.10. We could verify that R is a fuzzy order congruence with respect to the fuzzy partial order e on X.

Firstly, through direct computing, we have

$$R_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

And we can see that R_e is reflexive, but not transitive.

Secondly, we could get $R_e^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. It is easy to see that $R_e \neq R_e^T$.

Thirdly, $R = R_e^T \wedge (R_e^T)^{-1}$. So R is a fuzzy order congruence on the fuzzy poset (X,e). And the corresponding fuzzy quotient set is $X/R = \{R_{x_1}, R_{x_2}, R_{x_3} = R_{x_4}\}$, the fuzzy partial order on X/R is defined by:

$e_{X/R}(R_x, R_y) = R_e^T(x,y)$ and $e_{X/R} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \end{pmatrix}$.

The next lemma is the characterization of the fuzzy preorders that contain the fuzzy partial order. It is essential to obtain the relationship between the fuzzy order congruence and the fuzzy preorder that contains the fuzzy partial order.

Lemma 3.4. *Let (X, e) be a fuzzy poset, $\sigma \in L^{X\times X}$. Then σ is a fuzzy preorder that contains the fuzzy partial order if and only if there is a fuzzy poset (Y, e_Y) and a fuzzy order-preserving map $f : (X, e) \to (Y, e_Y)$ with $\sigma = H_f$, where $H_f(x, y) = \bigwedge_{a\in Y} e_Y(f(y), a) \to e_Y(f(x), a)$ is called the direct kernel of f.*

Proof. Assume $f : (X, e) \to (Y, e_Y)$ is a fuzzy order-preserving map. Obviously, we have

$$e_Y(f(x), f(y)) = H_f(x, y),\ \forall x, y \in X.$$

Since f is fuzzy order-preserving, it holds that $e(x, y) \le e_Y(f(x), f(y)) = H_f(x, y)$ for all $x, y \in X$. By the transitivity of e_Y, we have $H_f \circ H_f \le H_f$.

Conversely, assume σ is a fuzzy preorder containing the fuzzy partial order, we define $R = \sigma \wedge \sigma^{-1}$. Then $R \le R_e^T \wedge (R_e^T)^{-1} \le \sigma \wedge \sigma^{-1} = R$. That is, R is a fuzzy order congruence. And we get a fuzzy poset $(X/R, e_\sigma)$, where $e_\sigma(R_x, R_y) = \sigma(x, y)$. In the following, we show that $H_{\eta_R} = \sigma$. On the one hand,

$$\begin{aligned} H_{\eta_R}(x, y) &= \bigwedge_{u\in X/R} e_{X/R}(R_y, u) \to e_{X/R}(R_x, u) \\ &\ge \bigwedge_{z\in X} \sigma(y, z) \to \sigma(x, z) \\ &\ge \sigma(x, y). \end{aligned}$$

On the other hand, $H_{\eta_R}(x, y) \le 1 \to \sigma(x, y) = \sigma(x, y)$. □

According to the characterization of the special fuzzy preorder on the fuzzy poset, we obtain the theorem below.

Theorem 3.5. *Let (X, e) be a fuzzy poset, R a fuzzy equivalence relation on X. Then the following are equivalent:*
(i) R is a fuzzy order congruence on (X, e);
(ii) There exists a fuzzy preorder σ that contains the fuzzy partial order such that $R = \sigma \wedge \sigma^{-1}$;
(iii) There exists a fuzzy poset (Y, e_Y) and a fuzzy order-preserving map $f : (X, e) \to (Y, e_Y)$ with $R = k_f$.

Proof. $(ii) \iff (iii)$: This is a deduction of Lemma 3.4. We only need to prove $(i) \iff (ii)$.

$(i) \Longrightarrow (ii)$: From the second part of the proof of Lemma 3.4, H_{η_R} is the fuzzy preorder that we need.

$(ii) \Longrightarrow (i)$: Define $R = \sigma\wedge\sigma^{-1}$, we prove that R is a fuzzy order congruence. For all $x, y \in X$, $e(x, y) \le \sigma(x, y)$ and $R(x, y) \le \sigma(x, y)$, so we have $R \le R_e^T \wedge (R_e^T)^{-1} \le \sigma \wedge \sigma^{-1} = R$. Therefore, R is a fuzzy order congruence. □

Remark 3.6. Let (X, e) be a fuzzy poset, R a fuzzy order congruence on (X, e). Then R_1 is an order congruence on the underlying poset X_0.

Because for a given fuzzy order congruence R, the quotient set X/R may support several different compatible fuzzy orders, it is necessary to specify which fuzzy order is being considered on the quotient set X/R. A fuzzy preorder $\rho \ge e$ on (X, e) can generate a fuzzy order congruence $\overline{\rho} = \rho \wedge \rho^{-1}$. The induced fuzzy partial order $e_{X/\overline{\rho}}$ on the quotient fuzzy ordered set is defined by $e_{X/\overline{\rho}}(\overline{\rho}_x, \overline{\rho}_y) = \rho(x, y)$, which is denoted by e_ρ for convenience.

Let R be a fuzzy order congruence on a fuzzy poset (X, e). The fuzzy preorder that contains the fuzzy partial order e and R is not unique. From Theorem 3.5, we obtain the following corollary.

Corollary 3.7. *Let (X, e) be a fuzzy poset, R a fuzzy order congruence on (X, e). Then R_e^T is the least fuzzy preorder that contains e and R.*

Definition 3.8. [32] Let (X, e), (Y, e_Y) be fuzzy posets. A map $f : (X, e) \to (Y, e_Y)$ is said to be
(1) fuzzy order embedding if $e(x, y) = e_Y(f(x), f(y))$ for all $x, y \in X$;
(2) fuzzy order isomorphism if it is a surjective fuzzy order embedding.

The homomorphism theorem of fuzzy poset can be easily derived.

Theorem 3.9. *Let (X, e), (Y, e_Y) be fuzzy posets, $f : (X, e) \to (Y, e_Y)$ a surjective fuzzy order-preserving map. Then there exists a unique fuzzy order isomorphism $g : (X/k_f, e_{H_f}) \to (Y, e_Y)$ such that $g \circ \eta_{k_f} = f$, where $ker f = H_f \wedge H_f^{-1}$.*

Proof. For the sake of convenience, let $R = k_f$. Then R is a fuzzy order congruence in terms of Theorem 3.5 and $(X/R, e_{X/R})$ is the quotient fuzzy poset, where $e_{X/R}(R_x, R_y) = H_f(x, y)$.
1) Define a map $g : (X/R, e_{X/R}) \to (Y, e_Y)$ by $g(R_x) = f(x)$. We show that it is a well-defined map. Assume $R_x = R_y$, we have

$$1 = R(x, y) = H_f(x, y) \wedge H_f^{-1}(x, y) = e_Y(f(x), f(y)) \wedge e_Y(f(y), f(x)).$$

i.e., $e_Y(f(x), f(y)) = e_Y(f(y), f(x)) = 1$. Then we have $f(x) = f(y)$ from the antisymmetry of e_Y.
2) According to the construction of the fuzzy order on the quotient set, we have

$$e_{X/R}(R_x, R_y) = H_f(x, y) = e_Y(f(x), f(y)) = e_Y(g(R_x), g(R_y)),$$

i.e. g is fuzzy order embedding. The surjectivity and uniqueness of g is obvious. Therefore, g is a unique fuzzy order isomorphism and that $g : X/k_f \to Y$ such that $g \circ \eta_{k_f} = f$. $\square$

The next theorem is a generalization of Theorem 3.9, which is also known as the decomposition theorem.

