LOCAL BASES WITH STRATIFIED STRUCTURE IN
I-TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper, the concept of local base with stratified structure in I-topological vector spaces is introduced. We prove that every I-topological vector space has a balanced local base with stratified structure. Furthermore, a new characterization of I-topological vector spaces by means of the local base with stratified structure is given.

1. Introduction

The concept of fuzzy topological vector space was introduced rationally by Katsaras in 1981 [8]. According to the terminology of standardization in [5], it has now been renamed as I-topological vector space, where $I = [0, 1]$. In research of I-topological vector spaces, two different kind of neighborhood structures have been used, one being neighborhood introduced by Warren [11] (for short, $W$-neighborhood) and the other quasi-coincident neighborhood introduced by Pu and Liu [9] (for short, $Q$-neighborhood). In [2], the author discussed the relations between $W$-neighborhood (base) of a point $x$ and $Q$-neighborhood (base) of a fuzzy point $x_0$ in I-topological spaces and gave the characterizations of I-topological vector spaces in terms of such two kind of local bases.

In this paper, we continue to investigate the local bases of I-topological vector spaces. We first introduce the concept of local base with stratified structure in I-topological vector spaces and prove that every I-topological vector space has a balanced local base with stratified structure, and also show that $W$-neighborhood bases of $\theta$ in I-topological vector spaces are all local bases with stratified structure. Next, we give a new characterization of I-topological vector spaces by means of the local base with stratified structure, which is a main theorem in the present paper. As its a deduction, we obtain a new characterization of I-topological vector spaces in terms of $W$-neighborhood base of $\theta$. These results improve and simplify the corresponding results of Katsaras [8] and Fang [2]. In addition, as applications of these results, two example are given.
2. Preliminaries

Throughout this paper, let $X$ be a nonempty set, $\mathbb{R} = (-\infty, +\infty)$ and $I^X$ denote a family of all fuzzy subset on $X$. A fuzzy set which takes the constant value $r \in I$ on $X$ is denoted by $\underline{r}$. A fuzzy set on $X$ is called a fuzzy point, denoted by $x_\lambda$, if it takes the value $0$ for all $y \in X$ with $y \neq x$ and the value at $x$ is $\lambda \in (0, 1]$. We denote the set of all fuzzy points on $X$ by $\mathcal{P}(I^X)$. Let $U \subseteq I^X$, a fuzzy point $x_\lambda$ on $X$ is said to be quasi-coincident with $U$, denoted by $x_\lambda \in U$, if $U(x) > 1 - \lambda$.

An $I$-topology (i.e., fuzzy topology in Chang’s sense [1]) on $X$ is a collection of fuzzy sets of $X$ closed under arbitrary unions and finite intersections, which contains the characteristic functions of $\emptyset$ and $X$, denoted $\underline{0}$ and $\underline{1}$ respectively. The pair $(X, T)$ is called an $I$-topological space if $T$ is an $I$-topology on $X$. An $I$-topology $T$ on $X$ is said to be stratified if $\underline{r} \in T$ for each $r \in I$.

**Definition 2.1.** [1] Let $(X, T)$ be an $I$-topological space and $x \in X$. A fuzzy set $U$ on $X$ is called a $W$-neighborhood of $x$ if there exists $G \in T$ such that $G \subseteq U$ and $G(x) = U(x) > 0$.

**Definition 2.2.** [8] Let $(X, T)$ be an $I$-topological space and $x \in X$. A family $B_x$ of $W$-neighborhoods of $x$ is called a $W$-neighborhood base of $x$ if for each $W$-neighborhood $P$ of $x$ and $\alpha \in [0, P(x))$, there exists $U \in B_x$ such that $U \subseteq P$ and $U(x) > \alpha$.

**Definition 2.3.** [9] Let $(X, T)$ be an $I$-topological space and $x_\lambda$ be a fuzzy point on $X$. A fuzzy set $U$ of $X$ is called a $Q$-neighborhood of $x_\lambda$ if there exists $G \in T$ such that $x_\lambda \in G \subseteq U$.

A family $U_{x_\lambda}$ of $Q$-neighborhoods of $x_\lambda$ is called a $Q$-neighborhood base of $x_\lambda$ if for each $Q$-neighborhood $A$ of $x_\lambda$ there exists $U \in U_{x_\lambda}$ such that $U \subseteq A$.

Let $X$ be a vector space over the field $\mathbb{K}$ (or $\mathbb{C}$), $A, B \in I^X$ and $k \in \mathbb{K}$. For the definitions of $A + B$ and $kA$, see [7] or [6].

**Definition 2.4.** [8] Let $X$ be a vector space over the field $\mathbb{K}$ and $T$ be a stratified $I$-topology on $X$. If the following two mappings (the addition and the scalar multiplication):

$$f : X \times X \to X, \quad (x, y) \mapsto x + y \quad \text{and} \quad g : \mathbb{K} \times X \to X, \quad (k, x) \mapsto kx,$$

are both $I$-continuous, where $\mathbb{K}$ is equipped with the usual induced $I$-topology $\omega(J_\mathbb{K})$, $X \times X$ and $\mathbb{K} \times X$ are equipped with the corresponding product $I$-topologies $T \times T$ and $\omega(J_\mathbb{K}) \times T$, respectively. Then the pair $(X, T)$ is called an $I$-topological vector space (for short, $I$-tvs).

