LOCAL BASES WITH STRATIFIED STRUCTURE IN I-TOPOLOGICAL VECTOR SPACES

J. X. FANG

Abstract. In this paper, the concept of *local base with stratified structure* in I-topological vector spaces is introduced. We prove that every I-topological vector space has a balanced local base with stratified structure. Furthermore, a new characterization of I-topological vector spaces by means of the local base with stratified structure is given.

1. Introduction

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981 [8]. According to the terminology of standardization in [5], it has now been renamed as I-topological vector space, where $I = [0, 1]$. In research of I-topological vector spaces, two different kind of neighborhood structures have been used, one being neighborhood introduced by Warren [11] (for short, W-neighborhood) and the other quasi-coincident neighborhood introduced by Pu and Liu [9] (for short, Q-neighborhood). In [2], the author discussed the relations between W-neighborhood (base) of a point x and Q-neighborhood (base) of a fuzzy point x_λ in I-topological spaces and gave the characterizations of I-topological vector spaces in terms of such two kind of local bases.

In this paper, we continue to investigate the local bases of I-topological vector spaces. We first introduce the concept of *local base with stratified structure* in I-topological vector spaces and prove that every I-topological vector space has a balanced local base with stratified structure, and also show that W-neighborhood bases of θ in I-topological vector spaces are all local bases with stratified structure. Next, we give a new characterization of I-topological vector spaces by means of the local base with stratified structure, which is a main theorem in the present paper. As its a deduction, we obtain a new characterization of I-topological vector spaces in terms of W-neighborhood base of θ. These results improve and simplify the corresponding results of Katsaras [8] and Fang [2]. In addition, as applications of these results, two example are given.

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2. **Preliminaries**

Throughout this paper, let X be a nonempty set, $\mathbb{R} = (-\infty, +\infty)$ and I^X denote a family of all fuzzy subset on X. A fuzzy set which takes the constant value $r \in I$ on X is denoted by $\underline{r}$. A fuzzy set on X is called a fuzzy point, denoted by x_λ, if it takes the value 0 for all $y \in X$ with $y \neq x$ and the value at x is $\lambda \in (0, 1]$. We denote the set of all fuzzy points on X by $\mathrm{Pt}(I^X)$. Let $U \in I^X$, a fuzzy point x_λ on X is said to be quasi-coincident with U, denoted by $x_\lambda \widetilde{\in} U$, if $U(x) > 1 - \lambda$.

An I-topology (i.e., fuzzy topology in Chang's sense [1]) on X is a collection of fuzzy sets of X closed under arbitrary unions and finite intersections, which contains the characteristic functions of $\emptyset$ and X, denoted $\underline{0}$ and $\underline{1}$, respectively. The pair (X, T) is called an I-topological space if T is an I-topology on X. An I-topology T on X is said to be stratified if $\underline{r} \in \mathrm{T}$ for each $r \in I$.

Definition 2.1. [11] Let (X, T) be an I-topological space and $x \in X$. A fuzzy set U on X is called a W-neighborhood of x if there exists $G \in \mathrm{T}$ such that $G \subset U$ and $G(x) = U(x) > 0$.

Definition 2.2. [8] Let (X, T) be an I-topological space and $x \in X$. A family B_x of W-neighborhoods of x is called a W-neighborhood base of x if for each W-neighborhood P of x and $\alpha \in [0, P(x))$, there exists $U \in \mathrm{B}_x$ such that $U \subset P$ and $U(x) > \alpha$.

Definition 2.3. [9] Let (X, T) be an I-topological space and x_λ be a fuzzy point on X. A fuzzy set U of X is called a Q-neighborhood of x_λ if there exists $G \in \mathrm{T}$ such that $x_\lambda \widetilde{\in} G \subset U$.

A family U_{x_λ} of Q-neighborhoods of x_λ is called a Q-neighborhood base of x_λ if for each Q-neighborhood A of x_λ there exists $U \in \mathrm{U}_{x_\lambda}$ such that $U \subset A$.

Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $A, B \in I^X$ and $k \in \mathbb{K}$. For the definitions of $A + B$ and kA, see [7] or [6].

Definition 2.4. [8] Let X be a vector space over the filed $\mathbb{K}$ and T be a stratified I-topology on X. If the following two mappings (the addition and the scalar multiplication):

$$f : X \times X \to X, \quad (x, y) \mapsto x + y \quad \text{and} \quad g : \mathbb{K} \times X \to X, \quad (k, x) \mapsto kx,$$

are both I-continuous, where $\mathbb{K}$ is equipped with the usual induced I-topology $\omega(J_{\mathbb{K}})$, $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product I-topologies $\mathrm{T} \times \mathrm{T}$ and $\omega(J_{\mathbb{K}}) \times \mathrm{T}$, respectively. Then the pair (X, T) is called an I-topological vector space (for short, I-tvs).