Theorem 3.10. *Let (A, e_A), (B, e_B) and (C, e_C) be fuzzy posets. Assume $f : (A, e_A) \to (B, e_B)$ is a surjective fuzzy order-preserving map, $g : (A, e_A) \to (C, e_C)$ a fuzzy order-preserving with $H_f \leq H_g$. Then there exists a unique fuzzy order-preserving $h : (B, e_B) \to (C, e_C)$ such that $h \circ f = g$. Moreover, $H_f = H_g$ if and only if h is fuzzy order embedding; h is surjective if and only if g is surjective.*

Proof. Since f is surjective, for every $a \in B$, there exists $x \in A$ such that $f(x) = a$. We define $h : (B, e_B) \to (C, e_C)$ by $h(a) = g(x)$.

First, it is a well-defined map. Assume there exist $x, y \in A$ such that $f(x) = f(y) = a$, we have

$$1 = H_f(x,y) = e_B(f(x), f(y)) \leq H_g(x,y) = e_C(g(x), g(y)).$$

Similarly, we have $e_C(g(y), g(x)) = 1$. According to the antisymmetry of e_C, $g(x) = g(y)$.

Second, for every $a, b \in B$, there exist $x, y \in A$ such that $f(x) = a, f(y) = b$. Because

$$e_B(a,b) = e_B(f(x), f(y)) = H_f(x,y) \leq H_g(x,y) = e_C(g(x), g(y)) = e_C(h(a), h(b)),$$

h is fuzzy order-preserving. From the definition of h, $h \circ f = g$.

Third, assume there exists another $h' : (B, e_B) \to (C, e_C)$ such that $h' \circ f = g$. Since for every $a \in B$, there exists $x \in A$ such that $f(x) = a$. So we have $h(a) = h(f(x)) = g(x)$ and $h'(a) = h'(f(x)) = g(x)$. Hence $h(a) = h'(a)$. That is to say, h is unique.

Moreover, $H_f = H_g$ is equivalent to

$$e_B(a,b) = e_B(f(x), f(y)) = H_f(x,y) = H_g(x,y) = e_C(g(x), g(y)) = e_C(h(a), h(b)),$$

i.e., h is fuzzy order embedding.

Assume g is surjective, then for every $u \in C$, there exists $x \in A$ such that $g(x) = u$. Since $h \circ f = g$, h is surjective. □

Let (X, e) be a fuzzy poset, in the following, we will discuss a fuzzy order congruences generated by an arbitrary reflexive fuzzy relation. Let $H \in L^{X \times X}$ and H reflexive. Then $(H_e^T) \wedge (H_e^T)^{-1}$ is the least fuzzy order congruence that contains H. We denote it by $\overline{H}$. So we easily obtain the following proposition.

Proposition 3.11. *For arbitrary reflexive fuzzy relations H and K on a fuzzy poset (X, e), the map $f : X/\overline{H} \to X/\overline{K}$ defined by $f(\overline{H}_x) = \overline{K}_x$ is a fuzzy order isomorphism if and only if*
1. $H(x,y) \leq (K_e^T)(x,y)$;
2. $K(x,y) \leq (H_e^T)(x,y)$.

In the classical case of set theory, an equivalence relation on a set X can be deduced by a subset A of X by $R_A = \{(x,x) | x \in X \backslash A\} \cup \{(x,y) | x, y \in A\}$. In fuzzy case, a fuzzy equivalence relation [8] could also be induced by a fuzzy subset of X by

$$R_A(x,y) = A(x) \leftrightarrow A(y). \tag{4}$$

In the following theorem, we will discuss the fuzzy order congruence induced by a special fuzzy subset of the fuzzy poset (X, e). For convenience, the short form R of R_A is used. This will lead to no confusion.

Theorem 3.12. *Let (X, e) be a fuzzy poset. If $A \in L^X$ be a fuzzy lower subset of (X, e), then R defined by equation (3) is a fuzzy order congruence on (X, e).*

Proof. For the sake of convenience, let $\rho(x,y) = A(y) \to A(x)$. It is not difficult to verify that ρ is transitive. Because A is a fuzzy lower set of (X, e), we have

$A(y) * e(x,y) \leq A(x) \Leftrightarrow e(x,y) \leq A(y) \to A(x)$ for every $x, y \in X$. So $\rho \geq e$ is a fuzzy preorder on (X, e). Then we have $R \leq R_e = e \circ R \leq \rho \circ \rho \leq \rho$. So

$$R \leq R_e \wedge R_e^{-1} \leq \rho \wedge \rho^{-1} = R.$$

Therefore, R is a fuzzy order congruence induced by A on (X, e). □

Dually, given a fuzzy upper subset B of (X, e), a fuzzy order congruence determined by B is given by $B = \sigma \wedge \sigma^{-1}$, where σ is a fuzzy preorder determined by the fuzzy upper subset B $\sigma(x,y) = B(x) \to B(y)$. It can been seen that, the fuzzy preorders determined by the fuzzy lower subsets and the fuzzy upper subsets are different.

Fuzzy congruences play the vital role in the discussion of the colimit of the category of fuzzy posets. Next, we construct the coequalizer, pushout in terms of fuzzy order congruences.

Let (A, e_A) and (B, e_B) be fuzzy posets, $f, g : (A, e_A) \to (B, e_B)$ fuzzy order-preserving. Define a fuzzy relation K on (B, e_B), for every $x, y \in B$,

$$K(x,y) = \begin{cases} 1, & \text{if } x = y \text{ or } x = f(a), y = g(a) \text{ or } x = g(a), y = f(a); \\ 0, & \text{otherwise .} \end{cases}$$

It is easy to see that K is a fuzzy equivalence relation. The fuzzy order congruence $\nu(K)$ on (B, e_B) is generated by K.