**Proposition 2.5.** [12, 2] Let $(X, T)$ be an $I$-tvs.

1. The addition mapping $f$ is $I$-continuous iff for each fuzzy point $(x, y)_\lambda$ on $X \times X$ and each $Q$-neighborhood $W$ of $(x + y)_\lambda$, there exist $Q$-neighborhoods $U$ of $x_\lambda$ and $V$ of $y_\lambda$ such that $U + V \subseteq W$.

2. The scalar multiplication mapping $g$ is $I$-continuous iff for each fuzzy point $(k, x)_\lambda$ on $\mathbb{K} \times X$ and each $Q$-neighborhood $W$ of $kx_\lambda$, there exist a $Q$-neighborhood $U$ of $x_\lambda$ and $\delta > 0$ such that $tU \subseteq W$ for all $t \in \mathbb{K}$ with $|t - k| < \delta$. 
Proposition 2.6. [2] Let \((X, T)\) be an \(I\)-tvs and \(B \subseteq I^X\). Then \(B\) is a \(W\)-neighborhood of \(\theta\) iff \(U(\theta) > 0\) for each \(U \in B\), and for each \(\lambda \in (0, 1]\),

\[
U_\lambda = \{ U \in B \mid U(\theta) > 1 - \lambda \}
\]

is a \(Q\)-neighborhood base of \(\theta_\lambda\) in \((X, T)\).

Definition 2.7. [10] Let \(L_1\) and \(L_2\) be two completely distributive lattices with order-reversing involution \(\ell\). 0 and 1 are respectively their smallest and greatest elements. A mapping \(\phi: L_1 \to L_2\) is called an order-homomorphism, if the following conditions hold:

(OH-1) \(\phi\) is union-preserving, i.e., \(f(\bigvee a_k) = \bigvee f(a_k)\);

(OH-2) \(\phi\) is complement-preserving, i.e., for each \(b \in L_2\), \(\phi^*(b') = [\phi^*(b)]'\),

where the mapping \(\phi^*: L_2 \to L_1\) is given by \(\phi^*(b) = \bigvee \{a \in L_1 \mid \phi(a) \leq b\}\).

3. The Local Bases with Stratified Structure

Definition 3.1. Let \(U \subseteq I^X\). A family \(\{ U_\lambda \}_{\lambda \in [0, 1]}\) of nonempty subsets of \(U\) is called the stratified structure of \(U\) if the following two equalities hold: \(U = \bigcup_{\lambda \in [0, 1]} U_\lambda\) and \(U = \bigcup_{\lambda \in [0, 1]} U_\lambda\).

Remark 3.2. It is easy to prove that if \(\{ U_\lambda \}_{\lambda \in (0, 1]}\) is a family of nonempty subsets of \(U\) and \(U = \bigcup_{\lambda \in (0, 1]} U_\lambda\), then \(\{ U_\lambda \}_{\lambda \in (0, 1]}\) is the stratified structure of \(U\) if and only if the following conditions are satisfied:

1. \(0 < \mu < \lambda \leq 1\) implies that \(U_\mu \subseteq U_\lambda\);  
2. For each \(U \in U_\lambda\) there exists \(\mu_0 \in (0, \lambda)\) such that \(U \in U_{\mu_0}\).

Example 3.3. Let \(X = \mathbb{R}\), \(U = \{ (-a, a) \cap \mathbb{R} \mid a > 0, 0 < r \leq 1\}\). For each \(\lambda \in (0, 1]\), define a subset \(U_\lambda\) of \(U\) by \(U_\lambda = \{ (-a, a) \cap \mathbb{R} \mid a > 0, r \in (1 - \lambda, 1]\}\). Then it is easy to verify that \(\{ U_\lambda \}_{\lambda \in (0, 1]}\) is the stratified structure of \(U\).

Example 3.4. Let \((X, T)\) be an \(I\)-topological space and \(x \in X\), and let \(B_\lambda(x)\) be a \(Q\)-neighborhood base of \(x_\lambda\) for each \(\lambda \in (0, 1]\). We define

\[
U_\lambda(x) = \bigcup_{0 < \mu < \lambda} B_\mu(x) \quad \text{and} \quad U(x) = \bigcup_{\lambda \in (0, 1]} U_\lambda(x).
\]

Then \(\{ U_\lambda(x) \}_{\lambda \in [0, 1]}\) is the stratified structure of \(U(x)\).

To prove this conclusion, it suffices to show that \(U_\lambda(x) = \bigcup_{0 < \mu < \lambda} U_\mu(x)\). It is evident that \(\bigcup_{0 < \mu < \lambda} U_\mu(x) \subseteq U_\lambda(x)\). On the other hand, if \(U \in U_\lambda(x)\), then by the first equality of (3.1), there exists \(\mu \in (0, \lambda)\) such that \(U \in B_\mu(x)\). Taking \(\lambda_0 \in (\mu, \lambda]\), it follows that \(U \in U_{\lambda_0}(x)\), and so \(U \in \bigcup_{0 < \mu < \lambda} U_\mu(x)\). Therefore \(\{ U_\lambda(x) \}_{\lambda \in [0, 1]}\) is the stratified structure of \(U(x)\).

