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SOME CHARACTERIZATIONS OF HOPF GROUP ON FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, some fundamental concepts are given relating to fuzzy topological spaces. Then it is shown that there is a contravariant functor from the category of the pointed fuzzy topological spaces to the category of groups and homomorphisms. Also the fuzzy topological spaces which are Hopf spaces are investigated and it is shown that a pointed fuzzy toplogical space having the same homotopy type as an Hopf group is itself an Hopf group.

1. **Introduction and Preliminaries**

The notation of fuzzy sets and fuzzy set operations were introduced by Zadeh in his paper [16]. Subsequently, some basic concepts from general topology were applied to fuzzy sets, see e.g. [1, 12, 15]. Rosenfeld defined fuzzy groups in [13]. Foster introduced the notion of fuzzy topological groups in [5]. The consept of fuzzy topological groups was introduced in [3, 10, 11]. Then the consepts of fuzzy fundamental group and fuzzy homotopy in fuzzy topological spaces were defined and fuzzy topological and homotopic invariance of fuzzy fundamental groups was studied, see e.g [2, 4]. In the following years, the concept of algebraic topology was generalized to fuzzy topology. Fuzzy topological groups on the fuzzy continuous function space were studied in [6]. Gumus and Yildiz gave the consept of the pointed fuzzy topological spaces in [7]. The concept of Hopf space was introduced by Hopf, see [9]. Subsequently, a lot of scientists studied in this field. An Hopf space consists of a pointed topological space P together with a continuous multiplication $\mu : P \times P \to P$ for which the constant map $c : P \to P$ is a homotopy identity, i.e., $\mu \circ (1_P, c) \sim 1_P$ and $\mu \circ (c, 1_P) \sim 1_P$. A group structure can be established on an Hopf space by the homotopy group operations which are similar to group operations. This group is called Hopf group. More precisely, an Hopf group is an Hopf space which has homotopy associative multiplication and homotopy identity. The classical example to Hopf group is topological groups.

Throughout this paper, the symbol I will denote the unit interval [0, 1].

Definition 1.1. Let A be a fuzzy set in X. The set

$$SuppA = \{x \in X \mid \mu_A(x) > 0\}$$

is called the support of fuzzy set A.

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Definition 1.2. A fuzzy point α_r in X is a fuzzy set with the membership function,

$$\alpha_r(x) = \begin{cases} r & , x = \alpha \\ 0 & , x \neq \alpha \end{cases}$$

for all $x \in X$, where $0 < r \leq 1$.

We denote by k_λ the fuzzy set in X with constant membership fuction $\mu_{k_\lambda}(x) = \lambda$ for all $x \in X$.

Definition 1.3. A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:

i) $k_0, k_1 \in \tau$

ii) If $A_1, A_2, ..., A_n \in \tau$ then $\bigcap\limits_{i=1}^{n} A_i \in \tau$

iii) If $Aj \in \tau$ for all $j \in J$(where J is an index set) then $\bigcup\limits_{j \in J} Aj \in \tau$.

The pair (X, τ) is called a fuzzy topological space. Every member of τ is called a $\tau-$fuzzy open set.

Definition 1.4. Let (X, τ) be a fuzzy topological space and $A \in FP(X)$. The family $\tau_A = \{B \cap A \mid B \in \tau\}$ is called induced fuzzy topology on A and the pair (A, τ_A) is called fuzzy subspace of (X, τ).

Definition 1.5. Let (X, τ) be a fuzzy topological space. If all constant fuzzy sets in X are $\tau-$fuzzy open, then (X, τ) is called fully stratified space.

Definition 1.6. Let X and Y be two sets, $f : X \to Y$ be a function and A be a fuzzy set in X, B be a fuzzy set in Y. Then,

i) the image of A under f is the fuzzy set $f(A)$ in Y with the membership function,

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{A(x)} & , f^{-1}(y) \neq \varnothing \\ 0 & , otherwise \end{cases}$$

for all $y \in Y$.

ii) the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X with the membership function $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, for all $x \in X$.

Definition 1.7. Let (X, τ) and (Y, τ') be two fuzzy topological spaces. A function $f : (X, \tau) \to (Y, \tau')$ is fuzzy continuous iff the inverse image of any $\tau'-$fuzzy open set in Y is an $\tau-$fuzzy open set in X, i.e. $f^{-1}(V) \in \tau$, for all $V \in \tau'$.