Proposition 3.13. *Let (A, e_A), (B, e_B) be fuzzy posets, $f, g : (A, e_A) \to (B, e_B)$ fuzzy order-preserving. Then the coequalizer of f and g is the quotient fuzzy poset $(B/\nu(K), e_{\nu(K)})$.*

Proof. We only need to prove the following: for every fuzzy poset (C, e_C) and the fuzzy order-preserving map $h : (B, e_B) \to (C, e_C)$ with $h \circ f = h \circ g$, there exists exactly one fuzzy order-preserving map $m : (B/\nu(K), e_{\nu(K)}) \to (C, e_C)$ such that $m \circ \eta = h$. For the convenience, we let $R = \nu(K)$. Define $m : (B/\nu(K), e_{\nu(K)}) \to (C, e_C)$ for any $x \in B$, $m(R_x) = h(x)$.

First, m is well-defined. For any $x, y \in B$ and $R(x,y) = 1$, i.e., $e_R(R_x, R_y) = e_R(R_y, R_x) = 1$. Since $h \circ f = h \circ g$, we have $K(x,y) \leq H_h(x,y) = e_C(h(x), h(y))$. And because H_h is a fuzzy preorder with $e \leq H_h$, so $(\alpha(K))^T_{e_B}(x,y) = e_R(R_x, R_y) \leq H_h(x,y)$. Therefore, $e_C(h(x), h(y)) = e_C(h(y), h(x)) = 1$, i.e. $h(x) = h(y)$.

Second, it is easy to verify that m is the unique fuzzy order-preserving map such that $m \circ \eta = h$. □

The pushout of the pair $f : (A, e_A) \to (B, e_B)$ and $g : (A, e_A) \to (C, e_C)$ is constructed as bellow: $(B + C/R, e_R)$ is the quotient fuzzy poset, where $B + C$ is the coproduct of (B, e_B) and (C, e_C), R is the fuzzy order congruence generated by the following fuzzy relation on $B + C$,

$$K(x,y) = \begin{cases} 1, & \text{if } x = y \text{ or } x = (1, f(a)), y = (2, g(a)) \text{ or } x = (2, g(a)), y = (1, f(a)); \\ 0, & \text{otherwise .} \end{cases}$$

The verification is similar to Proposition 3.13.

Since the category of fuzzy posets **FPos** has all products, coproducts, equalizers and coequalizers, we come into conclusion that **FPos** is complete and cocomplete. For more details about the category theory, please refer to [1].

4. **Lattice Properties of Fuzzy Preorders containing the Fuzzy Partial Order**

In this section, we discuss the order properties of the set of fuzzy preorders containing the fuzzy partial order on a fuzzy poset. Let (X, e) be a fuzzy poset, ρ, σ two fuzzy preorders on (X, e) that contain e. If $\rho \leq \sigma$, we define a fuzzy relation σ/ρ on $(X/\overline{\rho}, e_\rho)$ as follows: for all $x, y \in X$,

$$(\sigma/\rho)(\overline{\rho}_x, \overline{\rho}_y) = \sigma(x, y). \tag{5}$$

Remark 4.1. σ/ρ is a fuzzy preorder on $(X/\overline{\rho}, e_\rho)$ that contain e_ρ. First, $e_\rho(\overline{\rho}_x, \overline{\rho}_y) = \rho(x, y) \leq \sigma(x, y) = (\sigma/\rho)(\overline{\rho}_x, \overline{\rho}_y)$ for all $x, y \in X$. Second, for all $x, y \in X$,

$$\begin{aligned}(\sigma/\rho) \circ (\sigma/\rho)(\overline{\rho}_x, \overline{\rho}_y) &= \bigvee_{u \in X/\overline{\rho}} (\sigma/\rho)(\overline{\rho}_x, u) * (\sigma/\rho)(u, \overline{\rho}_y) \\ &\leq \bigvee_{a \in X} \sigma(x, a) * \sigma(a, y) \leq \sigma(x, y) = (\sigma/\rho)(\overline{\rho}_x, \overline{\rho}_y)\end{aligned}$$

i.e. σ/ρ is transitive.

The fuzzy preorder σ/ρ can define a fuzzy order congruence on $X/\overline{\rho}$ by $(\sigma/\rho) \wedge (\sigma/\rho)^{-1}$. So in the following theorem, we discuss the relationship between $(X/\overline{\rho})/\overline{(\sigma/\rho)}$ and $X/\overline{\sigma}$.

Proposition 4.2. *Let (X, e) be a fuzzy poset, ρ, σ two fuzzy preorders on (X, e) with $e \leq \rho \leq \sigma$. Then $(X/\overline{\rho})/\overline{(\sigma/\rho)}$ and $X/\overline{\sigma}$ is fuzzy order isomorphic, where σ/ρ is defined by* (4).

Proof. We define a map $\phi : X/\overline{\rho} \to X/\overline{\sigma}$ via $\phi(\overline{\rho}_x) = \overline{\sigma}_x$ for all $x \in X$. Obviously, it is well-defined and surjective. Since $e_\rho(\overline{\rho}_x, \overline{\rho}_y) = \rho(x, y) \leq \sigma(x, y) = e_\sigma(\overline{\sigma}_x, \overline{\sigma}_y)$, we have ϕ is fuzzy order-preserving. And $H_\phi(\overline{\rho}_x, \overline{\rho}_y) = e_\sigma(\overline{\sigma}_x, \overline{\sigma}_y) = \sigma(x, y) = (\sigma/\rho)(\overline{\rho}_x, \overline{\rho}_y)$. According to Theorem 3.9, $X/\overline{\rho}/\overline{(\sigma/\rho)}$ and $X/\overline{\sigma}$ are fuzzy order isomorphic. □

Let (X, e) be a fuzzy poset, ρ a fuzzy preorder on (X, e). For a fuzzy preorder θ on $(X/\overline{\rho}, e_\rho)$ with $e_\rho \leq \theta$, we could define a fuzzy relation $\theta \bullet \rho$ on (X, e) as follows: for all $x, y \in X$:

$$(\theta \bullet \rho)(x, y) = \theta(\overline{\rho}_x, \overline{\rho}_y). \tag{6}$$

Moreover, it is a fuzzy preorder on (X, e). First, for all $x, y \in X$,

$$e(x, y) \leq \rho(x, y) = e_\rho(\overline{\rho}_x, \overline{\rho}_y) \leq \theta(\overline{\rho}_x, \overline{\rho}_y) = (\theta \bullet \rho)(x, y);$$

Second,

$$\begin{aligned}(\theta \bullet \rho) \circ (\theta \bullet \rho)(x, y) &= \bigvee_{a \in X} (\theta \bullet \rho)(x, a) * (\theta \bullet \rho)(a, y) \\ &= \bigvee_{a \in X} \theta(\overline{\rho}_x, \overline{\rho}_a) * \theta(\overline{\rho}_a, \overline{\rho}_y) \\ &\leq \theta(\overline{\rho}_x, \overline{\rho}_y) = (\theta \bullet \rho)(x, y);\end{aligned}$$

i.e. $\theta \bullet \rho$ is a fuzzy preorder on (X, e) satisfying $e \leq \theta \bullet \rho$. It is easy to obtain the following proposition.