Definition 3.5. Let \((X, T)\) be an \(I\)-tvs. A family \(U\) of fuzzy subsets on \(X\) is called a local base with the stratified structure \(\{ U_\lambda \}_{\lambda \in [0, 1]}\) in \((X, T)\) if \(\{ U_\lambda \}_{\lambda \in [0, 1]}\) is its stratified structure and \(U_\lambda\) is a \(Q\)-neighborhood base of \(\theta_\lambda\) in \((X, T)\) for each \(\lambda \in (0, 1]\).
Example 3.6. Let \((X, \| \cdot \|, L, R)\) be a fuzzy normed space (see [4] and [13]). For a fixed \( \varepsilon > 0 \), the fuzzy set \( B_\varepsilon \) on \( X \) is defined by
\[
B_\varepsilon(x) = \sup \{ 1 - \alpha : \| x \|_2 < \varepsilon \}.
\]
By Theorem 3.1 in [13], we know that if \( R \leq \max \), then there exists an \( I \)-topology \( T_{\| \cdot \|} \) on \( X \) such that \((X, T_{\| \cdot \|})\) is an \( I \)-topological vector space and
\[
U_\lambda = \{ B_\varepsilon \cap \mathbb{R}^n \mid \varepsilon > 0, \, r \in (1 - \lambda, 1) \}
\]
is a \( Q \)-neighborhood base of \( \theta_\lambda \) for each \( \lambda \in (0, 1) \). In addition, by using Remark 3.1, it is not difficult to prove that the family of fuzzy subsets
\[
U = \{ B_\varepsilon \cap \mathbb{R}^n \mid \varepsilon > 0, \, r \in (0, 1) \}
\]
has the stratified structure \( \{ U_\lambda \}_{\lambda \in (0, 1)} \). Therefore \( U \) is a local base with the stratified structure \( \{ U_\lambda \}_{\lambda \in (0, 1)} \) in \((X, T_{\| \cdot \|})\).

Theorem 3.7. Every \( I \)-tvs has a balanced local base with stratified structure.

Proof. We know that every \( I \)-tvs \((X, T)\) has a balanced \( Q \)-neighborhood base \( B_\lambda \) of \( \theta_\lambda \) for each \( \lambda \in (0, 1) \) (see [12]). Now define
\[
U_\lambda = \bigcup_{\alpha \in (0, \lambda)} B_\alpha \quad \text{and} \quad U = \bigcup_{\lambda \in (0, 1)} U_\lambda.
\]
By Example 3.4, we know that \( \{ U_\lambda \}_{\lambda \in (0, 1)} \) is the stratified structure of \( U \).
It remains to prove that for each \( \lambda \in (0, 1] \), \( U_\lambda \) is a \( Q \)-neighborhood base of \( \theta_\lambda \). It is not difficult to see that members of \( U_\lambda \) are all \( Q \)-neighborhoods of \( \theta_\lambda \). Now let \( W \) be an arbitrary \( Q \)-neighborhood of \( \theta_\lambda \). It is evident that there exists \( \mu \in (0, \lambda) \) such that \( W \) is also \( Q \)-neighborhood of \( \theta_\mu \). Since \( B_\mu \) is a \( Q \)-neighborhood base of \( \theta_\mu \), there exists \( U \subset B_\mu \) such that \( U \subset W \). Note that \( B_\mu \subset U_\lambda \). This shows that \( U_\lambda \) is a \( Q \)-neighborhood base of \( \theta_\lambda \). Therefore \( U \) is a local base with stratified structure in \((X, T)\).

Theorem 3.8. Every \( W \)-neighborhood base \( B \) of \( \theta \) in an \( I \)-tvs \((X, T)\) is a local base with the stratified structure \( \{ U_\lambda \}_{\lambda \in (0, 1]} \), where \( U_\lambda = \{ U \in B \mid U(\theta) > 1 - \lambda \} \).

Proof. Since \( B \) is a \( W \)-neighborhood base of \( \theta \), by Proposition 2.6 we know that \( U_\lambda = \{ U \in B \mid U(\theta) > 1 - \lambda \} \) is a \( Q \)-neighborhood base of \( \theta_\lambda \) for each \( \lambda \in (0, 1] \), and \( B = \bigcup_{\lambda \in (0, 1]} U_\lambda \). In the following, we need only to show that \( U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu \).

By the definition of \( U_\lambda \), it is evident that \( \mu < \lambda \) implies that \( U_\mu \subset U_\lambda \). Hence \( \bigcup_{\mu \in (0, \lambda)} U_\mu \subset U_\lambda \). On the other hand, if \( U \in U_\lambda \), then \( U \in B \) and \( U(\theta) > 1 - \lambda \). Taking \( \mu \in (1 - U(\theta), \lambda] \), it is not difficult to see that \( U \in U_\mu \), and so \( U \in \bigcup_{\mu \in (0, \lambda)} U_\mu \). This shows that \( U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu \). Therefore \( B \) is a local base with the stratified structure \( \{ U_\lambda \}_{\lambda \in (0, 1]} \) in \((X, T)\).

As a special case of Theorem 3.8, we have the following example.
Example 3.9. Let \((X, T)\) be an I-tvs and \(U = \{ G \in I^X \mid G \in T, G(\theta) > 0 \}\). It is evident that \(U\) is an open \(W\)-neighborhood base of \(\theta\) in \((X, T)\). Hence, by Theorem 3.8, we conclude that \(U\) is a local base with the stratified structure \(\{U_\lambda\}_{\lambda \in (0, 1]}\) in \((X, T)\), where \(U_\lambda = \{ U \in T \mid U(\theta) > 1 - \lambda \}\).