The set of all fuzzy continuous functions from X to Y is denoted by $FC(X, Y)$.

Definition 1.8. Let (X, τ) and (Y, τ') be two fuzzy topological spaces, (A, τ_A), (B, τ_B) be fuzzy subspaces of X and Y, respectively, and $f \in FC(X, Y)$ such that $f(A) \subset B$. If for all $U \in \tau_B$, $f^{-1}(U) \cap A \in \tau_A$ then f is called relative F-continuous.

Definition 1.9. Let $(G, \cdot)$ be a group and $A \in FP(X)$. If for all $x, y \in G$,

i) $\mu_A(x \cdot y) \geq min\{\mu_A(x), \mu_A(y)\}$

ii) $\mu_A(x^{-1}) \geq \mu_A(x)$

then A is called a fuzzy subgroup of G.

Definition 1.10. Let $(X, \cdot)$ be a group, (X, τ) be a fully stratified space and G be a fuzzy subgroup of X. Then $(G, \tau_G, \cdot)$ is called fuzzy topological group in X if,

i) $f : (G, \tau_G) \times (G, \tau_G) \to (G, \tau_G)$, $f : (x, y) \to x \cdot y$ is relative F-continuous
ii) $g : (G, \tau_G) \to (G, \tau_G)$, $f : x \to x^{-1}$ is relative F-continuous.

Theorem 1.11. [6] *Let (X, τ) be a fuzzy topological space, $(Y, \tau', \cdot)$ be fuzzy topological group with the identity element "e" and $f, g \in FC(X, Y)$. Then the maps*

$$f * g : x \to f(x) \cdot g(x), \ \ f^{-1} : x \to (f(x))^{-1} \ \text{and} \ e' : x \to e$$

from fuzzy topological space X to fuzzy topological space Y are fuzzy continuous.

Theorem 1.12. [7] *Let (X, τ) be a fully stratified fuzzy topological space and $(Y, \tau', \cdot)$ be a fuzzy topological group. Then the pair $(FC(X, Y), *)$ is a group. Also if the group $(Y, \cdot)$ is abelian, then the group $(FC(X, Y), *)$ is abelian.*

2. **Fuzzy Homotopy and Pointed Fuzzy Topological Spaces**

In this section we have shown that there is a contravariant functor from the homotopy category of the pointed fuzzy topological spaces to the category of groups and homomorphisms. So we provide a transition from fuzzy topological structure to topological structure.

Definition 2.1. [8] Let (X, T) be a topological space. Then

$$\widetilde{\tau} = \{A \in FP(X) \mid SuppA \in T\}$$

is a fuzzy topology on X, called the fuzzy topology on X introduced by T and $(X, \widetilde{\tau})$ is called the fuzzy topological space introduced by (X, T).

Let ε_I denote the Euclidean subspace topology on I and $(I, \widetilde{\varepsilon}_I)$ denote the fuzzy topological space introduced by the topological space (I, ε_I).

Definition 2.2. Let (X, τ), (Y, τ') be fuzzy topological spaces and $f, g \in FC(X, Y)$. If there exist a fuzzy continuous function

$$F : (X, \tau) \times (I, \widetilde{\varepsilon}_I) \to (Y, \tau')$$

such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$, then f and g are fuzzy homotopic. The mapping F is called fuzzy homotopy from f to g and we write $f \overset{F}{\sim} g$. In addition, if there exist a subset X_0 of X such that, for all $x' \in X_0$,

$$F(x', t) = f(x') = g(x')$$

then f ang g are called fuzzy homotopic $(relX_0)$. In this case the mapping F is called fuzzy relative homotopy from f to g and we write $f \overset{F}{\sim} g(relX_0)$.

The fuzzy homotopy relation " $\sim$ " is an equivalence relation. Thus the set $FC(X,Y)$ is partitioned into equivalence classes under the relation " $\sim$ ". The equivalence classes are called fuzzy homotopy classes and the set of all fuzzy homotopy classes of the fuzzy continuous functions from (X,τ) to (Y,τ') is denoted by $[(X,\tau);(Y,\tau')]$. The fuzzy homotopy class of a function f is denoted by $[f]$.