Proposition 4.3. *The fuzzy poset* $((X/\overline{\rho})/\overline{\theta}, e_\theta)$ *and the fuzzy poset* $(X/\overline{\theta \bullet \rho}, e_{\theta\bullet\rho})$ *are fuzzy order isomorphic.*

Proof. We define a map $\psi : X/\overline{\rho} \to (X/\overline{\rho})/\overline{\theta}$ by $\psi(\overline{\rho}_x) = \overline{\theta}_{\overline{\rho}_x}$. Obviously, it is well-defined and surjective fuzzy order-preserving. From the proof of Lemma 3.4, $H_\psi(\overline{\rho}_x, \overline{\rho}_y) = \theta(\overline{\rho}_x, \overline{\rho}_y) = \overline{(\theta \bullet \rho)}(x, y)$. So we have $(X/\overline{\theta \bullet \rho}, e_{\theta\bullet\rho})$ is fuzzy order isomorphic to $((X/\overline{\rho})/\overline{\theta}, e_\theta)$ according to Theorem 3.9. □

We use $FPO(X, e)$ to denote the set of all fuzzy preorders on the fuzzy poset (X, e) that contain e. There is a natural order structure on (X, e): $\rho_1 \leq \rho_2$ if $\rho_1(x, y) \leq \rho_2(x, y)$ for $\rho_1, \rho_2 \in FPO(X, e)$, $x, y \in X$.

Proposition 4.4. *Let* (X, e) *be a fuzzy poset. Then* $FPO(X, e)$ *is a complete lattice.*

Proof. The constant fuzzy relation $\overline{1}_{X\times X}$ is obviously the greatest fuzzy preorder on (X, e) with $e \leq \overline{1}_{X\times X}$. For any $\rho_i, i \in I$, we prove $\bigwedge_{i\in I} \rho_i$ is a fuzzy preorder with $e \leq \bigwedge_{i\in I} \rho_i$. Then from [9], it implies that $FPO(X, e)$ is a complete lattice. For every $i \in I$,

$$(\bigwedge_{i\in I} \rho_i) \circ (\bigwedge_{i\in I} \rho_i) \leq \rho_i \circ \rho_i \leq \rho_i;$$

Therefore, it is transitive. It is obviously $e(x, y) \leq (\bigwedge_{i\in I} \rho_i)(x, y)$, because $e(x, y) \leq \rho_i(x, y)$ for every $i \in I, x, y \in X$. □

Obviously, the set of fuzzy order congruences is also a complete lattice.

Let ρ be a fuzzy preorder on (X, e) with $e \leq \rho$. The next theorem will establish a connection between $FPO(X/\overline{\rho}, e_{\overline{\rho}})$ and the set of all fuzzy preorders on (X, e) that contain ρ, and is denoted by $FP^\rho_\leq = \{\sigma \in FPO(X, e)|\ \rho \leq \sigma\}$.

Theorem 4.5. $FPO(X/\overline{\rho}, e_{\overline{\rho}})$ *and* $FP^\rho_\leq$ *are order isomorphic with the classical ordered structure.*

Proof. We define $f : FPO(X/\overline{\rho}, e_{\overline{\rho}}) \to \{\sigma \in FPO(X, e)|\ \rho \leq \sigma\}$ by $f(\theta) = \theta \bullet \rho$ and $g : \{\sigma \in FPO(X, e)|\ \rho \leq \sigma\} \to FPO(X/\overline{\rho}, e_{\overline{\rho}})$ by $g(\sigma) = \sigma/\rho$ for all $\theta \in FPO(X/\overline{\rho}, e_{\overline{\rho}})$, $\sigma \in \{\sigma \in FPO(X, e)|\ \rho \leq \sigma\}$. It is not difficult to verify that f, g are order-preserving in terms of their definition. Furthermore, they are inverse to each other. □

5. Fuzzy Complete Congruences on Fuzzy Complete Lattice

In this section, we introduce the fuzzy complete congruences of the fuzzy complete lattice and characterize them in terms of the fuzzy complete lattice homomorphisms.

Definition 5.1. Let (X, e) be a fuzzy complete lattice, R a fuzzy equivalence relation on X. R is called a fuzzy complete congruence on (X, e) if it satisfies
1. R is a fuzzy order congruence on (X, e);
2. $R^T_e(\bigsqcup A, y) = \bigwedge_{x\in X} A(x) \to R^T_e(x, y)$ and $R^T_e(y, \bigsqcap A) = \bigwedge_{x\in X} A(x) \to R^T_e(y, x)$ hold for every $A \in L^X, y \in X$.

Remark 5.2. Let (X,e) be a fuzzy complete lattice and R a fuzzy complete congruence on (X,e). Then the classical relation R_1 is a complete congruence for the underlying complete lattice X_0.

Naturally, we have the following proposition discussing relationship between the fuzzy complete congruence and the quotient fuzzy poset.

Proposition 5.3. *Let (X,e) be a fuzzy complete lattice, R a fuzzy complete congruence. Then the quotient fuzzy poset is a fuzzy complete lattice and the fuzzy quotient map is a fuzzy complete lattice homomorphism.*

Proof. For every $\mathscr{U} \in L^{X/R}, x \in X$, we could define $A \in L^X$ by $A(x) = \mathscr{U}(R_x)$. Since (X,e) is a fuzzy complete lattice, we may assume $\bigsqcup A = x_0$. We prove $\bigsqcup \mathscr{U} = R_{x_0}$. This is an easy verification in terms of the Definitions 2.5 and 5.1. The meet of every fuzzy set of $L^{X/R}$ can be found similarly.