Proposition 2.5. [12, 2] *Let (X, T) be an I-tvs.*

(1) *The addition mapping f is I-continuous iff for each fuzzy point $(x, y)_\lambda$ on $X \times X$ and each Q-neighborhood W of $(x + y)_\lambda$, there exist Q-neighborhoods U of x_λ and V of y_λ such that $U + V \subset W$.*

(2) *The scalar multiplication mapping g is I-continuous iff for each fuzzy point $(k, x)_\lambda$ on $\mathbb{K} \times X$ and each Q-neighborhood W of kx_λ, there exist a Q-neighborhood U of x_λ and $\delta > 0$ such that $tU \subset W$ for all $t \in \mathbb{K}$ with $|t - k| < \delta$.*

Proposition 2.6. [2] *Let* (X, T) *be an* I*-tvs and* $\mathrm{B} \subset I^X$. *Then* B *is a* W*-neighborhood base of* θ *iff* $U(\theta) > 0$ *for each* $U \in \mathrm{B}$, *and for each* $\lambda \in (0,1]$,

$$\mathrm{U}_\lambda = \{U \in \mathrm{B} \mid U(\theta) > 1 - \lambda\}$$

is a Q*-neighborhood base of* θ_λ *in* (X, T).

Definition 2.7. [10] Let L_1 and L_2 be two completely distributive lattices with order-reversing involution $'$, 0 and 1 are respectively their smallest and greatest elements. A mapping $\phi : L_1 \to L_2$ is called an order-homomorphism, if the following conditions hold:

(OH-1) ϕ is union-preserving, i.e., $f(\bigvee a_t) = \bigvee f(a_t)$;

(OH-2) ϕ^* is complement-preserving, i.e., for each $b \in L_2$, $\phi^*(b') = [\phi^*(b)]'$, where the mapping $\phi^* : L_2 \to L_1$ is given by $\phi^*(b) = \bigvee\{a \in L_1 \mid \phi(a) \le b\}$.

3. The Local Bases with Stratified Structure

Definition 3.1. Let $\mathrm{U} \subset I^X$. A family $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ of nonempty subsets of U is called the stratified structure of U if the following two equalities hold: $\mathrm{U} = \bigcup_{\lambda \in (0,1]} \mathrm{U}_\lambda$ and $\mathrm{U}_\lambda = \bigcup_{\mu \in (0,\lambda)} \mathrm{U}_\mu$.

Remark 3.2. It is easy to prove that if $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ be a family of nonempty subsets of U and $\mathrm{U} = \bigcup_{\lambda \in (0,1]} \mathrm{U}_\lambda$, then $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U if and only if the following conditions are satisfied:

(1) $0 < \mu < \lambda \le 1$ implies that $\mathrm{U}_\mu \subseteq \mathrm{U}_\lambda$;

(2) For each $U \in \mathrm{U}_\lambda$ there exists $\mu_0 \in (0, \lambda)$ such that $U \in \mathrm{U}_{\mu_0}$.

Example 3.3. Let $X = \mathbb{R}$, $\mathrm{U} = \{(-a, a) \cap \underline{r} \mid a > 0,\ 0 < r \le 1\}$. For each $\lambda \in (0,1]$, define a subset U_λ of U by $\mathrm{U}_\lambda = \{(-a, a) \cap \underline{r} \mid a > 0,\ r \in (1 - \lambda, 1]\}$. Then it is easy to verify that $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ is the stratified structure of U.

Example 3.4. Let (X, T) be an I-topological space and $x \in X$, and let $\mathrm{B}_\lambda(x)$ be a Q-neighborhood base of x_λ for each $\lambda \in (0,1]$. We define

$$\mathrm{U}_\lambda(x) = \bigcup_{0<\mu<\lambda} \mathrm{B}_\mu(x) \quad \text{and} \quad \mathrm{U}(x) = \bigcup_{\lambda \in (0,1]} \mathrm{U}_\lambda(x). \tag{3.1}$$

Then $\{\mathrm{U}_\lambda(x)\}_{\lambda \in (0,1]}$ is the stratified structure of $\mathrm{U}(x)$.

To prove this conclusion, it suffices to show that $\mathrm{U}_\lambda(x) = \bigcup_{0<\mu<\lambda} \mathrm{U}_\mu(x)$. It is evident that $\bigcup_{o<\mu<\lambda} \mathrm{U}_\mu(x) \subseteq \mathrm{U}_\lambda(x)$. On the other hand, if $U \in \mathrm{U}_\lambda(x)$, then by the first equality of (3.1), there exists $\mu \in (0, \lambda)$ such that $U \in \mathrm{B}_\mu(x)$. Taking $\lambda_0 \in (\mu, \lambda)$, it follows that $U \in \mathrm{U}_{\lambda_0}(x)$, and so $U \in \bigcup_{0<\mu<\lambda} \mathrm{U}_\mu(x)$. This shows that $\mathrm{U}_\lambda(x) = \bigcup_{0<\mu<\lambda} \mathrm{U}_\mu(x)$. Therefore $\{\mathrm{U}_\lambda(x)\}_{\lambda \in (0,1]}$ is the stratified structure of $\mathrm{U}(x)$.

Definition 3.5. Let (X, T) be an I-tvs. A family U of fuzzy subsets on X is called a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ in (X, T) if $\{\mathrm{U}_\lambda\}_{\lambda \in (0,1]}$ is its stratified structure and U_λ is a Q-neighborhood base of θ_λ in (X, T) for each $\lambda \in (0,1]$.