Definition 2.3. Let (X,τ), (Y,τ') be fuzzy topological spaces and $f \in FC(X,Y)$, $g \in FC(Y,X)$. If $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$, then (X,τ) and (Y,τ') are called fuzzy homotopic equivalent or having the same homotopy type. Also f and g are called fuzzy equivalences.

Definition 2.4. Let (X,τ) be a fuzzy topological space and α_λ be a fuzzy point in X. The pair (X,α_λ) is called a pointed fuzzy topological space and α_λ is called the base point of (X,α_λ).When we are concerned with pointed fuzzy topological spaces $(X,\alpha_\lambda),(Y,\beta_\lambda)$, etc. we always require that all fuzzy continuous functions $f:(X,\alpha_\lambda)\to(Y,\beta_\lambda)$ shall preserve base point, i.e. $f(\alpha)=\beta$ and that all fuzzy homotopies be relative to the base point. The homotopy classes of base point preserving functions $f:(X,\alpha_\lambda)\to(Y,\beta_\lambda)$ is denoted by $[(X,\alpha_\lambda);(Y,\beta_\lambda)]$.

If $(Y,\tau',\cdot)$ is a fuzzy topological group with the base point e_λ then it is denoted by $(Y,\cdot,e_\lambda)$.

Theorem 2.5. [6] *Let (X,α_λ) be a pointed fuzzy topological space, $(Y,\cdot,e_\lambda)$ be a fuzzy topological group. Then $[(X,\alpha_\lambda);(Y,\cdot,e_\lambda)]$ is a group under the product*

$$[f]\bullet[g]=[f*g]$$

for all $f,g\in FC((X,\alpha_\lambda),(Y,\cdot,e_\lambda))$. Also if $(Y,\cdot)$ is an abelian group, then the group $([(X,\alpha_\lambda);(Y,\cdot,e_\lambda)],\bullet)$ is abelian.

Definition 2.6. [14] A category may be thought of intuitively as consisting of sets and functions. More precisely, a category C consists of

a) A class of objects

b) *For every ordered pair of objects X and Y, a set $hom(X,Y)$ of morphisms with domain X and range Y. If $f\in hom(X,Y)$, we write $f:X\to Y$ or $X\xrightarrow{f}Y$*

c) *For every ordered triple of objects X,Y and Z, a function associating to a pair of morphisms $f:X\to Y$ and $g:Y\to Z$ their composite*

$$gf=g\circ f:X\to Z.$$

These satisfy the following two axioms:

c$_1$) *Associativity: If $f:X\to Y, g:Y\to Z$ and $h:Z\to W$ then $h(gf)=(hg)f:X\to W$.*

c$_2$) *Identity: For every object Y, there is a morphism $1_Y:Y\to Y$ such that if $f:X\to Y$, then $1_Y f=f$ and if $h:Y\to Z$, then $h1_Y=h$.*

Definition 2.7. Let C and D be two categories. A contravariant functor T from C to D is a mapping which associates to every object X of C an object $T(X)$ of D and associates to every morphism $f : X \to Y$ of C a morphism $T(f) : T(Y) \to T(X)$ of D such that

i) $T(1_X) = 1_{T(X)}$
ii) $T(gf) = T(f)T(g)$.

Theorem 2.8. [8] *For any category C and object Y of C, there is a contravariant functor Π^Y from C to the category of sets and functions which associates to an object X of C the set $\Pi^Y(X) = hom(X, Y)$ and to a morphism $f : X \to X'$ the function $\Pi^Y(f) = f^* : hom(X', Y) \to hom(X, Y)$ defined by $f^*(g') = g' \circ f$, for $g' : X' \to Y$.*

Definition 2.9. The category whose objects are pointed fuzzy topological spaces and the set of morphisms is

$$hom((X, \alpha_\lambda), (Y, \beta_\lambda)) = [(X, \alpha_\lambda); (Y, \beta_\lambda)]$$

is called the homotopy category of the pointed fuzzy topological spaces. The composition of morphisms is the operation "$\circ$" identified such that $[g] \in [(X, \alpha_\lambda); (Y, \beta_\lambda)]$, $[f] \in [(Y, \beta_\lambda); (Z, \delta_\lambda)]$,

$$[f] \circ [g] = [f \circ g]$$

is well defined and satisfies c_1) and c_2).