For every $B \in L^X$, we may assume $\bigsqcup B = y_0$. For every $y \in X$, we have

$$\begin{aligned}
\bigwedge_{a\in X} \widetilde{\eta_{R*}}^{\rightarrow}(B)(R_a) \to e_R(R_a,R_y) &= \bigwedge_{a\in X}(\bigvee_{x\in X}(B(x) * R_e^T(a,x))) \to R_e^T(a,y),\\
&= \bigwedge_{x\in X} B(x) \to (\bigwedge_{a\in X} R_e^T(a,x) \to R_e^T(a,y)),\\
&= \bigwedge_{x\in X} B(x) \to R_e^T(x,y),\\
&= R_e^T(\bigsqcup B, y).
\end{aligned}$$

That is to say, the fuzzy quotient map is fuzzy join-preserving. Similarly, we could prove that it is fuzzy meet-preserving. □

Example 5.4. Let $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$ and $L = \{0, 0.5, 1\}$.
(1) A fuzzy partial order $e : X \times X \to L$ is defined as follows:

$$e = (e_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 1 & 1\\ 0 & 0.5 & 1 & 0.5 & 1 & 1\\ 0 & 0.5 & 0.5 & 1 & 1 & 1\\ 0 & 0.5 & 0.5 & 0.5 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And (X,e) is verified to be a fuzzy complete lattice[32].
(2) We define $R : X \times X \to L$ as follows:

$$R = (R_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 0.5 & 0.5 & 0\\ 1 & 1 & 1 & 0.5 & 0.5 & 0\\ 1 & 1 & 1 & 0.5 & 0.5 & 0\\ 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0\\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

According to the definition, R is a fuzzy equivalence relation. We illustrate that it is a fuzzy order congruence above all. Through direct computation, we have

$$R_e = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 0.5 & 1 & 0.5 & 1 & 1\\ 0.5 & 0.5 & 0.5 & 1 & 1 & 1\\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } R_e^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & 1 & 1\\ 0.5 & 0.5 & 0.5 & 1 & 1 & 1\\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that $R_e \neq R_e^T$ and $R = R_e^T \wedge (R_e^T)^{-1}$. So R is a fuzzy order congruence on the fuzzy poset (X,e). And the corresponding fuzzy quotient set is $X/R = \{R_{x_1} = R_{x_2} = R_{x_3}, R_{x_4}, R_{x_5}, R_{x_6}\}$, the fuzzy partial order on X/R is defined by $e_{X/R}(R_x, R_y) = R_e^T(x, y)$, where $e_{X/R}$ is described by the following matrix:

$$e_{X/R} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0.5 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, we can verify that it is a fuzzy complete congruence. According to Definition 5.1, we have to show that

$$R_e^T(\bigsqcup A, y) = \bigwedge_{x\in X} A(x) \to R_e^T(x, y) \text{ and } R_e^T(y, \bigsqcap A) = \bigwedge_{x\in X} A(x) \to R_e^T(y, x)$$

hold for every $A \in L^X, y \in X$. By definition of R_e^T, this is equivalent to prove the quotient fuzzy poset is a fuzzy complete lattice. Let $\phi : X/R \to L$ be any fuzzy set of X/R, we have to show that the join exists for every ϕ in terms of Proposition 2.7. In the following, we denote $X/R = \{a_1, a_2, a_3, a_4\}$ for convenience and ϕ by a vector of dimension 4 as follows:

$\phi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$
$\phi_2 = \begin{pmatrix} 0 & 0.5 & 0 & 0 \end{pmatrix}$
$\phi_3 = \begin{pmatrix} 0 & 0 & 0.5 & 0 \end{pmatrix}$
$\phi_4 = \begin{pmatrix} 0 & 0 & 0 & 0.5 \end{pmatrix}$
$\phi_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$
$\phi_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$
$\phi_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$
$\phi_8 = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \end{pmatrix}$
$\phi_9 = \begin{pmatrix} 0 & 0 & 0.5 & 1 \end{pmatrix}$
$\phi_{10} = \begin{pmatrix} 0 & 0 & 1 & 0.5 \end{pmatrix}$
$\phi_{11} = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$
$\phi_{12} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \end{pmatrix}$
$\phi_{13} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \end{pmatrix}$
$\phi_{14} = \begin{pmatrix} 0 & 0.5 & 1 & 0 \end{pmatrix}$
$\phi_{15} = \begin{pmatrix} 0 & 0.5 & 0 & 1 \end{pmatrix}$
$\phi_{16} = \begin{pmatrix} 0 & 1 & 0.5 & 0 \end{pmatrix}$
$\phi_{17} = \begin{pmatrix} 0 & 1 & 0 & 0.5 \end{pmatrix}$
$\phi_{18} = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$
$\phi_{19} = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$
$\phi_{20} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{21} = \begin{pmatrix} 0 & 0.5 & 0.5 & 1 \end{pmatrix}$
$\phi_{22} = \begin{pmatrix} 0 & 0.5 & 1 & 0.5 \end{pmatrix}$
$\phi_{23} = \begin{pmatrix} 0 & 0.5 & 1 & 1 \end{pmatrix}$
$\phi_{24} = \begin{pmatrix} 0 & 1 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{25} = \begin{pmatrix} 0 & 1 & 0.5 & 1 \end{pmatrix}$
$\phi_{26} = \begin{pmatrix} 0 & 1 & 1 & 0.5 \end{pmatrix}$
$\phi_{27} = \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix}$
$\phi_{28} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \end{pmatrix}$
$\phi_{29} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \end{pmatrix}$
$\phi_{30} = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \end{pmatrix}$
$\phi_{31} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \end{pmatrix}$
$\phi_{32} = \begin{pmatrix} 0.5 & 1 & 0 & 0 \end{pmatrix}$
$\phi_{33} = \begin{pmatrix} 0.5 & 0 & 1 & 0 \end{pmatrix}$
$\phi_{34} = \begin{pmatrix} 0.5 & 0 & 0 & 1 \end{pmatrix}$
$\phi_{35} = \begin{pmatrix} 0.5 & 0 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{36} = \begin{pmatrix} 0.5 & 0 & 0.5 & 1 \end{pmatrix}$
$\phi_{37} = \begin{pmatrix} 0.5 & 0 & 1 & 0.5 \end{pmatrix}$
$\phi_{38} = \begin{pmatrix} 0.5 & 0 & 1 & 1 \end{pmatrix}$
$\phi_{39} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0.5 \end{pmatrix}$
$\phi_{40} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0 \end{pmatrix}$
$\phi_{41} = \begin{pmatrix} 0.5 & 0.5 & 1 & 0 \end{pmatrix}$
$\phi_{42} = \begin{pmatrix} 0.5 & 0.5 & 0 & 1 \end{pmatrix}$
$\phi_{43} = \begin{pmatrix} 0.5 & 1 & 0.5 & 0 \end{pmatrix}$
$\phi_{44} = \begin{pmatrix} 0.5 & 1 & 0 & 0.5 \end{pmatrix}$
$\phi_{45} = \begin{pmatrix} 0.5 & 1 & 1 & 0 \end{pmatrix}$
$\phi_{46} = \begin{pmatrix} 0.5 & 1 & 0 & 1 \end{pmatrix}$
$\phi_{47} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{48} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}$
$\phi_{49} = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 \end{pmatrix}$
$\phi_{50} = \begin{pmatrix} 0.5 & 0.5 & 1 & 1 \end{pmatrix}$
$\phi_{51} = \begin{pmatrix} 0.5 & 1 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{52} = \begin{pmatrix} 0.5 & 1 & 0.5 & 1 \end{pmatrix}$
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$\phi_{54} = \begin{pmatrix} 0.5 & 1 & 1 & 1 \end{pmatrix}$
$\phi_{55} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$
$\phi_{56} = \begin{pmatrix} 1 & 0.5 & 0 & 0 \end{pmatrix}$
$\phi_{57} = \begin{pmatrix} 1 & 0 & 0.5 & 0 \end{pmatrix}$
$\phi_{58} = \begin{pmatrix} 1 & 0 & 0 & 0.5 \end{pmatrix}$
$\phi_{59} = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$
$\phi_{60} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$
$\phi_{61} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$
$\phi_{62} = \begin{pmatrix} 1 & 0 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{63} = \begin{pmatrix} 1 & 0 & 0.5 & 1 \end{pmatrix}$
$\phi_{64} = \begin{pmatrix} 1 & 0 & 1 & 0.5 \end{pmatrix}$
$\phi_{65} = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}$
$\phi_{66} = \begin{pmatrix} 1 & 0.5 & 0 & 0.5 \end{pmatrix}$
$\phi_{67} = \begin{pmatrix} 1 & 0.5 & 0.5 & 0 \end{pmatrix}$
$\phi_{68} = \begin{pmatrix} 1 & 0.5 & 1 & 0 \end{pmatrix}$
$\phi_{69} = \begin{pmatrix} 1 & 0.5 & 0 & 1 \end{pmatrix}$
$\phi_{70} = \begin{pmatrix} 1 & 1 & 0.5 & 0 \end{pmatrix}$
$\phi_{71} = \begin{pmatrix} 1 & 1 & 0 & 0.5 \end{pmatrix}$
$\phi_{72} = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}$
$\phi_{73} = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}$
$\phi_{74} = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{75} = \begin{pmatrix} 1 & 0.5 & 0.5 & 1 \end{pmatrix}$
$\phi_{76} = \begin{pmatrix} 1 & 0.5 & 1 & 0.5 \end{pmatrix}$
$\phi_{77} = \begin{pmatrix} 1 & 0.5 & 1 & 1 \end{pmatrix}$
$\phi_{78} = \begin{pmatrix} 1 & 1 & 0.5 & 0.5 \end{pmatrix}$
$\phi_{79} = \begin{pmatrix} 1 & 1 & 0.5 & 1 \end{pmatrix}$
$\phi_{80} = \begin{pmatrix} 1 & 1 & 1 & 0.5 \end{pmatrix}$
$\phi_{81} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