4. Main Results

Theorem 4.1. Let \((X, T)\) be an I-tvs. Then the local base \(U\) with stratified structure \(\{U_\lambda\}_{\lambda \in (0, 1]}\) in \((X, T)\) has the following properties:

1. If \(U \in U_\lambda\), \(V \in U_\lambda\) or \(V = \emptyset (1 - \lambda < r \leq 1)\), then there exists \(W \in U_\lambda\) such that \(W \subseteq U \cap V\);
2. If \(U \in U_\lambda\), then there exists \(V \in U_\lambda\) such that \(V + V \subseteq U\);
3. If \(U \in U_\lambda\), then there exists \(V \in U_\lambda\) such that \(kV \subseteq U\) for all \(|k| \leq 1\); 
4. If \(U \in U_\lambda\), then for each \(x \in X\), there is an \(\alpha > 0\) such that \(x_\lambda \in \alpha U\).

Conversely, let \(X\) be a vector space over \(\mathbb{K}\) and \(U\) be a family of fuzzy subsets on \(X\) with the stratified structure \(\{U_\lambda\}_{\lambda \in (0, 1]}\) which satisfies the above conditions (1)–(4). Then there exists a unique I-topology \(T\) on \(X\) such that \((X, T)\) is an I-tvs and \(U\) is a local base with the stratified structure \(\{U_\lambda\}_{\lambda \in (0, 1]}\) in \((X, T)\).

Proof. (1) is evident. The proofs of (2), (3) and (4) are easy by Proposition 2.5 (see also the proof of Theorem 4.1 in [2]).

Conversely, without loss of generality, we can suppose that every member \(U\) in \(U\) satisfies \(U(\theta) = \sup_{x \in X} U(x)\). Otherwise, we substitute \(\tilde{U} = \bigcup_{\lambda \in (0, 1]} \tilde{U}_\lambda\) for \(U\), where \(\tilde{U}_\lambda = \{ U \cap \tilde{U} \mid U \in U_\lambda \}\). Obviously, \(\{ \tilde{U}_\lambda \}_{\lambda \in (0, 1]}\) is the stratified structure of \(\tilde{U}\) and satisfies the conditions (1)–(4). We set

\[ T = \{ G \in I^X \mid \forall x_\mu \in G, \exists V \in U_\mu\ such\ that \ x + V \subseteq G \}. \]

Then it is not difficult to verify that \(T\) is a stratified I-topology on \(X\).

In fact, for each \(r \in (0, 1]\), if \(x_\lambda \in U_\lambda\), i.e., \(r > 1 - \lambda\), then the condition (1) implies that there exists \(W \in U_\lambda\) such that \(W \subseteq \emptyset\) and so \(x + W \subseteq x + \emptyset = x\). This shows \(x \in T\). Moreover, it is easy to show that \(T\) is closed under arbitrary unions and finite intersections. Therefore \(T\) is a stratified I-topology on \(X\).

For each \(U \in U\), define a fuzzy set \(G_U\) on \(X\) as follows:

\[ G_U(x) = \sup \{ 1 - \mu \mid x + V \subseteq U \text{ for some } V \in U_\mu \}. \]  

We can prove the following conclusions:

(a) \(G_U \in T\); (b) \(G_U \subset U\); (c) \(U \in U_\lambda\) \(\Rightarrow \theta_\lambda \subseteq G_U\).

(a) Let \(x_\mu \in G_U\), i.e., \(G_U(x) > 1 - \mu\). By (4.1), there exist \(\nu \in (0, \mu)\) and \(V \in U_\nu\) such that \(x + V \subseteq U\). Note that \(\nu < \mu\) implies \(U_\nu \subseteq U_\mu\). Hence \(V \in U_\mu\). By (2), there exists \(P \in U_\mu\) such that \(P + P \subseteq V\), and so \(x + P + P \subseteq x + V \subseteq U\). Since \(P \in U_\mu = \bigcup_{\alpha \in (0, \mu)} U_\alpha\), there exists \(\delta \in (0, \mu)\) such that \(P \in U_\mu - \delta\). By (1),
for each $\varepsilon > 0$, there exists $W \in U_{\mu - \delta}$ such that $W \subseteq P \cap (1 - \mu + \delta + \varepsilon)$. We can prove that $x + W \subseteq G_U$.

In fact, if $(x + W)(z) = \alpha > 0$, then $W(z - x) = \alpha$, and so

$$(x + W + W)(w) = (W + W)(w - x) \geq W(w - z) \wedge W(z - x)$$
$$= W(w - z) \wedge \alpha = (z + W \cap \alpha)(w), \quad \forall w \in X,$$

which implies that $z + W \cap \alpha \subseteq x + W \subseteq x + P + P \subseteq U$. Note that

$$1 - \mu + \delta + \varepsilon \geq W(z - x) = \alpha \implies 1 - \alpha + \varepsilon \geq \mu - \delta \implies U_{\mu - \delta} \subseteq U_{1 - \alpha + \varepsilon}.$$

Hence $W \in U_{1 - \alpha + \varepsilon}$. Since $\alpha > 1 - (1 - \alpha + \varepsilon)$, by (1) there exists $W_0 \in U_{1 - \alpha + \varepsilon}$ such that $W_0 \subseteq W \cap \alpha$. So, we have $z + W_0 \subseteq z + W \cap \alpha \subseteq U$. It follows from (4.1) that $G_U(z) \geq \alpha - \varepsilon$. By the arbitrariness of $\varepsilon$, we get $G_U(z) \geq \alpha$. This shows that $x + W \subseteq G_U$. Note that $W \in U_{\mu - \delta} \subseteq U_{\mu}$. Hence $G_U \subseteq T$.