Theorem 2.10. *If $(Y, \cdot, e_\lambda)$ be a fuzzy topological group, then Π^Y is a contravariant functor from the homotopy category of the pointed fuzzy topological spaces to the category of groups and homomorphisms.*

Proof. We show that if (X, α_λ), (X', α'_λ) are two objects and

$$[f] \in [(X, \alpha_\lambda) ; (X', \alpha'_\lambda)]$$

is a morphism of the homotopy category of the pointed fuzzy topological spaces, then $\Pi^Y((X, \alpha_\lambda))$ is a group and $\Pi^Y([f])$ is an homomorphism.

$$\Pi^Y((X, \alpha_\lambda)) = hom((X, \alpha_\lambda), (Y, \cdot, e_\lambda)) = [(X, \alpha_\lambda) ; (Y, \cdot, e_\lambda)].$$

So $\Pi^Y((X, \alpha_\lambda))$ is a group.

$$\begin{aligned}\Pi^Y([f]) &= f^* : hom((X', \alpha'_\lambda), (Y, \cdot, e_\lambda)) \to hom((X, \alpha_\lambda), (Y, \cdot, e_\lambda)) \\ &\Rightarrow f^* : [(X', \alpha'_\lambda) ; (Y, \cdot, e_\lambda)] \to [(X, \alpha_\lambda) ; (Y, \cdot, e_\lambda)]\end{aligned}$$

is a function such that, for any $[g] \in [(X', \alpha'_\lambda) ; (Y, \cdot, e_\lambda)]$,

$$f^*([g]) = [g] \circ f = [g \circ f].$$

As $([(X', \alpha'_\lambda) ; (Y, \cdot, e_\lambda)], \bullet)$ and $([(X, \alpha_\lambda) ; (Y, \cdot, e_\lambda)], \bullet)$ are groups, f^* is a morphism between groups.

Let $[g'], [g''] \in [(X', \alpha'_\lambda) ; (Y, \cdot, e_\lambda)]$. Then,

$$\begin{aligned}f^*([g'] \bullet [g'']) &= f^*([g' * g'']) \\ &= [(g' * g'') \circ f] \\ &= [(g' \circ f) * (g'' \circ f)] \\ &= [g' \circ f] \bullet [g'' \circ f] \\ &= f^*([g']) \bullet f^*([g'']).\end{aligned}$$

Consequently f^* is homomorphism. To prove that Π^Y is a contravariant functor, let $[1_X] \in [(X,\alpha_\lambda);(X,\alpha_\lambda)]$ be the unit morphism of the homotopy category of the pointed fuzzy topological spaces. Then,

$$\Pi^Y([1_X]) = 1_X^* : [(X,\alpha_\lambda);(Y,\cdot,e_\lambda)] \to [(X,\alpha_\lambda);(Y,\cdot,e_\lambda)]$$

and for any morphism $[h] \in [(X,\alpha_\lambda);(Y,\cdot,e_\lambda)]$,

$$1_X^*([h]) = [h \circ 1_X] = [h].$$

Hence the morphism $\Pi^Y([1_X])$ is unit.

Let $\varphi \in [(X',\alpha'_\lambda);(X'',\alpha''_\lambda)]$. For any morphism $h' \in [(X'',\alpha''_\lambda);(Y,\cdot,e_\lambda)]$,

$$\begin{aligned} \Pi^Y([\varphi])([h']) &= [h' \circ \varphi] \\ \Pi^Y([f])([h' \circ \varphi]) &= [(h' \circ \varphi) \circ f] \\ &= [h' \circ (\varphi \circ f)] \\ &= \Pi^Y([\varphi \circ f])([h']) . \end{aligned}$$

Thus

$$\Pi^Y([f])([h' \circ \varphi]) = \Pi^Y([f])\left(\Pi^Y([\varphi])([h'])\right) = \left(\Pi^Y([f]) \circ \Pi^Y([\varphi])\right)([h']) .$$

So

$$\Pi^Y([\varphi \circ f]) = \Pi^Y([f]) \circ \Pi^Y([\varphi]) .$$

Consequently, since Π^Y preserves the identity and the composition, Π^Y is a contravariant functor. □

3. **Pointed Fuzzy Topological Spaces Having Hopf Group Structure**

In this section pointed fuzzy topological spaces having Hopf group structure are investigated and it is shown that being Hopf group is homotopic invariant.