Obviously, we have

$$\begin{aligned} e(a_1, \_\_) &= \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, & e(a_2, \_\_) &= \begin{pmatrix} 0.5 & 1 & 1 & 1 \end{pmatrix}, \\ e(a_3, \_\_) &= \begin{pmatrix} 0.5 & 0.5 & 1 & 1 \end{pmatrix}, & e(a_4, \_\_) &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Although it is not obvious but easy to calculate that

• $\bigwedge_{a\in X/R} \phi_i(a) \to e_{X/R}(a, \_\_) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = e_{X/R}(a_1, \_\_)$
for $i = 1, 2, 3, 13, 28, 29, 30, 40, 55, 56, 57, 67$, so $\bigsqcup \phi_i = a_1$ for these i;

• $\bigwedge_{a\in X/R}\phi_i(a)\to e_{X/R}(a,\text{--})=\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}=e_{X/R}(a_4,\text{--})$
for $i=4,7-12,15,17,19-27,31,34-39,42,44,46-54,58,61-66,69,71,73-81$, so $\bigsqcup\phi_i=a_4$ for these i;
• $\bigwedge_{a\in X/R}\phi_i(a)\to e_{X/R}(a,\text{--})=\begin{pmatrix}0.5 & 1 & 1 & 1\end{pmatrix}=e_{X/R}(a_2,\text{--})$
for $i=5,16,32,43,59,70$, so $\bigsqcup\phi_i=a_2$ for these i;
• $\bigwedge_{a\in X/R}\phi_i(a)\to e_{X/R}(a,\text{--})=\begin{pmatrix}0.5 & 0.5 & 1 & 1\end{pmatrix}=e_{X/R}(a_3,\text{--})$
for $i=6,14,18,33,41,45,60,68,72$, so $\bigsqcup\phi_i=a_3$ for these i.

Then we come to the conclusion that the join of every fuzzy set ϕ_i exists and $(X_R,e_{X/R})$ is a fuzzy complete lattice in terms of Proposition 2.7. So we have verified that the fuzzy order congruence R defined above is a fuzzy complete congruence.

Proposition 5.5. *Let $(X,e),(Y,e_Y)$ be fuzzy complete lattices and $f:(X,e)\to(Y,e_Y)$ a fuzzy complete lattice homomorphism. Then $(X/k_f,e_{k_f})$ is a fuzzy complete lattice and the fuzzy quotient map is a fuzzy complete lattice homomorphism. Moreover, there exists a fuzzy complete lattice homomorphism $h:(X/k_f,e_{k_f})\to(Y,e_Y)$ such that $f=h\circ\eta_{k_f}$. If f is surjective, then $(X/k_f,e_{k_f})$ is fuzzy order isomorphic with (Y,e_Y).*

Proof. We make $R=k_f$ for the convenience.

(1). For every $\mathscr{U}\in L^{X/R}$ and $x\in X$, we could define $A\in L^X$ by $A(x)=\mathscr{U}(R_x)$. Since (X,e) is a fuzzy complete lattice, we may assume $\bigsqcup A=x_0$. Next we prove $\bigsqcup\mathscr{U}=R_{x_0}$, i.e. for $\forall y\in X$,

$$\bigwedge_{x\in X}\mathscr{U}(R_x)\to e_R(R_x,R_y)=e_R(R_{x_0},R_y).$$

We only have to prove $\bigwedge_{x\in X}A(x)\to e_Y(f(x),f(y))=e_Y(f(x_0),f(y))$.

Since f is fuzzy join-preserving, we have $f(x_0)=f(\bigsqcup A)=\bigsqcup\widetilde{f}_*^{\rightarrow}(A)$. Then for every $u\in Y$,

$$\begin{aligned}e_Y(f(x_0),u) &= \bigwedge_{v\in Y}\widetilde{f}_*^{\rightarrow}(A)(v)\to e_Y(v,u),\\ &= \bigwedge_{v\in Y}(\bigvee_{x\in X}A(x)*e_Y(v,f(x)))\to e_Y(v,u).\end{aligned}$$

We have

$$\begin{aligned}e_Y(f(x_0),f(y)) &= \bigwedge_{v\in Y}(\bigvee_{x\in X}A(x)*e_Y(v,f(x)))\to e_Y(v,f(y)),\\ &= \bigwedge_{x\in X}A(x)\to(\bigwedge_{v\in Y}e_Y(v,f(x))\to e_Y(v,f(y))),\\ &= \bigwedge_{x\in X}A(x)\to e_Y(f(x),f(y)).\end{aligned}$$

Therefore, $(X/R,e_R)$ is a fuzzy complete lattice.