Example 3.6. Let $(X, \|\cdot\|, L, R)$ be a fuzzy normed space (see [4] and [13]). For a fixed $\varepsilon > 0$, the fuzzy set B_ε on X is defined by

$$B_\varepsilon(x) = \sup\{1-\alpha : \|\|x\|\|_2^\alpha < \varepsilon\}.$$

By Theorem 3.1 in [13], we know that if $R \leq \max$, then there exists an I-topology $T_{\|\cdot\|}$ on X such that $(X, T_{\|\cdot\|})$ is an I-topological vector space and

$$\mathrm{U}_\lambda = \{B_\varepsilon \cap \underline{r} \mid \varepsilon > 0,\ r \in (1-\lambda, 1]\}$$

is a Q-neighborhood base of θ_λ for each $\lambda \in (0,1]$. In addition, by using Remark 3.1, it is not difficult to prove that the family of fuzzy subsets

$$\mathrm{U} = \{B_\varepsilon \cap \underline{r} \mid \varepsilon > 0,\ r \in (0,1]\}$$

has the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$. Therefore U is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in $(X, T_{\|\cdot\|})$.

Theorem 3.7. *Every I-tvs has a balanced local base with stratified structure.*

Proof. We know that every I-tvs (X, T) has a balanced Q-neighborhood base B_λ of θ_λ for each $\lambda \in (0,1]$ (see [12]). Now define

$$\mathrm{U}_\lambda = \bigcup_{\alpha\in(0,\lambda)} \mathrm{B}_\alpha \quad \text{and} \quad \mathrm{U} = \bigcup_{\lambda\in(0,1)} \mathrm{U}_\lambda.$$

By Example 3.4, we know that $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ is the stratified structure of U.

It remains to prove that for each $\lambda \in (0,1]$, U_λ is a Q-neighborhood base of θ_λ. It is not difficult to see that members of U_λ are all Q-neighborhoods of θ_λ. Now let W be an arbitrary Q-neighborhood of θ_λ. It is evident that there exists $\mu \in (0,\lambda)$ such that W is also Q-neighborhood of θ_μ. Since B_μ is a Q-neighborhood base of θ_μ, there exists $U \in \mathrm{B}_\mu$ such that $U \subset W$. Note that $\mathrm{B}_\mu \subset \mathrm{U}_\lambda$. Hence $U \in \mathrm{U}_\lambda$. This shows that U_λ is a Q-neighborhood base of θ_λ. Therefore U is a local base with stratified structure in (X, T). □

Theorem 3.8. *Every W-neighborhood base* B *of θ in an I-tvs (X, T) is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$, where $\mathrm{U}_\lambda = \{U \in \mathrm{B} \mid U(\theta) > 1-\lambda\}$.*

Proof. Since B is a W-neighborhood base of θ, by Proposition 2.6 we know that $\mathrm{U}_\lambda = \{U \in \mathrm{B} \mid U(\theta) > 1-\lambda\}$ is a Q-neighborhood base of θ_λ for each $\lambda \in (0,1]$, and $\mathrm{B} = \bigcup_{\lambda\in(0,1]} \mathrm{U}_\lambda$. In the following, we need only to show that $\mathrm{U}_\lambda = \bigcup_{\mu\in(0,\lambda)} \mathrm{U}_\mu$.

By the definition of U_λ, it is evident that $\mu < \lambda$ implies that $\mathrm{U}_\mu \subset \mathrm{U}_\lambda$. Hence $\bigcup_{\mu\in(0,\lambda)} \mathrm{U}_\mu \subset \mathrm{U}_\lambda$. On the other hand, if $U \in \mathrm{U}_\lambda$, then $U \in \mathrm{B}$ and $U(\theta) > 1-\lambda$. Taking $\mu \in (1-U(\theta), \lambda)$, it is not difficult to see that $U \in \mathrm{U}_\mu$, and so $U \in \bigcup_{\mu\in(0,\lambda)} \mathrm{U}_\mu$. This shows that $\mathrm{U}_\lambda = \bigcup_{\mu\in(0,\lambda)} \mathrm{U}_\mu$. Therefore B is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in (X, T). □

As a special case of Theorem 3.8, we have the following example.

Example 3.9. Let (X,T) be an I-tvs and $\mathrm{U}=\{G\in I^X \mid G\in \mathrm{T},\ G(\theta)>0\}$. It is evident that U is an open W-neighborhood base of θ in (X,T). Hence, by Theorem 3.8, we conclude that U is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in (X,T), where $\mathrm{U}_\lambda=\{U\in \mathrm{T}\mid U(\theta)>1-\lambda\}$.

4. Main Results

Theorem 4.1. *Let (X,T) be an I-tvs. Then the local base* U *with stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in (X,T) has the following properties:*

(1) *If $U\in\mathrm{U}_\lambda$, $V\in\mathrm{U}_\lambda$ or $V=\underline{r}$ $(1-\lambda<r\le 1)$, then there exists $W\in\mathrm{U}_\lambda$ such that $W\subset U\cap V$;*

(2) *If $U\in\mathrm{U}_\lambda$, then there exists $V\in\mathrm{U}_\lambda$ such that $V+V\subset U$;*

(3) *If $U\in\mathrm{U}_\lambda$, then there exists $V\in\mathrm{U}_\lambda$ such that $kV\subset U$ for all $|k|\le 1$;*

(4) *If $U\in\mathrm{U}_\lambda$, then for each $x\in X$, there is an $\alpha>0$ such that $x_\lambda\widetilde{\in}\,\alpha U$.*

Conversely, let X be a vector space over $\mathbb{K}$ and U *be a family of fuzzy subsets on X with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ which satisfies the above conditions $(1)-(4)$. Then there exists a unique I-topology* T *on X such that (X,T) is an I-tvs and* U *is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in (X,T).*

Proof. (1) is evident. The proofs of (2), (3) and (4) are easy by Proposition 2.5 (see also the proof of Theorem 4.1 in [2]).