(b) If $x \in X$ and $G_U(x) = \alpha > 0$, then by (4.1) for each $\varepsilon \in (0, \alpha)$, there exist $\mu \in (0, 1]$ with $1 - \mu > \alpha - \varepsilon$ and $V \in U_{\mu}$ such that $x + V \subseteq U$. This implies that $U(x) \supseteq V(\theta) > 1 - \mu > \alpha - \varepsilon$. By the arbitrariness of $\varepsilon$, we get $U(x) \supseteq \alpha = G_U(x)$. This shows that $G_U \subseteq U$.

(c) Let $U \in U_\lambda$. Note that $U_\lambda = \bigcup_{\mu \in (0, \lambda)} U_\mu$. Then there exists $\mu \in (0, \lambda)$ such that $U \subseteq U_\mu$. Since $\theta + U = U$, by (4.1) we have $G_U(\theta) \geq 1 - \mu > 1 - \lambda$, i.e., $\lambda \in G_U$.

From (a)–(c), it is easy to see that every member in $U_\lambda$ is a $Q$-neighborhood of $\theta_\lambda$ in $(X, T)$. Again, by the definition of $T$, we conclude that $U_\lambda$ is a $Q$-neighborhood base of $\theta_\lambda$ and $T$ is a translation invariant, i.e., $G \in T$ implies that $x + G \in T$ for each $x \in X$. Therefore $U_{\lambda x} = \{x + U \mid U \subseteq U_\lambda\}$ is a $Q$-neighborhood base of $x_\lambda$.

Finally, we prove that the addition $f$ and the scalar multiplication $g$ are both $T$-continuous.

Let $(x, y)_\lambda$ be a fuzzy point on $X \times X$ and $W$ be a $Q$-neighborhood of $(x + y)_\lambda$. Then there exists $U \subseteq U_{\lambda x}$ such that $(x + y) + U \subseteq W$. By (2), there exists $V \subseteq U_\lambda$ such that $V + V \subseteq U$, and so we have $(x + V) + (y + V) \subseteq (x + y) + U \subseteq W$. Note that $x + V \subseteq U_{\lambda x}$, $y + V \subseteq U_{\lambda y}$. By Proposition 2.5, we infer that $f$ is $T$-continuous.

Let $(x_0, x)_\lambda$ be a fuzzy point on $\mathbb{K} \times X$ and $W$ be a $Q$-neighborhood of $(x_0 x)_\lambda$. Then there exists $U \subseteq U_{\lambda x}$ such that $k_0 x + U \subseteq W$. By (2), there exists $V \subseteq U_\lambda$ such that $V + V \subseteq U$. By (3), there exists $V_1 \subseteq U_{\lambda x}$ such that $V_1 \subseteq V$ for all $|t| \leq 1$. By (4), there exists a $\beta > 0$ such that $x_\lambda \in \beta V_1$. Put $s = (\beta V_1)(x)$, then $s > 1 - \lambda$, and so we have

$$r x_\lambda \in r \beta V_1 \subseteq V \quad \text{for all } r \in \mathbb{K} \text{ with } |r| \leq 1/\beta.$$  

(4.2)

Put $N = \lceil |k_0| \rceil + 1$, where $\lceil |k_0| \rceil$ is the largest integral part of $|k_0|$. Take $n \in \mathbb{N}$ with $2^n \geq N$. Repeatedly make use of (2), there exists $V_2 \subseteq U_{\lambda x}$ such that $V_2 + \cdots + V_2 \subseteq V$. Again by (3), there exists $V_3 \subseteq U_{\lambda x}$ such that $t V_3 \subseteq V_2$ for all $|t| \leq 1$. Put $\delta_1 = N - \lceil |k_0| \rceil$, then when $|k - k_0| \leq \delta_1$ we have
\[ kV_2 \subset \bigwedge_{V_2 \subset V_0 + \cdots + V_2} \subset V \quad \text{(because } V_2(\theta) = \sup_{x \in X} V_2(x)\text{).} \quad (4.3) \]

Let \( \delta = \min\{\delta_1, 1/\beta\} \). It follows from (4.2) and (4.3) that

\[ (k - k_0) x + (kV_3) \cap \check{\mathfrak{a}} = (k - k_0) x + (kV_3) \cap \check{\mathfrak{a}} \subset V + V \subset U \quad (4.4) \]

whenever \(|k - k_0| \leq \delta\), which implies that

\[ k(x + V_3 \cap \check{\mathfrak{a}}) = kx + kV_3 \cap \check{\mathfrak{a}} \subset k_0 x + U \subset W. \quad (4.5) \]

whenever \(|k - k_0| \leq \delta\). Note that \( x + V_3 \cap \check{\mathfrak{a}} \) is a \( Q \)-neighborhood of \( x_\lambda \). By (4.5) and Proposition 2.5, we conclude that \( g \) is \( I \)-continuous. This shows that \( (X, T) \) is an \( I \)-topological vector space and \( U_\lambda \) is a \( Q \)-neighborhood base of \( \theta \) for each \( \lambda \in (0, 1] \). Note that \( \{U_\lambda\}_{\lambda \in (0, 1]} \) is the stratified structure of \( U \). Therefore \( U \) is a local base with the stratified structure \( \{U_\lambda\}_{\lambda \in (0, 1]} \) in \( (X, T) \). \( \square \)

**Remark 4.2.** Theorem 4.1 is an improvement of Theorem 4.1 in [2]. Here, it is necessary to point out that in the proof of Theorem 4.1 in [2], the argument for continuity of \( g \) is incorrect.