Definition 3.1. Let (X,x_0) be a pointed topological space, $\gamma : X \times X \to X$ be a continuous multiplication and $c : X \to X, c : x \to x_0$ be a constant function. If

$$\gamma \circ (c, 1_X) \sim 1_X \sim \gamma \circ (1_X, c)$$

then (X,x_0) is called a Hopf space and c is called homotopy identity. Here, $(c,1_X)(x) = (c(x),1_X(x)) = (x_0,x)$, for all $x \in X$.

Theorem 3.2. *A pointed fuzzy topological space having the same homotopy type as an Hopf space is itself a Hopf space.*

Proof. Let a pointed fuzzy topological space (X,α_λ) be an Hopf space with the fuzzy continuous multiplication γ, (Y,β_λ) has the same homotopy type as (X,α_λ) and $f \in FC(X,Y)$, $g \in FC(Y,X)$ be fuzzy homotopy equivalences. Let the function γ' be defined such that

$$\gamma' = f \circ \gamma \circ (g \times g) : Y \times Y \to Y.$$

Then γ' is a fuzzy continuous multiplication of Y. Let $c : X \to X$, $c : x \to \alpha$ and $c' : Y \to Y, c' : y \to \beta$ be constant functions. So

$$(g \times g) \circ (1_Y, c') = (1_X, c) \circ g.$$

Since (X, α_λ) is a Hopf space, $\gamma \circ (1_X, c) \sim 1_X$. Therefore,

$$\begin{aligned} \gamma' \circ (1_Y, c') &= (f \circ \gamma \circ (g \times g)) \circ (1_Y, c') \\ &= f \circ \gamma \circ (1_X, c) \circ g \sim f \circ 1_X \circ g = f \circ g. \end{aligned}$$

Similarly,

$$\gamma' \circ (c', 1_Y) = f \circ \gamma \circ (c, 1_X) \circ g \sim f \circ g.$$

As $f \circ g \sim 1_Y$,

$$\gamma' \circ (1_Y, c') \sim 1_Y \sim \gamma' \circ (c', 1_Y).$$

Consequently the pointed fuzzy topological space (Y, β_λ) is a Hopf space. □

Definition 3.3. Let (X, α_λ) and (Y, β_λ) be Hopf spaces with the fuzzy continuous multiplications γ and γ', respectively. Then a function $f : (X, \alpha_\lambda) \to (Y, \beta_\lambda)$ is called Hopf homomorphism if $f \circ \gamma \sim \gamma' \circ (f \times f)$.

Theorem 3.4. *Let the pointed fuzzy topological spaces (X, α_λ) and (Y, β_λ) be Hopf spaces with the same homotopy type. Then the fuzzy equivalences are Hopf homomorphisms.*

Proof. Let $f \in FC(X, Y)$, $g \in FC(Y, X)$ be fuzzy homotopy equivalences, γ and $\gamma' = f \circ \gamma \circ (g \times g)$ be fuzzy continuous multiplications of X and Y, respectively. As $g \circ f \sim 1_X$,

$$g \circ \gamma' = g \circ (f \circ \gamma \circ (g \times g)) \sim 1_X \circ \gamma \circ (g \times g) = \gamma \circ (g \times g).$$

Thereby $g \circ \gamma' \sim \gamma \circ (g \times g)$. So g is a Hopf homomorphism. Also,

$$g \circ f \sim 1_X \Rightarrow (g \times g) \circ (f \times f) \sim 1_{X \times X}.$$

So $\gamma' \circ (f \times f) = f \circ \gamma \circ (g \times g) \circ (f \times f) \sim f \circ \gamma \circ 1_{X \times X} = f \circ \gamma$. Therefore

$$f \circ \gamma \sim \gamma' \circ (f \times f).$$

Consequently f is a Hopf homomorphism. □

Definition 3.5. Let the pointed topological space (X, x_0) be a Hopf space with the continuous multiplication γ. If $\gamma \circ (\gamma, 1_X) \sim \gamma \circ (1_X, \gamma)$, then γ is called homotopy associative. If there exist a continuous function $\varphi : X \to X$ such that

$$\gamma \circ (\varphi, 1_X) \sim 1_X \sim \gamma \circ (1_X, \varphi).$$

Then φ is called homotopy invers. If there exist a function $T : X \times X \to X \times X$, $T : (x, x') \to (x', x)$ such that $\gamma \circ T \sim \gamma$, then γ is called homotopy commutative.