Conversely, without loss of generality, we can suppose that every member U in U satisfies $U(\theta)=\sup_{x\in X}U(x)$. Otherwise, we substitute $\widetilde{\mathrm{U}}=\bigcup_{\lambda\in(0,1]}\widetilde{\mathrm{U}}_\lambda$ for U, where $\widetilde{\mathrm{U}}_\lambda=\{U\cap\underline{U(\theta)}\mid U\in\mathrm{U}_\lambda\}$. Obviously, $\left\{\widetilde{\mathrm{U}}_\lambda\right\}_{\lambda\in(0,1]}$ is the stratified structure of $\widetilde{\mathrm{U}}$ and satisfies the conditions $(1)-(4)$. We set

$$\mathrm{T}=\{G\in I^X\mid \forall\, x_\mu\widetilde{\in}\, G,\ \exists\, V\in\mathrm{U}_\mu \text{ such that } x+V\subset G\}.$$

Then it is not difficult to verify that T is a stratified I-topology on X.

In fact, for each $r\in(0,1]$, if $x_\lambda\widetilde{\in}\,\underline{r}$, i.e., $r>1-\lambda$, then the condition (1) implies that there exists $W\in\mathrm{U}_\lambda$ such that $W\subset\underline{r}$, and so $x+W\subset x+\underline{r}=\underline{r}$. This shows $\underline{r}\in\mathrm{T}$. Moreover, it is easy to show that T is closed under arbitrary unions and finite intersections. Therefore T is a stratified I-topology on X.

For each $U\in\mathrm{U}$, define a fuzzy set G_U on X as follows:

$$G_U(x)=\sup\{1-\mu\mid x+V\subset U \text{ for some } V\in\mathrm{U}_\mu\}. \tag{4.1}$$

We can prove the following conclusions:

(a) $G_U\in\mathrm{T}$; (b) $G_U\subset U$; (c) $U\in\mathrm{U}_\lambda\ \Rightarrow\ \theta_\lambda\,\widetilde{\in}\,G_U$.

(a) Let $x_\mu\widetilde{\in}\,G_U$, i.e., $G_U(x)>1-\mu$. By (4.1), there exist $\nu\in(0,\mu)$ and $V\in\mathrm{U}_\nu$ such that $x+V\subset U$. Note that $\nu<\mu$ implies $\mathrm{U}_\nu\subset\mathrm{U}_\mu$. Hence $V\in\mathrm{U}_\mu$. By (2), there exists $P\in\mathrm{U}_\mu$ such that $P+P\subset V$, and so $x+P+P\subset x+V\subset U$. Since $P\in\mathrm{U}_\mu=\bigcup_{\alpha\in(0,\mu)}\mathrm{U}_\alpha$, there exists $\delta\in(0,\mu)$ such that $P\in\mathrm{U}_{\mu-\delta}$. By (1), for each $\varepsilon > 0$, there exists $W \in \mathrm{U}_{\mu-\delta}$ such that $W \subset P \cap \underline{(1-\mu+\delta+\varepsilon)}$. We can prove that $x + W \subset G_U$.

In fact, if $(x+W)(z) = \alpha > 0$, then $W(z-x) = \alpha$, and so

$$\begin{aligned}(x+W+W)(w) &= (W+W)(w-x) \geq W(w-z) \wedge W(z-x) \\ &= W(w-z) \wedge \alpha = (z + W \cap \underline{\alpha})(w), \quad \forall\, w \in X,\end{aligned}$$

which implies that $z + W \cap \underline{\alpha} \subset x + W + W \subset x + P + P \subset U$. Note that

$$1-\mu+\delta+\varepsilon \geq W(z-x) = \alpha \implies 1-\alpha+\varepsilon \geq \mu-\delta \implies \mathrm{U}_{\mu-\delta} \subset \mathrm{U}_{1-\alpha+\varepsilon}.$$

Hence $W \in \mathrm{U}_{1-\alpha+\varepsilon}$. Since $\alpha > 1-(1-\alpha+\varepsilon)$, by (1) there exists $W_0 \in \mathrm{U}_{1-\alpha+\varepsilon}$ such that $W_0 \subset W \cap \underline{\alpha}$. So, we have $z + W_0 \subset z + W \cap \underline{\alpha} \subset U$. It follows from (4.1) that $G_U(z) \geq \alpha - \varepsilon$. By the arbitrariness of ε, we get $G_U(z) \geq \alpha$. This shows that $x + W \subset G_U$. Note that $W \in \mathrm{U}_{\mu-\delta} \subset \mathrm{U}_\mu$. Hence $G_U \in \mathrm{T}$.

(b) If $x \in X$ and $G_U(x) = \alpha > 0$, then by (4.1) for each $\varepsilon \in (0,\alpha)$, there exist $\mu \in (0,1]$ with $1-\mu > \alpha - \varepsilon$ and $V \in \mathrm{U}_\mu$ such that $x + V \subset U$. This implies that $U(x) \geq V(\theta) > 1-\mu > \alpha-\varepsilon$. By the arbitrariness of ε, we get $U(x) \geq \alpha = G_U(x)$. This shows that $G_U \subset U$.