In fact, to prove the continuity of \( g \), it suffices to show that for each fuzzy point \((k_0, x)_\lambda\) on \( K \times X \) and each \( Q \)-neighborhood \( W \) of \( k_0 x_\lambda \), there exist \( \delta > 0 \) and \( V_0 \in U_\lambda \) such that \( k \varepsilon + V_0 \subset W \) for all \( k \in K \) with \(|k - k_0| < \delta\). Please notice that the mentioned \( V_0 \) is relative to \( k \). However, in the proof of [2] (see page 344-345), the arisen \( V^{(2)} \) in (4.2) is relative to \( k \) and \( V_0 \) is relative to \( V^{(2)} \) and \( V^{(4)} \), \( V_0 \subset V^{(2)} \cap V^{(4)} \), and so \( V_0 \) is also relative to \( k \). Obviously, we need not such a \( V_0 \).

Imitating the proof of Theorem 4.1, we easily correct the faults in [2].

As a consequence of Theorem 4.1, we obtain the following theorem:

**Theorem 4.3.** Let \((X, T)\) be an \( I \)-fuzzy and \( B \) be a \( W \)-neighborhood base of \( \theta \). The \( B \) has the following properties:

- (i) \( U \in B \) implies that \( U(\theta) > 0 \).
- (ii) If \( U \in B, V \in B \) or \( V = \bigwedge_0 \frac{1}{\alpha} < U(\theta) \wedge V(\theta) \), then there exists \( W \in B \) with \( W(\theta) > \alpha \) such that \( W \subset U \cap V \).
- (iii) If \( U \in B \) with \( U(\theta) > \alpha \geq 0 \), then there exists \( V \in B \) with \( V(\theta) > \alpha \) such that \( V + V \subset U \).
- (iv) If \( U \in B \) with \( U(\theta) > \alpha \geq 0 \), there exists \( V \in B \) with \( V(\theta) > \alpha \) such that \( tV \subset U \) for all \( t \in K \) with \(|t| \leq 1 \).
- (v) If \( U \in B \) with \( U(\theta) > \alpha \geq 0 \), then for any \( x \in X \), there exists \( t > 0 \) such that \( \langle U \rangle(x) \alpha > \alpha \).

Conversely, let \( X \) be a vector space over \( K \) and \( B \) be a family of fuzzy sets on \( X \) satisfying the above conditions (i)-(v). Then there exists a unique \( I \)-topology \( T \) on \( X \) such that \((X, T)\) is an \( I \)-fuzzy and \( B \) is a \( W \)-neighborhood base of \( \theta \).

**Proof.** Put \( U_\lambda = \{ U \in B \mid U(\theta) > 1 - \lambda \}, \alpha = 1 - \lambda \). Then it is easy to see that (ii)-(iv) are equivalent to (1)-(4) in Theorem 4.1. Moreover, if \( B \) is a \( W \)-neighborhood base of \( \theta \), then by Theorem 3.8 we know that \( B \) is a local base.
with the stratified structure \( \{U_{\lambda}\}_{\lambda \in (0,1]} \). Therefore the conclusion can follows from Theorem 4.1 and Proposition 2.6 immediately. \( \square \)

**Remark 4.4.** Theorem 4.3 is an improvement of Theorem 4.2 in [2] and Theorem 4.3 in [8]. The conditions (i)-(vii) of Theorem 4.3 in [8] have been simplified.

As applications of Theorems 4.1 and 4.3, we give the following two examples.

**Example 4.5.** Let \( C(\mathbb{R}) \) denote the set of all continuous real valued functions on \( \mathbb{R} \). For each \( \alpha \in (0,1] \), we define a mapping \( \| \cdot \|_\alpha : C(\mathbb{R}) \to [0, \infty) \) by

\[
\|x\|_\alpha = \int_{-\infty}^{\infty} |x(t)| \, dt.
\]

(4.6)

Obviously, \( \| \cdot \|_\alpha \) has the following properties:

- (a) \( 0 < \beta < \alpha \leq 1 \) implies \( \|x\|_\alpha \leq \|x\|_\beta \) for all \( x \in C(\mathbb{R}) \);
- (b) \( \|x + y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha \) for all \( x, y \in C(\mathbb{R}) \).

For each \( \alpha \in (0,1] \) and \( r > 0 \), we define a fuzzy set \( U_{\alpha,r} \) on \( X \) by

\[
U_{\alpha,r}(x) = \begin{cases} 
\alpha & \text{if } \|x\|_\alpha < r, \\
0 & \text{otherwise}
\end{cases}
\]

(4.7)

where \( \|x\|_\alpha \) is defined by (4.6). Then by Theorem 4.1, we can prove that there exists a unique \( I \) topology \( T \) on \( C(\mathbb{R}) \) such that \( (C(\mathbb{R}), T) \) is an \( I \)-tvS and the family of fuzzy sets \( U = \{U_{\alpha,r} \mid \alpha, r \in (0,1]\} \) is a local base with the stratified structure \( \{U_{\lambda}\}_{\lambda \in (0,1]} \) in \( (C(\mathbb{R}), T) \), where

\[
U_{\lambda} = \{U_{\alpha,r} \mid \alpha \in (1-\lambda, 1], \ r > 0\}.
\]

(4.8)

In fact, it is easy to prove that \( \{U_{\lambda}\}_{\lambda \in (0,1]} \) is the stratified structure of \( U \). It remains to prove that \( \{U_{\lambda}\}_{\lambda \in (0,1]} \) satisfies the conditions (1)-(4) in Theorem 4.1.