(2). We prove that the fuzzy quotient map η_R is fuzzy complete lattice homomorphism. We just prove it is fuzzy join-preserving. For every $A\in L^X$, we assume $\bigsqcup A=x_0$. We have to verify $R_{x_0}=\bigsqcup\widetilde{\eta_{R*}}^{\rightarrow}(A)$. For every $a\in X$,

$$\widetilde{\eta_{R*}}^{\rightarrow}(A)(R_a) = \bigvee_{x\in X}A(x)*e_R(R_a,R_x),$$

$$
\begin{aligned}
&= \bigvee_{x\in X} A(x) * e_Y(f(a), f(x)),\\
&= \widetilde{f_*}^{\rightarrow}(A)(f(a)).
\end{aligned}
$$

For every $y \in X$, we have

$$
\begin{aligned}
\bigwedge_{a\in X} \widetilde{\eta_{R*}}^{\rightarrow}(A)(R_a) \to e_R(R_a, R_y) &= \bigwedge_{a\in X} \widetilde{f_*}^{\rightarrow}(A)(f(a)) \to e_Y(f(a), f(y)),\\
&= \bigwedge_{x\in X} A(x) \to e_Y(f(x), f(y)),\\
&= e_Y(f(x_0), f(y)) = e_R(R_{x_0}, R_y).
\end{aligned}
$$

That is to say $R_{x_0} = \bigsqcup \widetilde{\eta_{R*}}^{\rightarrow}(A)$.

We define $h(R_x) = f(x)$ for every $x \in X$. It is tedious but easy to verify that h is a fuzzy complete lattice homomorphism. If f is surjective, then $(X/k_f, e_{k_f})$ is fuzzy order isomorphic to (Y, e_Y) in terms of Theorem 3.9. □

From Propositions 5.3 and Propositions 5.5, we could obtain the following theorem that characterizes the fuzzy complete congruence on fuzzy complete lattice.

Theorem 5.6. *Let (X, e) be a fuzzy complete lattice, R a fuzzy equivalence relation on X. Then the following are equivalent:*
i. R is a fuzzy complete congruence on (X, e);
ii. There exists a fuzzy preorder σ with $e \leq \sigma$ such that
(1). $R = \sigma \wedge \sigma^{-1}$;
(2). $\sigma(\bigsqcup A, y) = \bigwedge_{x\in X} A(x) \to \sigma(x, y)$, $\sigma(y, \bigsqcap A) = \bigwedge_{x\in X} A(x) \to \sigma(y, x)$ hold for every $A \in L^X, y \in X$;
iii. There exists a fuzzy complete lattice (Y, e_Y) and a fuzzy complete lattice homomorphism $f : (X, e) \to (Y, e_Y)$ with $R = k_f$.

Proof. (i⇔ii) It follows from Theorem 3.5 and the definition of fuzzy complete congruence.

(ii⇔iii) This can be verified by Proposition 5.3 and Proposition 5.5. □

In the following, we will discuss the correspondence between the subcontext and the fuzzy complete congruence.

Definition 5.7. If (X, Y, I) is an L-context and if $H \subseteq X, N \subseteq Y$, then $(H, N, I|_{H\times N})$ is called an L-subcontext.

In the sequel, we focus on how the concept system of (X, Y, I) is related to the L-subcontext.

Proposition 5.8. *If $N \subseteq Y$, then every extent of $(X, N, I \cap X \times N)$ is an extent of (X, Y, I).*

This means that the omission of attributes is equivalent to coarsen the closure system of the extents. Certainly, the similar result holds for the case of the omission of objects.

Proposition 5.9. *If $N \subseteq Y$, then the map*

$$\begin{array}{rcl} \mathscr{B}(X, N, I \cap X \times N) & \to & \mathscr{B}_I \\ \langle A, B\rangle & \mapsto & \langle A, A^{\uparrow}\rangle \end{array}$$

is a fuzzy meet-preserving order embedding. Dually, if $H \subseteq X$, the map

$$\begin{array}{rcl} \mathscr{B}(H, Y, I \cap H \times Y) & \to & \mathscr{B}_I \\ \langle A, B\rangle & \mapsto & \langle B^{\downarrow}, B\rangle \end{array}$$

is a fuzzy join-preserving order embedding.

According to the Propositions 5.8 and 5.9, we have the following proposition.

Proposition 5.10. *If $H \subseteq X, N \subseteq Y$, then the map*

$$\begin{array}{rcl} \mathscr{B}(H, N, I|_{H\times N}) & \to & \mathscr{B}_I \\ \langle A, B\rangle & \mapsto & \langle (i_H^{\rightarrow}(A))^{\uparrow\downarrow}, (i_H^{\rightarrow}(A))^{\uparrow}\rangle \end{array}$$

is a fuzzy order-embedding, and so is the map

$$\begin{array}{rcl} \mathscr{B}(H, N, I|_{H\times N}) & \to & \mathscr{B}_I \\ \langle A, B\rangle & \mapsto & \langle (i_N^{\rightarrow}(B))^{\downarrow}, (i_N^{\rightarrow}(B))^{\downarrow\uparrow}\rangle, \end{array}$$

where $i_H : H \to X, i_N : N \to Y$.

Corollary 5.11. *If $\langle A, B\rangle \in \mathscr{B}(H, N, I|_{H\times N})$, then $B = (i_H^{\rightarrow}(A))^{\uparrow}|_N$, $\quad A = (i_N^{\rightarrow}(B))^{\downarrow}|_H$.*

An L-subcontext $(H, N, I|_{H\times N})$ is called a dense L-subcontext of (X, Y, I) if γH is join-dense in $\mathscr{B}_I$ and μN is meet-dense in $\mathscr{B}(X, Y, I)$. The maps in Proposition 5.10 are bijective if $(H, N, I|_{H\times N})$ is a dense L-subcontext of (X, Y, I).

Proposition 5.12. *A fuzzy order-embedding of $\mathscr{B}_I$ in a given fuzzy complete lattice (V, e_V) exists if and only if there exist $\alpha : X \to V$ and $\beta : Y \to V$ with $I(x, y) = e_V(\alpha(x), \beta(y))$ for every $x \in X, y \in Y$.*

Remark 5.13. The fuzzy concept lattice of an L-subcontext is isomorphic to a sub fuzzy poset of the entire fuzzy concept lattice.