(c) Let $U \in \mathrm{U}_\lambda$. Note that $\mathrm{U}_\lambda = \bigcup_{\mu\in(0,\lambda)} \mathrm{U}_\mu$. Then there exists $\mu \in (0,\lambda)$ such that $U \in \mathrm{U}_\mu$. Since $\theta + U = U$, by (4.1) we have $G_U(\theta) \geq 1-\mu > 1-\lambda$, i.e., $\theta_\lambda \widetilde{\in} G_U$.

From (a)$-$(c), it is easy to see that every member in U_λ is a Q-neighborhood of θ_λ in (X,T). Again, by the definition of T, we conclude that U_λ is a Q-neighborhood base of θ_λ and T is a translation invariant, i.e., $G \in \mathrm{T}$ implies that $x + G \in \mathrm{T}$ for each $x \in X$. Therefore $\mathrm{U}_{x_\lambda} = \{x + U \mid U \in \mathrm{U}_\lambda\}$ is a Q-neighborhood base of x_λ.

Finally, we prove that the addition f and the scalar multiplication g are both I-continuous.

Let $(x,y)_\lambda$ be a fuzzy point on $X \times X$ and W be a Q-neighborhood of $(x+y)_\lambda$. Then there exists $U \in \mathrm{U}_\lambda$ such that $(x+y)+U \subset W$. By (2), there exists $V \in \mathrm{U}_\lambda$ such that $V + V \subset U$, and so we have $(x+V)+(y+V) \subset (x+y)+U \subset W$. Note that $x+V \in \mathrm{U}_{x_\lambda}$, $y+V \in \mathrm{U}_{y_\lambda}$. By Proposition 2.5, we infer that f is I-continuous.

Let $(k_0,x)_\lambda$ be a fuzzy point on $\mathbb{K} \times X$ and W be a Q-neighborhood of $(k_0x)_\lambda$. Then there exists $U \in \mathrm{U}_\lambda$ such that $k_0x + U \subset W$. By (2), there exists $V \in \mathrm{U}_\lambda$ such that $V+V \subset U$. By (3), there exists $V_1 \in \mathrm{U}_\lambda$ such that $tV_1 \subset V$ for all $|t| \leq 1$. By (4), there exists a $\beta > 0$ such that $x_\lambda \widetilde{\in} \beta V_1$. Put $s = (\beta V_1)(x)$, then $s > 1-\lambda$, and so we have

$$rx_s \in r\beta V_1 \subset V \quad \text{ for all } r \in \mathbb{K} \text{ with } |r| \leq 1/\beta. \tag{4.2}$$

Put $N = [|k_0|]+1$, where $[|k_0|]$ is the largest integral part of $|k_0|$. Take $n \in \mathbb{N}$ with $2^n \geq N$. Repeatedly make use of (2), there exists $V_2 \in \mathrm{U}_\lambda$ such that $\underbrace{V_2 + \cdots + V_2}_{2^n} \subset V$. Again by (3), there exists $V_3 \in \mathrm{U}_\lambda$ such that $tV_3 \subset V_2$ for all $|t| \leq 1$. Put $\delta_1 = N - |k_0|$, then when $|k-k_0| \leq \delta_1$ we have

$$kV_3 \subset NV_2 \subset \underbrace{V_2 + \cdots + V_2}_{2^n} \subset V \quad (\text{because } V_2(\theta) = \sup_{x\in X} V_2(x)). \tag{4.3}$$

Let $\delta = \min\{\delta_1, 1/\beta\}$. It follows from (4.2) and (4.3) that

$$(k-k_0)x + (kV_3) \cap \underline{s} = (k-k_0)x_s + (kV_3) \cap \underline{s} \subset V + V \subset U \tag{4.4}$$

whenever $|k-k_0| \le \delta$, which implies that

$$k(x + V_3 \cap \underline{s}) = kx + kV_3 \cap \underline{s} \subset k_0 x + U \subset W. \tag{4.5}$$

whenever $|k-k_0| \le \delta$. Note that $x + V_3 \cap \underline{s}$ is a Q-neighborhood of x_λ. By (4.5) and Proposition 2.5, we conclude that g is I-continuous. This shows that (X, T) is an I-topological vector space and U_λ is a Q-neighborhood base of θ_λ for each $\lambda \in (0,1]$. Note that $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ is the stratified structure of U. Therefore U is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in (X, T). □

Remark 4.2. Theorem 4.1 is an improvement of Theorem 4.1 in [2]. Here, it is necessary to point out that in the proof of Theorem 4.1 in [2], the argument for continuity of g is incorrect.

In fact, to prove the continuity of g, it suffices to show that for each fuzzy point $(k_0, x)_\lambda$ on $\mathbb{K} \times X$ and each Q-neighborhood W of $k_0 x_\lambda$, there exist $\delta > 0$ and $V_0 \in \mathrm{U}_\lambda$ such that $k(x + V_0) \subset W$ for all $k \in \mathbb{K}$ with $|k - k_0| < \delta$. Please notice that the mentioned V_0 is irrelative to k. However, in the proof of [2] (see page 344−345), the arisen $V^{(2)}$ in (4.2) is relative to k and V_0 is relative to $V^{(2)}$ and $V^{(4)}$, $V_0 \subset V^{(2)} \cap V^{(4)}$, and so V_0 is also relative to k. Obviously, we need not such a V_0.