By (4.7) it is easy to see that \( U_{\alpha,r} = B_{\alpha,r} \cap \tilde{\alpha} \) where

\[
B_{\alpha,r} = \{x \in X \mid \|x\|_\alpha < r\}.
\]

(4.9)

(1) Let \( U_{\alpha,r_i} \in U_{\lambda} \ (i = 1, 2) \). Then \( U_{\alpha,r_1} = B_{\alpha,r_1} \cap \alpha_i \), \( \alpha_i > 1 - \lambda \ (i = 1, 2) \). Put \( \alpha = \min\{\alpha_1, \alpha_2\} \) and \( r = \min\{r_1, r_2\} \). It is obvious that \( U_{\alpha,r} \in U_{\lambda} \) and \( U_{\alpha,r} = B_{\alpha,r} \cap \tilde{\alpha} \subseteq [B_{\alpha,r_1} \cap B_{\alpha,r_2}] \cap [B_{\alpha,r_1} \cap \tilde{\alpha}] = U_{\alpha_1,r_1} \cap U_{\alpha_2,r_2} \).

Let \( U_{\alpha,r} \in U_{\lambda} \) and \( \sigma \in (1-\lambda, 1] \). Then \( \beta = \min\{\alpha, \sigma\} \in (1-\lambda, 1] \), and so \( U_{\beta,r} \in U_{\lambda} \) and \( U_{\beta,r} = B_{\beta,r} \cap \tilde{\beta} \subseteq B_{\alpha,r} \cap \tilde{\alpha} \cap \tilde{\sigma} = U_{\alpha,r} \cap \tilde{\sigma} \).

(2) Let \( U_{\alpha,r} \in U_{\lambda} \). Then \( U_{\alpha,r} \subseteq U_{\lambda} \) and

\[
U_{\alpha,r} = \alpha + B_{\alpha,r} \cap \tilde{\alpha} \subseteq B_{\alpha,r} \cap \tilde{\alpha} = U_{\lambda}.
\]

(3) Let \( U_{\alpha,r} = B_{\alpha,r} \cap \tilde{\alpha} \in U_{\lambda} \). By (4.6) and (4.9), it is not difficult to prove that \( kB_{\alpha,r} \subseteq B_{\alpha,r} \) for all \( |k| \leq 1 \). Hence, we have \( kU_{\alpha,r} \subseteq U_{\alpha,r} \) for all \( |k| \leq 1 \).

(4) Let \( U_{\alpha,r} = B_{\alpha,r} \cap \tilde{\alpha} \in U_{\lambda} \). For each \( x \in X \), we take \( t > \frac{\|x\|_\alpha}{\|\tilde{\alpha}\|_\alpha} \). It follows that \( \|(1/t)x\|_\alpha < r \), i.e., \( (1/t)x \in B_{\alpha,r} \). Note that \( \alpha > 1 - \lambda \). Then we have \( (1/t)x_{\lambda} \subseteq B_{\alpha,r} \cap U_{\alpha,r} \), i.e., \( x_{\lambda} \in U_{\alpha,r} \).
Thus, by Theorem 4.1 we know that the conclusion of Example 4.5 holds.

**Remark 4.6.** Similarly, it is not difficult to verify that $U$ in Example 4.5 satisfies the conditions (i)–(v) of Theorem 4.3. Therefore, by Theorem 4.3, $U$ is a $W$-neighborhood base of $\theta$ in the I-tvs $(G(\mathbb{R}), T)$.

Let $X$ be a real vector space and $\tilde{f}$ be a fuzzy linear functional on $\text{Pt}(I^X)$, i.e., a fuzzy linear operator from $\text{Pt}(I^X)$ into $\text{Pt}(I^\mathbb{R})$ (See [3, Definition 4] or [14, Definition 2.3]). By [3, Theorem 4] or [14, Lemma 2.5], we know that $\tilde{f}$ is a fuzzy linear functional on $\text{Pt}(I^X)$ if and only if there exist an ordinary linear functional on $X$ and an order-homomorphism preserving finite meets $\phi : I \rightarrow I$ such that

$$\tilde{f}(x_\lambda) = [f(x)]_{\phi}(\lambda) \quad \text{for all } x_\lambda \in \text{Pt}(I^X).$$

In the following, for convenience, we use $(f, \phi)^{-\star}$ instead of the fuzzy linear functional $\tilde{f}$. Moreover, we define the mapping $|\cdot| : \text{Pt}(I^\mathbb{R}) \rightarrow [0, +\infty)$ by $|y_\mu| = (1/\mu)|\bar{y}|$ for all $y_\mu \in \text{Pt}(I^\mathbb{R})$.

**Example 4.7.** Let $X$ be a real vector space and $\tilde{C}_\phi(X)$ denote the set of all fuzzy linear functionals $(f, \phi)^{-\star}$ on $\text{Pt}(I^X)$ with the same $\phi$, i.e., $\tilde{C}_\phi(X) = \{(f, \phi)^{-\star} : f \in X^\#\}$, where $X^\#$ denotes the set of all ordinary linear functionals on $X$ and $\phi$ is a fixed order-homomorphism preserving finite meets from $I$ into $I$. For each $f \in X^\#$ and $t > 0$, we define a fuzzy set $B_{f,t}$ on $X$ by

$$B_{f,t}(x) = \text{sup}\{1 - \lambda : |(f, \phi)^{-\star}(x_\lambda)| < t\} \quad \text{for all } x \in X. \quad (4.10)$$

and put $U(f_1, \cdots, f_n, t) = \bigcap_{i=1}^n B_{f_i,t}$. Then by Theorem 4.3, we can prove that there exists a unique $I$-topology $T$ on $X$ such that $(X, T)$ is an I-tvs and the family of fuzzy sets

$$B = \{U(f_1, \cdots, f_n; t) \cap \mathfrak{L} : f_i \in X^\#, \; i = 1, \cdots, n, \; n \in \mathbb{N}; \; t > 0, \; r \in (0, 1)\}$$

is a $W$-neighborhood base of $\theta$ in $(X, T)$.