Definition 3.6. A Hopf group is a Hopf space with a homotopy associative multiplication and a homotopy invers.

Theorem 3.7. *A pointed fuzzy topological space with the same homotopy type as a Hopf group is itself a Hopf group.*

Proof. Let a pointed fuzzy topological space (X, α_λ) be a Hopf group, (Y, β_λ) has the same homotopy type with X and γ and $\gamma' = f \circ \gamma \circ (g \times g)$ be fuzzy continuous multiplications of (X, α_λ) and (Y, β_λ), respectively. Then (Y, β_λ) is a Hopf space. Let $f \in FC(X, Y)$, $g \in FC(Y, X)$ be fuzzy homotopy equivalences. As (X, α_λ) is a Hopf group, it has a homotopy invers and γ is homotopy associative. As $f \circ g \sim 1_Y$,

$$\begin{aligned} \gamma' \times 1_Y &= (f \circ \gamma \circ (g \times g)) \times 1_Y \sim (f \times f) \circ (\gamma \times 1_X) \circ (g \times g \times g) \\ 1_Y \times \gamma' &= 1_Y \times (f \circ \gamma \circ (g \times g)) \sim (f \times f) \circ (1_X \times \gamma) \circ (g \times g \times g). \end{aligned}$$

Also, as γ is homotopy associative,

$$(f \times f) \circ (\gamma \times 1_X) \circ (g \times g \times g) \sim (f \times f) \circ (1_X \times \gamma) \circ (g \times g \times g).$$

Thus

$$\gamma' \times 1_Y \sim 1_Y \times \gamma' \Rightarrow \gamma' \circ (\gamma' \times 1_Y) \sim \gamma' \circ (1_Y \times \gamma').$$

So γ' is homotopy associative. Let $\varphi \in FC(X, X)$ be homotopy invers and $\varphi' \in FC(Y, Y)$ be a function such that $\varphi' = f \circ \varphi \circ g$. In this case,

$$\begin{aligned} (g \times g) \circ (1_Y, \varphi') &= (g, g \circ \varphi') \\ &= (g, g \circ (f \circ \varphi \circ g)) \sim (g, \varphi \circ g) = (1_X, \varphi) \circ g \\ &\Rightarrow (f \circ \gamma \circ (g \times g))(1_Y, \varphi') \sim f \circ \gamma \circ (1_X, \varphi) \circ g. \end{aligned}$$

Thus

$$\gamma' \circ (1_Y \times \varphi') \sim f \circ \gamma \circ (1_X, \varphi) \circ g \sim f \circ 1_X \circ g = f \circ g \sim 1_Y.$$

Similarly

$$\gamma' \circ (\varphi' \times 1_Y) \sim 1_Y.$$

Hence φ' is a homotopy invers for (Y, β_λ). Consequently (Y, β_λ) is a Hopf group.

In Theorem 3.7, if γ is homotopy commutative then γ' is homotopy commutative. More precisely, as γ homotopy commutative, there is a function

$$T : X \times X \to X \times X,\ T : (x, x') \to (x', x)$$

for all $x, x' \in X$ such that $\gamma \circ T \sim \gamma$. Now consider the function

$$T' : Y \times Y \to Y \times Y,\ T' : (y, y') \to (y', y)$$

for all $y, y' \in Y$. Then, since $(g \times g) \circ T' \sim T \circ (g \times g)$,

$$f \circ \gamma \circ (g \times g) \circ T' \sim f \circ \gamma \circ T \circ (g \times g) \Rightarrow f \circ \gamma \circ (g \times g) \circ T' \sim f \circ \gamma \circ (g \times g)$$

Therefore $\gamma' \circ T' \sim \gamma'$. So γ' is homotopy commutative. □

Theorem 3.8. *Let a pointed fuzzy topological space (X, α_λ) be a Hopf space with the fuzzy continuous multiplication γ and (Y, β_λ) be a pointed fuzzy topological space. Let the operation " $\odot$ " on $[(Y, \beta_\lambda) ; (X, \alpha_\lambda)]$ be identified such that,*

$$[f] \odot [g] = [\gamma \circ (f, g)]$$

for all $[f], [g] \in [(Y, \beta_\lambda) ; (X, \alpha_\lambda)]$. If (X, α_λ) is a Hopf group, $([(Y, \beta_\lambda) ; (X, \alpha_\lambda)], \odot)$ is a group.