Imitating the proof of Theorem 4.1, we easily correct the faults in [2].

As a consequence of Theorem 4.1, we obtain the following theorem:

Theorem 4.3. *Let* (X, T) *be an* I*-tvs and* B *be a* W*-neighborhood base of* θ*. The* B *has the following properties:*

(i) $U \in \mathrm{B}$ *implies that* $U(\theta) > 0$.

(ii) *If* $U \in \mathrm{B}$, $V \in \mathrm{B}$ *or* $V = \underline{r}$ *with* $0 \le \alpha < U(\theta) \wedge V(\theta)$, *then there exists* $W \in \mathrm{B}$ *with* $W(\theta) > \alpha$ *such that* $W \subset U \cap V$.

(iii) *If* $U \in \mathrm{B}$ *with* $U(\theta) > \alpha \ge 0$, *then there exists* $V \in \mathrm{B}$ *with* $V(\theta) > \alpha$ *such that* $V + V \subset U$.

(iv) *If* $U \in \mathrm{B}$ *with* $U(\theta) > \alpha \ge 0$, *there exists* $V \in \mathrm{B}$ *with* $V(\theta) > \alpha$ *such that* $tV \subset U$ *for all* $t \in \mathbb{K}$ *with* $|t| \le 1$.

(v) *If* $U \in \mathrm{B}$ *with* $U(\theta) > \alpha \ge 0$, *then for any* $x \in X$, *there exists* $t > 0$ *such that* $(tU)(x) > \alpha$.

Conversely, let X *be a vector space over* $\mathbb{K}$ *and* B *be a family of fuzzy sets on* X *satisfying the above conditions* (i)-(v). *Then there exists a unique* I*-topology* T *on* X *such that* (X, T) *is an* I*-tvs and* B *is a* W*-neighborhood base of* θ.

Proof. Put $\mathrm{U}_\lambda = \{U \in \mathrm{B} \mid U(\theta) > 1 - \lambda\}$, $\alpha = 1 - \lambda$. Then it is easy to see that (ii)−(iv) are equivalent to (1)−(4) in Theorem 4.1. Moreover, if B is a W-neighborhood base of θ, then by Theorem 3.8 we know that B is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$. Therefore the conclusion can follows from Theorem 4.1 and Proposition 2.6 immediately. □

Remark 4.4. Theorem 4.3 is an improvement of Theorem 4.2 in [2] and Theorem 4.3 in [8]. The conditions (i)−(vii) of Theorem 4.3 in [8] have been simplified.

As applications of Theorems 4.1 and 4.3, we give the following two examples.

Example 4.5. Let $C(\mathbb{R})$ denote the set of all continuous real valued functions on $\mathbb{R}$. For each $\alpha \in (0,1]$, we define a mapping $\|\cdot\|_\alpha : C(\mathbb{R}) \to [0,\infty)$ by

$$\|x\|_\alpha = \int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}} |x(t)|dt. \tag{4.6}$$

Obviously, $\|\cdot\|_\alpha$ has the following properties:

(a) $0 < \beta < \alpha \le 1$ implies $\|x\|_\alpha \le \|x\|_\beta$ for all $x \in C(\mathbb{R})$;

(b) $\|x+y\|_\alpha \le \|x\|_\alpha + \|y\|_\alpha$ for all $x, y \in C(\mathbb{R})$.

For each $\alpha \in (0,1]$ and $r > 0$, we define a fuzzy set $U_{\alpha,r}$ on X by

$$U_{\alpha,r}(x) = \begin{cases} \alpha & \text{if } \|x\|_\alpha < r, \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

where $\|x\|_\alpha$ is defined by (4.6). Then by Theorem 4.1, we can prove that there exists a unique I-topology T on $C(\mathbb{R})$ such that $(C(\mathbb{R}), \mathrm{T})$ is an I-tvs and the family of fuzzy sets $\mathrm{U} = \{U_{\alpha,r} \mid \alpha,\ r \in (0,1]\}$ is a local base with the stratified structure $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ in $(C(\mathbb{R}), \mathrm{T})$, where

$$\mathrm{U}_\lambda = \{U_{\alpha,r} \mid \alpha \in (1-\lambda, 1],\ r > 0\}. \tag{4.8}$$

In fact, it is easy to prove that $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ is the stratified structure of U. It remains to prove that $\{\mathrm{U}_\lambda\}_{\lambda\in(0,1]}$ satisfies the conditions (1)−(4) in Theorem 4.1.

By (4.7) it is easy to see that $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha}$, where

$$B_{\alpha,r} = \{x \in X \mid \|x\|_\alpha < r\}. \tag{4.9}$$

(1) Let $U_{\alpha_i,r_i} \in \mathrm{U}_\lambda$ $(i = 1,2)$. Then $U_{\alpha_i,r_i} = B_{\alpha_i,r_i} \cap \underline{\alpha_i}$, $\alpha_i > 1-\lambda$ $(i = 1,2)$. Put $\alpha = \min\{\alpha_1, \alpha_2\}$ and $r = \min\{_1, r_2\}$. It is obvious that $U_{\alpha,r} \in \mathrm{U}_\lambda$ and $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \subset [B_{\alpha_1,r_1}\underline{\alpha_1}] \cap [B_{\alpha_1,r_2} \cap \underline{\alpha_2}] = U_{\alpha_1,r_1} \cap U_{\alpha_2,r_2}$.