Next, we will introduce the compatible L-subcontext and investigate the connection between the fuzzy complete congruences and compatible L-subcontexts.

Definition 5.14. An L-subcontext $(H, N, I|_{H\times N})$ is said to be compatible if $\langle A|_H, B|_N\rangle \in \mathscr{B}(H, N, I|_{H\times N})$ for every concept $\langle A, B\rangle \in \mathscr{B}_I$.

Proposition 5.15. *An L-subcontext $(H, N, I|_{H\times N})$ of (X, Y, I) is compatible if and only if the map*

$$\begin{array}{rcl} \Pi_{H,N} : \mathscr{B}_I & \to & \mathscr{B}(H, N, I|_{H\times N}) \\ \langle A, B\rangle & \mapsto & \langle A|_H, B|_N\rangle \end{array}$$

is a surjective fuzzy complete lattice homomorphism.

Proof. According to Definition 5.14, $(H, N, I|_{H\times N})$ is compatible if and only if $\Pi_{H,N}$ is a map. Now, we prove that the $\Pi_{H,N}$ surjective. We assume $\langle A, (i_H^{\rightarrow}(A))^{\uparrow}|_N\rangle \in \mathscr{B}(H, N, I|_{H\times N})$ in terms of Corollary 5.11. Then we have $\langle (i_H^{\rightarrow}(A))^{\uparrow\downarrow}, (i_H^{\rightarrow}(A))^{\uparrow}\rangle \in \mathscr{B}_I$ and $\Pi_{H,N}(\langle (i_H^{\rightarrow}(A))^{\uparrow\downarrow}, (i_H^{\rightarrow}(A))^{\uparrow}\rangle) = \langle (i_H^{\rightarrow}(A))^{\uparrow\downarrow}|_H, (i_H^{\rightarrow}(A))^{\uparrow}\rangle|_N$. And the two concepts have the same intent, so they are the same.

Assume $\forall \mathscr{U}\,\mathrm{L}^{\mathscr{B}_I}$, we prove $\Pi_{H,N}$ is fuzzy meet-preserving, i.e.

$$\Pi_{H,N}(\bigsqcap \mathscr{U}) = \bigsqcap \widetilde{\Pi}^{*\rightarrow}_{H,N}(\mathscr{U}).$$

For every $\langle A', B'\rangle \in \mathscr{B}(H, N, I|_{H\times N})$,

$$\begin{aligned}
&\bigwedge_{\langle C',D'\rangle\in\mathscr{B}_{I|_{H\times N}}} \widetilde{\Pi}^{*\rightarrow}_{H,N}(\mathscr{U})(\langle C', D'\rangle) \to \widetilde{e}_{H,N}(A', C')\\
&= \bigwedge_{\langle C,D\rangle\in\mathscr{B}_I} \mathscr{U}(\langle C, D\rangle) \to \widetilde{e}_{H,N}(\langle A', C|_H\rangle)\\
&= \widetilde{e}_{H,N}(A', (\bigwedge_{\langle A,B\rangle\in\mathscr{B}_I} \mathscr{U}(\langle A, B\rangle) \to A)|_H)\\
&= E_{H,N}(\langle A', B'\rangle, \Pi_{H,N}(\bigsqcap \mathscr{U})).
\end{aligned}$$

The last equation is due to Equation (2). Similarly, we can prove $\Pi_{H,N}$ is fuzzy join-preserving. □

The next proposition characterizes the compatible L-subcontext.

Proposition 5.16. *An L-subcontext $(H, N, I|_{H\times N})$ is a compatible L-subcontext of (X, Y, I) if and only if:*
(1) $(i_N^{\rightarrow}(A^{\uparrow}|_N))^{\downarrow}|_H \subseteq A^{\uparrow\downarrow}$ for all $A \in L^X$;
(2) $(i_H^{\rightarrow}(B^{\downarrow}|_H))^{\uparrow}|_N \subseteq B^{\downarrow\uparrow}$ for all $B \in L^Y$

Proof. According to Corollary 5.11, the necessity is obvious. For the sufficiency, let $\langle A, B\rangle \in \mathscr{B}_I$, then $(i_H^{\rightarrow}(A|_H))^{\uparrow}|_N \supseteq A^{\uparrow}|_N = B|_N$. According to condition (2), $(i_H^{\rightarrow}(A|_H))^{\uparrow}|_N = (i_H^{\rightarrow}(B^{\uparrow}|_H))^{\uparrow}|_N \subseteq B^{\downarrow\uparrow}$. So we have $(i_H^{\rightarrow}(A|_H))^{\uparrow}|_N = B|_N$. Dually, $(i_N^{\rightarrow}(B|_N))^{\downarrow}|_H = A|_H$. Therefore, $\langle A|_H, B|_N\rangle \in \mathscr{B}_{I|_{H\times N}}$. □

We naturally obtain the fuzzy complete congruences induced by compatible L-subcontexts.

Corollary 5.17. *Let $(H, N, I|_{H\times N})$ be a compatible L-subcontext of (X, Y, I). Then the map $\Pi_{H,N}$ induces a fuzzy complete congruence $\Theta_{H,N} = k_{\Pi_{H,N}}$ on $\mathscr{B}_I$, and $\mathscr{B}_I/\Theta_{H,N}$ is fuzzy order-isomorphic to $\mathscr{B}_{I|_{H\times N}}$.*

For $\langle A_i, B_i\rangle \in \mathscr{B}_I, i = 1, 2$, $\Theta_{H,N}(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) = 1 \Leftrightarrow A_1|_H = A_2|_H \Leftrightarrow B_1|_N = B_2|_N$ according to the Corollary 5.17.

6. Concluding Remarks

In this paper, we propose fuzzy order congruence on fuzzy poset, which is a generalization of order congruence. Then we characterize them in terms of fuzzy preorders. Moreover, we apply this to more particular structure–fuzzy complete lattice and introduce fuzzy complete congruence. Since the fuzzy complete congruence is closely related to fuzzy concept lattice, we obtain some connection between the fuzzy complete congruence and the compatible L-subcontext.

In the future, we will focus on investigating the connection between the fuzzy complete lattice and fuzzy concept analysis. The theory of fuzzy complete congruence maybe applied to more practical areas, so we try to study their applications in other fields. Other fields will also be studied, such as the fuzzy partially ordered algebras and we will discuss the role that fuzzy order congruences play in them.

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Jing Hao, College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou, 450000, China
E-mail address: haojingzy@gmail.com

Qingguo Li*, College of Mathematics and Econometrics, Hunan University, Changsha, 410082, China
E-mail address: liqingguoli@aliyun.com

Lankun Guo, College of Information Science and Engineering, Hunan University, Changsha, 410082, China
E-mail address: lankun.guo@gmail.com

*Corresponding author