Proof. Let (X, α_λ) be a Hopf group. For $[g_1], [g_2], [g_3] \in [(Y, \beta_\lambda); (X, \alpha_\lambda)]$,

$$(1_X \times \gamma) \circ (g_1, (g_2, g_3)) = (g_1, \gamma \circ ((g_2, g_3)))\,.$$

So $[\gamma \circ (1_X \times \gamma) \circ (g_1, (g_2, g_3))] = [\gamma \circ (g_1, \gamma \circ (g_2, g_3)]$. Also

$$(\gamma \times 1_X) \circ (g_1, (g_2, g_3)) = (\gamma \circ (g_1, g_2)\,, g_3)\,.$$

Hence $[\gamma \circ (\gamma \times 1_X) \circ (g_1, (g_2, g_3))] = [\gamma \circ (\gamma \circ (g_1, g_2)\,, g_3)]$. Since γ is homotopy associative,

$$\gamma \circ (1_X \times \gamma) \sim \gamma \circ (\gamma \times 1_X) \Rightarrow (\gamma \circ (1_X \times \gamma)) \circ (g_1, (g_2, g_3)) \sim (\gamma \circ (\gamma \times 1_X)) \circ (g_1, (g_2, g_3)).$$

Thus

$$[\gamma \circ (g_1, \gamma \circ (g_2, g_3)] = [\gamma \circ (\gamma \circ (g_1, g_2)\,, g_3)].$$

Also,

$$[\gamma \circ (g_1, \gamma \circ (g_2, g_3)] = [g_1] \odot ([\gamma \circ (g_2, g_3)]) = [g_1] \odot ([g_2] \odot [g_3])$$

and

$$[\gamma \circ (\gamma \circ (g_1, g_2)\,, g_3)] = [\gamma \circ (g_1, g_2)] \odot [g_3] = ([g_1] \odot [g_2]) \odot [g_3].$$

Consequently

$$[g_1] \odot ([g_2] \odot [g_3]) = ([g_1] \odot [g_2]) \odot [g_3].$$

Let $h : (Y, \beta_\lambda) \to (X, \alpha_\lambda)$, $h : y \to \alpha$ be a constant function. Then

$$[g] \odot [h] = (\gamma \circ (g, h))$$

for all $[g] \in [(Y, \beta_\lambda); (X, \alpha_\lambda)]$. Also

$$\gamma \circ (g, h) = \gamma \circ (1_X \times c) \circ g.$$

As (X, α_λ) is a Hopf space, $\gamma \circ (1_X \times c) \sim 1_X$. So $\gamma \circ (1_X \times c) \circ g \sim 1_X \circ g = g$. Thus

$$[\gamma \circ (1_X \times c) \circ g] = [g] \Rightarrow [\gamma \circ (g, h)] = [g] \Rightarrow [g] \odot [h] = [g]\,.$$

Consequently, $[h] \in [(Y, \beta_\lambda); (X, \alpha_\lambda)]$ is the unit element. Let φ be the homotopy invers of (X, α_λ). Then for any function $g : (Y, \beta_\lambda) \to (X, \alpha_\lambda)$

$$[\varphi \circ g] \in [(Y, \beta_\lambda); (X, \alpha_\lambda)].$$

Also,

$$[g] \odot [\varphi \circ g] = [\gamma \circ (g, \varphi \circ g)] = [\gamma \circ (1_X \times \varphi) \circ g]\,.$$

I it is clear that $c \circ g = h$. Therefore, as $\gamma \circ (1_X \times \varphi) \sim c$,

$$\gamma \circ (1_X \times \varphi) \circ g \sim c \circ g = h \Rightarrow [g] \odot [\varphi \circ g] = [h].$$