Let $U_{\alpha,r} \in \mathrm{U}_\lambda$ and $\sigma \in (1-\lambda, 1]$. Then $\beta = \min\{\alpha,\sigma\} \in (1-\lambda, 1]$, and so $U_{\beta,r} \in \mathrm{U}_\lambda$ and $U_{\beta,r} = B_{\beta,r} \cap \underline{\beta} \subset B_{\alpha,r} \cap \underline{\alpha} \cap \underline{\sigma} = U_{\alpha,r} \cap \underline{\sigma}$.

(2) Let $U_{\alpha,r} \in \mathrm{U}_\lambda$. Then $U_{\alpha,\frac{r}{2}} \in \mathrm{U}_\lambda$ and

$$U_{\alpha,\frac{r}{2}} + U_{\alpha,\frac{r}{2}} = B_{\alpha,\frac{r}{2}} \cap \underline{\alpha} + B_{\alpha,\frac{r}{2}} \cap \underline{\alpha} \subset B_{\alpha,r} \cap \underline{\alpha} = U_{\alpha,r}.$$

(3) Let $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \in \mathrm{U}_\lambda$. By (4.6) and (4.9), it is not difficult to prove that $kB_{\alpha,r} \subset B_{\alpha,r}$ for all $|k| \le 1$. Hence, we have $kU_{\alpha,r} \subset U_{\alpha,r}$ for all $|k| \le 1$.

(4) Let $U_{\alpha,r} = B_{\alpha,r} \cap \underline{\alpha} \in \mathrm{U}_\lambda$. For each $x \in X$, we take $t > \frac{\|x\|_\alpha}{r}$. It follows that $\|(1/t)x\|_\alpha < r$, i.e., $(1/t)x \in B_{\alpha,r}$. Note that $\alpha > 1-\lambda$. Then we have $(1/t)x_\lambda \,\widetilde{\in}\, B_{\alpha,r} \cap \underline{\alpha} = U_{\alpha,r}$, i.e., $x_\lambda \,\widetilde{\in}\, tU_{\alpha,r}$.

Thus, by Theorem 4.1 we know that the conclusion of Example 4.5 holds.

Remark 4.6. Similarly, it is not difficult to verify that U in Example 4.5 satisfies the conditions (i)−(v) of Theorem 4.3. Therefore, by Theorem 4.3, U is a W-neighborhood base of θ in the I-tvs $(C(\mathbb{R}), \mathrm{T})$.

Let X be a real vector space and $\tilde{f}$ be a fuzzy linear functional on $\mathrm{Pt}(I^X)$, i.e., a fuzzy linear operator from $\mathrm{Pt}(I^X)$ into $\mathrm{Pt}(I^{\mathbb{R}})$ (See [3, Definition 4] or [14, Definition 2.3]). By [3, Theorem 4] or [14, Lemma 2.5], we know that $\tilde{f}$ is a fuzzy linear functional on $\mathrm{Pt}(I^X)$ if and only if there exist an ordinary linear functional on X and an order-homomorphism preserving finite meets $\phi : I \to I$ such that

$$\tilde{f}(x_\lambda) = [f(x)]_{\phi(\lambda)} \quad \text{for all } x_\lambda \in \mathrm{Pt}(I^X).$$

In the following, for convenience, we use $(f, \phi)^{\to}$ instead of the fuzzy linear functional $\tilde{f}$. Moreover, we define the mapping $|\cdot| : \mathrm{Pt}(I^{\mathbb{R}}) \to [0, +\infty)$ by $|y_\mu| = (1/\mu)|y|$ for all $y_\mu \in \mathrm{Pt}(I^{\mathbb{R}})$.

Example 4.7. Let X be a real vector space and $\widetilde{C}_\phi(X)$ denote the set of all fuzzy linear functionals $(f, \phi)^{\to}$ on $\mathrm{Pt}(I^X)$ with the same ϕ, i.e., $\widetilde{C}_\phi(X) = \{(f, \phi)^{\to} \mid f \in X^{\#}\}$, where $X^{\#}$ denotes the set of all ordinary linear functionals on X and ϕ is a fixed order-homomorphism preserving finite meets from I into I. For each $f \in X^{\#}$ and $t > 0$, we define a fuzzy set $B_{f,t}$ on X by

$$B_{f,t}(x) = \sup\{1 - \lambda : \ |(f, \phi)^{\to}(x_\lambda)| < t\} \quad \text{for all } x \in X. \tag{4.10}$$

and put $U(f_1, \cdots, f_n; t) = \bigcap_{i=1}^n B_{f_i,t}$. Then by Theorem 4.3, we can prove that there exists a unique I-topology T on X such that (X, T) is an I-tvs and the family of fuzzy sets

$$\mathrm{B} = \left\{U(f_1, \cdots, f_n; t) \cap \underline{r} \ \middle|\ f_i \in X^{\#},\ i = 1, \cdots, n,\ n \in \mathbb{N};\ t > 0,\ r \in (0, 1]\right\}$$

is a W-neighborhood base of θ in (X, T).