To prove this conclusion, we first prove that

$$x_\lambda \in B_{f,t} \iff |(f, \phi)^{-\star}(x_\lambda)| < t, \quad \text{i.e., } |f(x)| < \phi(\lambda)t. \quad (4.11)$$

In fact, if $x_\lambda \in B_{f,t}$, i.e., $B_{f,t}(x) > 1 - \lambda$, then by (4.10) there exists $\lambda_0 \in (0, \lambda)$ such that $|f(x)| < \phi(\lambda_0)t$. Note that $\phi$ is nondecreasing. Hence we have $|f(x)| < \phi(\lambda)t$. Conversely, let $|f(x)| < \phi(\lambda)t$. Note that $\phi$ is union-preserving. Taking $\lambda_n \in (0, \lambda) \wedge \lambda$, we have

$$|f(x)| < \phi \left( \bigvee_{n \in \mathbb{N}} \lambda_n \right) t = \bigvee_{n \in \mathbb{N}} \phi(\lambda_n)t.$$

Thus, there exists some $n_0 \in \mathbb{N}$ such that $|f(x)| < \phi(\lambda_{n_0})t$, i.e., $|(f, \phi)^{-\star}(x_{\lambda_{n_0}})| < t$. By (4.10) we get $B_{f,t}(x) > 1 - \lambda_{n_0} > 1 - \lambda$, i.e., $x_\lambda \in B_{f,t}$. (4.11) is proved.

By using (4.10) and (4.11), it is easy to prove that $B_{f,t}$ has the following properties:

(a) $B_{f,t}(\theta) = 1$;
(b) \(0 < s < t\) implies that \(B_{f,s} \subseteq B_{f,t}\);

(c) \(B_{f,t} + B_{f,t} \subseteq B_{f,t}\);

(d) \(B_{f,t}\) is balanced, i.e., \(kB_{f,t} \subseteq B_{f,t}\) for all \(|k| \leq 1\);

now we prove that \(B\) satisfies the conditions (i)–(v)

(i) Let \(W = U(f_1, \ldots, f_n; t) \cap \mathbb{B}\). By (a), we know that \(W(\theta) = r > 0\).

(ii) By (b), we have that \(U(f, g; t \wedge s) \subseteq B_{f,t} \cap B_{g,s}\) for all \(f, g \in \mathbb{X}^r\) and \(s > 0\).

So, if \(W_1 = U(f_1, \ldots, f_n; t) \cap r_1, W_2 = U(g_1, \ldots, g_m; s) \cap r_2 \in \mathbb{B}\) with \(W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0\), then there exists \(W = U(f_1, \ldots, f_n, g_1, \ldots, g_m; t \wedge s) \cap r_1 \wedge r_2 \in \mathbb{B}\) such that \(W(\theta) > \alpha\) and \(W \subseteq W_1 \cap W_2\).

If \(W_1 = U(f_1, \ldots, f_n; t) \cap r_1 \in \mathbb{B}\) and \(W_2 = r_2 > 0\) for \(W_1(\theta) \wedge W_2(\theta) > \alpha \geq 0\), then there exists \(W = U(f_1, \ldots; t) \cap r_1 \wedge r_2 \in \mathbb{B}\) such that \(W(\theta) > \alpha\) and \(W \subseteq W_1 \cap W_2\).

(iii) If \(W = U(f_1, \ldots, f_n; t) \cap \mathbb{B}\) and \(W(\theta) = r > \alpha \geq 0\), then it follows from (c) that

\[U(f_1, \ldots, f_n; \frac{t}{2}) + U(f_1, \ldots, f_n; \frac{t}{2}) \subseteq U(f_1, \ldots, f_n; t),\]

and so there exists \(V = U(f_1, \ldots, f_n; t) \cap \mathbb{B}\) such that \(V(\theta) = r > \alpha\) and \(V + V \subseteq W\).

(iv) Let \(W = U(f_1, \ldots, f_n, g_1, \ldots, g_m; t) \cap \mathbb{B}\) and \(W(\theta) > \alpha \geq 0\). By (d), we know that each \(B_{i,t}\) is balanced \((i = 1, \ldots, n)\). Hence \(U(f_1, \ldots, f_n, g_1, \ldots, g_m; t)\) is also balanced, and so we have \(W \subseteq W\) for all \(|k| \leq 1\).

(v) Let \(W = U(f_1, \ldots, f_n; t) \cap \mathbb{B}\) and \(W(\theta) > \alpha \geq 0\). For any \(x \in \mathbb{X}\), taking \(\beta > \frac{\max\{i \leq n \mid f_i(\theta)\}}{r(1-\alpha)}\), then we have \(|f_i(x)/\beta| < \phi(1-\alpha)t, i = 1, 2, \ldots, n\). It follows from (4.11) that \((x/\beta), r \in \bigcap_{i=1}^n B_{i,t} = U(f_1, \ldots, f_n; t)\). Note that \(r = W(\theta) > \alpha\). Hence \(W(x/\beta) = U(f_1, \ldots, f_n; t) \cap (x/\beta) > \alpha\) i.e., \((\beta W)(x) > \alpha\).

Thus, By Theorem 4.3, we have the conclusion of Example 4.7.

References


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