Hence $[\varphi \circ g]$ is the invers of $[g]$. □

Theorem 3.9. *Let a pointed fuzzy topological space* $\left(Y, \beta_y\right)$ *be a Hopf group. Then* Π^Y *is a contravariant functor from the homotopy category of the fuzzy pointed topological spaces to the category of groups and homomorphisms.*

Proof. Let $\gamma : Y \times Y \to Y$ be the fuzzy continuous multiplication, (X, α_λ) and (Z, δ_λ) be two objects of the homotopy category of pointed fuzzy topological spaces and $f : (X, \alpha_\lambda) \to (Z, \delta_\lambda)$ be any morphism. Then the morphism $\Pi^Y(f) = f^*$ is a function from $\left[(Z, \delta_\lambda); \left(Y, \beta_y\right)\right]$ to $\left[(X, \alpha_\lambda); \left(Y, \beta_y\right)\right]$ such that

$$f^*([g]) = [g] \circ f = [g \circ f].$$

Let $[g_1], [g_2] \in \left[(Z, \delta_\lambda); \left(Y, \beta_y\right)\right]$ .Then

$$\begin{aligned} f^*([g_1] \odot [g_2]) &= f^*([\gamma \circ (g_1, g_2)]) \\ &= [(\gamma \circ (g_1, g_2)) \circ f] \\ &= [\gamma \circ (g_1 \circ f, g_2 \circ f)] \\ &= [g_1 \circ f] \odot [g_2 \circ f] \\ &= ([g_1] \circ f) \odot ([g_2] \circ f) \\ &= (f^*([g_1])) \odot (f^*[g_2])). \end{aligned}$$

Consequently f^* is a homomorphism. It is shown that Π^Y is a contravariant functor in the same way by Theorem 2.10. □

References

[1] C. L. Chang, *Fuzzy topological space*, J. Math. Anal. Appl., **24** (1968), 182-190.

[2] Z. Chang You and L. Wang Jin, *Fuzzy homotopy type invariance of fundamental group of fuzzy topological space*, Fuzzy Math., **4** (1984), 53-56.

[3] I. Chon, *Some properties of fuzzy topological groups*, Fuzzy Sets and Systems, **13** (2001), 197-201.

[4] S. Chuanlin, *Homotopy and fundamental group in fuzzy topology*, Dongbei Shida Xuebao, **2** (1985), 25-31.

[5] D. H. Foster, *Fuzzy topological groups*, J. Math. Anal. Appl., **67** (1979), 549-564.

[6] M. Ganster, D. N. Georgiou and S. Jafari, *On fuzzy topological groups and fuzzy continuous function*, Hacettepe Journal of Mathematics and Statistics, **34 S** (2005),35-43.

[7] S. Gumus and C. Yildiz, *The fuzzy sheaf of the groups formed by fuzzy topological space*, Int. Jorn. of Pure and Appl. Math., **37(1)** (2007),101-112.

[8] A. Hatcher, *Algebraic topology, "H-spaces and Hopf Algebras"*, Cambridge University Press, Cambridge, New York, 2001.

[9] H. Hopf, *Uber die topologie der gruppen-mannigfaltigkeiten und Ihre verallgemeinerungen*, The Annals of Mathematics Second Series, **42(1)** (1941), 22-52

[10] P. K. Jha and T. B. Singh, *Some local properties of fuzzy topology and fuzzy topological groups*, Bull. Cal. Math. Soc., **87** (1995), 441-448.

[11] P. K. Jha and T. B. Singh, *Fuzzy topological groups*,Bull. Cal. Math. Soc. , **91(5)** (1999), 351-358.

[12] R. Lowen, *Fuzzy Topological spaces and fuzzy compactness*, J. Math. Anal. Appl., **56** (1976), 621-633.

[13] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., **35** (1971), 512-517.

[14] E. H. Spainer, *Algebraic topology*, Springer, Verlag., 1994.

[15] C. K. Wong, *Fuzzy topology-product and quotient theorems*, J. Math. Anal. Appl., **45** (1974), 512-521.

[16] L. A. Zadeh, *Fuzzy sets,* Informations and control, **8** (1965), 338-353.

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