To prove this conclusion, we first prove that

$$x_\lambda \,\widetilde{\in}\, B_{f,t} \iff |(f, \phi)^{\to}(x_\lambda)| < t, \ \text{ i.e., } \ |f(x)| < \phi(\lambda)t. \tag{4.11}$$

In fact, if $x_\lambda \,\widetilde{\in}\, B_{f,t}$, i.e., $B_{f,t}(x) > 1 - \lambda$, then by (4.10) there exists $\lambda_0 \in (0, \lambda)$ such that $|f(x)| < \phi(\lambda_0)t$. Note that ϕ is nondecreasing. Hence we have $|f(x)| < \phi(\lambda)t$. Conversely, let $|f(x)| < \phi(\lambda)t$. Note that ϕ is union-preserving. Taking $\lambda_n \in (0, \lambda)$ with $\lambda_n \nearrow \lambda$, we have

$$|f(x)| < \phi\left(\bigvee_{n \in \mathbb{N}} \lambda_n\right) t = \bigvee_{n \in \mathbb{N}} \phi(\lambda_n) t.$$

Thus, there exists some $n_0 \in \mathbb{N}$ such that $|f(x)| < \phi(\lambda_{n_0})t$, i.e., $|(f, \phi)^{\to}(x_{\lambda_{n_0}})| < t$. By (4.10) we get $B_{f,t}(x) \geq 1 - \lambda_{n_0} > 1 - \lambda$, i.e., $x_\lambda \,\widetilde{\in}\, B_{f,t}$. (4.11) is proved.

By using (4.10) and (4.11), it is easy to prove that $B_{f,t}$ has the following properties:

(a) $B_{f,t}(\theta) = 1$;

(b) $0 < s < t$ implies that $B_{f,s} \subset B_{f,t}$;

(c) $B_{f,\frac{t}{2}} + B_{f,\frac{t}{2}} \subset B_{f,t}$;

(d) $B_{f,t}$ is balanced, i.e., $kB_{f,t} \subset B_{f,t}$ for all $|k| \leq 1$;

now we prove that B satisfies the conditions (i)−(v)

(i) Let $W = U(f_1, \cdots, f_n; t) \cap \underline{r} \in$ B. By (a), we know that $W(\theta) = r > 0$.

(ii) By (b), we have that $U(f, g; t \wedge s) \subset B_{f,t} \cap B_{g,s}$ for all $f, g \in X^{\#}$ and $t, s > 0$. So, if $W_1 = U(f_1, \cdots, f_n; t) \cap \underline{r_1}$, $W_2 = U(g_1, \cdots, g_m; s) \cap \underline{r_2} \in$ B with $W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0$, then there exists $W = U(f_1, \cdots, f_n, g_1, \cdots, g_m; t \wedge s) \cap \underline{r_1 \wedge r_2} \in$ B such that $W(\theta) > \alpha$ and $W \subset W_1 \cap W_2$.

If $W_1 = U(f_1, \cdots, f_n; t) \cap \underline{r_1} \in$ B and $W_2 = \underline{r_2}$ $(r_2 > 0)$ with $W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0$, then there exists $W = U(f_1, \cdots; t) \cap \underline{r_1 \wedge r_2} \in$ B such that $W(\theta) > \alpha$ and $W \subset W_1 \cap W_2$.

(iii) If $W = U(f_1, \cdots, f_n; t) \cap \underline{r} \in$ B and $W(\theta) = r > \alpha \geq 0$, then it follows from (c) that

$$U(f_1, \cdots, f_n; \tfrac{t}{2}) + U(f_1, \cdots, f_n; \tfrac{t}{2}) \subset U(f_1, \cdots, f_n; t),$$

and so there exists $V = U(f_1, \cdots, f_n; t) \cap \underline{r} \in$ B such that $V(\theta) = r > \alpha$ and $V + V \subset W$.

(iv) Let $W = U(f_1, \cdots, f_n; t) \cap \underline{r} \in$ B and $W(\theta) > \alpha \geq 0$. By (d), we know that each $B_{f_i,t}$ is balanced $(i = 1, \cdots, n)$. Hence $U(f_1, \cdots, f_n; t)$ is also balanced, and so we have $kW \subset W$ for all $|k| \leq 1$.

(v) Let $W = U(f_1, \cdots, f_n; t) \cap \underline{r} \in$ B and $W(\theta) > \alpha \geq 0$. For any $x \in X$, taking $\beta > \frac{\max_{1 \leq i \leq n} |f(x)|}{t\phi(1-\alpha)}$, then we have $|f_i(x/\beta)| < \phi(1-\alpha)t$, $i = 1, 2, \cdots, n$. It follows from (4.11) that $(x/\beta)_{1-\alpha} \widetilde{\in} \bigcap_{i=1}^{n} B_{f_i,t} = U(f_1, \cdots, f_n; t)$. Note that $r = W(\theta) > \alpha$. Hence $W(x/\beta) = [U(f_1, \cdots, f_n; t) \cap \underline{r}](x/\beta) > \alpha$ i.e., $(\beta W)(x) > \alpha$.

Thus, By Theorem 4.3, we have the conclusion of Example 4.7.

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Jin-Xuan Fang, School of Mathematical Science, Nanjing Normal University, Nanjing, Jiangsu 210097, P. R. China
E-mail address: jxfang@njnu.edu.cn